A note on Cauchy completeness for preorders

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Abstract

In this paper, we study the notion of Cauchy-complete preorder in a regular category, following work in [CS86], introducing the logic of a regular category. We give a different, stronger characterization than in *loc.cit.* for those preorders. Using this, we provide a new construction of the Cauchy-completion in a exact category.

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Introduction

The notion of a Cauchy-complete category was introduced in [Law73] and has proved to be of crucial importance in category theory. As the notion is of a 2-categorical nature the study of Cauchy-completeness has been approached in several different contexts, see [CS86, BD86] and the references therein.

In this paper we strengthen an existence theorem of [CS86], which provides a construction for the Cauchy-completion of a preorder in an arbitrary topos, to the more general case of an exact category, not only extending the proof of the cited result, but also showing that a Cauchy-completion may be obtained by a different construction which does not require any higher-order structure.

In section 1 we review some basic notions and well-known results about regular categories, and in section 2 we prove a characterization of Cauchy-complete preorders in a regular category which leads to the proof of the main result.

1 Basic notions

Definitions follow basically [CS86], but we shall diverge from those in at least one case. So we recall all notions we use in the present paper for sake of completeness.

The basic notion of the paper is that of a *regular* category (*à la* Barr, [Bar71]). We take the slightly more general definition proposed in [CS86]: a category with finite wide pullbacks and stable coequalizers of intersections of kernel pairs. The main property in a regular category is that every pair (n-ple) of maps can be factored as a coequalizer followed by a jointly monic pair (n-ple):

$$\begin{tikzcd}
& A \arrow[dl, "f_1"'] \arrow[d, "e"] \arrow[dr, "f_2"] & \\
B_1 & I \arrow[l, "i_1"'] \arrow[r, "i_2"] & B_2
\end{tikzcd}$$

with the pair (i_1, i_2) jointly monic. (Note that most of the common instances of regular categories have representation of n-ples, *i.e.* finite products, yet we find that it is that assuming only existence of wide pullbacks is a useful level of generality.) The crucial part of the proof of the above is to show that a coequalizer in a regular category *is* a coequalizer of its kernel pair. Because of this, very often one calls *regular* an epimorphism which appears as a coequalizer.

It is notationally convenient to introduce the Kripke-Joyal semantics for the $\exists \,\&\, {=}$-fragment of logic in a regular category $\mathbf{C}$. Consider a language with (some) objects of $\mathbf{C}$ as types, and (some) maps of $\mathbf{C}$ as sorted function symbols; terms are built from (typed) variables using the symbols. For a formula ϕ in the $\exists \,\&\, {=}$-fragment with free variables among $\mathtt{x}_1$ of type A_1,..., $\mathtt{x}_n$ of type A_n, define satisfaction/forcing

$$X \Vdash_{\mathtt{x}_1,\ldots,\mathtt{x}_n} \phi[x_1,\ldots,x_n]$$

at an arbitrary object X in $\mathbf{C}$ with respect to a suitable string of maps $(x_i\colon X \longrightarrow A_i)_{i=1,\dots,n}$ from X into the appropriate types/objects, by the following clauses:

- $X \Vdash_{\mathtt{x}_1,\dots,\mathtt{x}_n} (t = t')[x_i]_i$ if the composite in $\mathbf{C}$ of the string $t[x_i/\mathtt{x}_i]$ equals that of the string $t'[x_i/\mathtt{x}_i]$,
- $X \Vdash_{\mathtt{x}_1,\dots,\mathtt{x}_n} (\psi \,\&\, \theta)[x_i]_i$ if $X \Vdash_{\mathtt{x}_1,\dots,\mathtt{x}_n} \psi[x_i]_i$ and $X \Vdash_{\mathtt{x}_1,\dots,\mathtt{x}_n} \theta[x_i]_i$,
- $X \Vdash_{\mathtt{x}_1,\dots,\mathtt{x}_n} (\exists \mathrm{y} \in A.\psi)[x_i]_i$ if there are a regular epi $e\colon Y \longrightarrow\!\!\triangleright\, X$ and a map $y\colon Y \longrightarrow A$ such that $Y \Vdash \psi_{\mathtt{x}_1,\dots,\mathtt{x}_n,\mathtt{y}}[x_1 \circ e, \dots, x_n \circ e, y]$.

We shall say that ϕ holds at X with respect to $(x_i)_i$, and usually drop the subscript variables when they are clear from the context. When $X \Vdash_{\mathtt{x}_1,\dots,\mathtt{x}_n} \phi[x_1,\dots,x_n]$ at all objects X with respect to all strings $(x_i)_i$, we say that ϕ holds in $\mathbf{C}$, and write $\mathbf{C} \Vdash \phi$.

The properties of the regular category $\mathbf{C}$ allow also to define the *extension* of a formula in the $\exists \,\&\, =$-fragment in a reasonable way. It need not apply in the general case to all formulae but only to those which are *hereditarily connected*. First introduce the relation of connectedness between variables in a formula as the reflexive and transitive closure of the relation that links two variables occuring in the same atomic formula. A formula is *connected* if the relation of connectedness it induces on its (free and bound) variables is total—note that this requires that a bound variable *must* appear in an atomic subformula. Finally, a formula is *hereditarily connected* if all of its subformulae are connected.

Suppose ϕ is a hereditarily connected formula in the $\exists \,\&\, =$-fragment with free variables among $\mathtt{x}_1$ of type A_1,…, $\mathtt{x}_n$ of type A_n; the *extension* of the formula with respect to the list $\mathtt{x}_1,\dots,\mathtt{x}_n$ is a jointly monic n-ple of maps

$$\llbracket\phi\rrbracket_{\mathtt{x}_1,\dots,\mathtt{x}_n}\colon \quad \begin{tikzcd} & & R \arrow[dll] \arrow[dl] \arrow[drr] & & \\ A_1 & A_2 & & \dots & A_n \end{tikzcd}$$

- for the case of an atomic formula $(t = t')$, one must consider the two cases depending on the number of distinct variables (1 or 2):

  one variable: $\llbracket t = t'\rrbracket_{\mathtt{x}_1,\dots,\mathtt{x}_n}$ is the equalizer of the composites of the strings $t[\mathrm{id}_{A_i}/\mathtt{x}_i]$ and $t'[\mathrm{id}_{A_i}/\mathtt{x}_i]$,

  two variables: $\llbracket t = t'\rrbracket_{\mathtt{x}_1,\dots,\mathtt{x}_n}$ is the pullback of the composites of the strings $t[\mathrm{id}_{A_i}/\mathtt{x}_i]$ and $t'[\mathrm{id}_{A_i}/\mathtt{x}_i]$—note that this clause gives rise to a jointly monic pair,
- $\llbracket\psi \,\&\, \theta\rrbracket_{\mathtt{x}_1,\dots,\mathtt{x}_n}$ is the intersection (= the diagonal composite in the wide pullback) of $\llbracket\psi\rrbracket_{\mathtt{x}_1,\dots,\mathtt{x}_n}$ and $\llbracket\theta\rrbracket_{\mathtt{x}_1,\dots,\mathtt{x}_n}$—note the crucial use of the hypothesis of connectedness,
- $\llbracket\exists \mathrm{y} \in A.\psi\rrbracket_{\mathtt{x}_1,\dots,\mathtt{x}_n}$ is the jointly monic n-ple obtained factorizing all maps in $\llbracket\psi\rrbracket_{\mathtt{x}_1,\dots,\mathtt{x}_n,\mathtt{y}}$ but that into A.

The two ways of interpreting a formula in the $\exists \,\&\, =$-fragment are related by the following.

1.1 THEOREM *Suppose ϕ is a hereditarily connected formula with free variables among $\mathtt{x}_1$ of type A_1,…, $\mathtt{x}_n$ of type A_n, and $(x_i\colon X \longrightarrow A_i)_{i=1,\dots,n}$ is a sequence of maps. Then $X \Vdash \phi[x_i]$ if and only if the n-ple of maps $(x_i)_i$ has a common factor through the n-ple $\llbracket\phi\rrbracket$, the extension of ϕ with respect to $\mathtt{x}_1,\dots,\mathtt{x}_n$.*

There is an obvious expansion of the interpretation to sequents in the $\exists \,\&\, =$-logic: $X \Vdash_{\mathtt{x}_1,\dots,\mathtt{x}_n} (\phi \Rightarrow \psi)[x_1,\dots,x_n]$ if $X \Vdash_{\mathtt{x}_1,\dots,\mathtt{x}_n} \psi[x_1,\dots,x_n]$ whenever $X \Vdash_{\mathtt{x}_1,\dots,\mathtt{x}_n} \phi[x_1,\dots,x_n]$. And $\phi \Rightarrow \psi$ *holds in* $\mathbf{C}$ with respect to $\mathtt{x}_1,\dots,\mathtt{x}_n$, when $X \Vdash_{\mathtt{x}_1,\dots,\mathtt{x}_n} (\phi \Rightarrow \psi)[x_1,\dots,x_n]$ for all X and all $(x_i)_i$.

The relationship between forcing of sequents in $\mathbf{C}$ and extensions of formulae is a direct application of 1.1.

1.2 COROLLARY *Suppose ϕ and ψ are hereditarily connected. Then $\phi \Rightarrow \psi$ holds in $\mathbf{C}$ with respect to $\mathtt{x}_1, \ldots, \mathtt{x}_n$ if and only if $[\![\phi]\!]_{\mathtt{x}_1,\ldots,\mathtt{x}_n}$ factors through $[\![\psi]\!]_{\mathtt{x}_1,\ldots,\mathtt{x}_n}$.*

All axioms of $\exists\,\&\,{=}$-logic hold in $\mathbf{C}$ and all rules of $\exists\,\&\,{=}$-logic preserve truth in $\mathbf{C}$.

1.3 EXAMPLES As a first example, we recall the definition of relational composition in a regular category. Recall now that a *relation* $[r]\colon A \mathrel{\mapstochar\!\!\longrightarrow} B$ in $\mathbf{C}$ is the equivalence class of jointly monic pairs under isomorphism of the domains, compatible with the pairs. First, note that each jointly monic pair is the extension of a connected formula: more specifically, the pair r

$$\begin{tikzcd} & R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & \\ A & & B \end{tikzcd}$$

is isomorphic to the extension of the formula $\exists \mathtt{x} \in R.(\mathtt{a} = r_1(\mathtt{x}) \,\&\, \mathtt{b} = r_2(\mathtt{x}))$ because the factoring regular epi in the diagram

$$\begin{tikzcd} & {[\![\exists \mathtt{x} \in R.(\mathtt{a} = r_1(\mathtt{x}) \,\&\, \mathtt{b} = r_2(\mathtt{x}))]\!]} \arrow[ddl] \arrow[d, two heads] \arrow[ddr] & \\ & R \arrow[dl, "r_1"] \arrow[dr, "r_2"'] & \\ A & & B \end{tikzcd}$$

is monic since the pair of the extension is jointly monic. We shall abbreviate the formula above as $\mathtt{a}\ \langle r \rangle\ \mathtt{b}$ where we find it comfortable to use an infix shorthand symbol.

Consider then two jointly monic pairs

$$r = \begin{tikzcd} & R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & \\ A & & B \end{tikzcd} \quad \begin{tikzcd} & S \arrow[dl, "s_1"'] \arrow[dr, "s_2"] & \\ B & & C \end{tikzcd} = s$$

The composite of the two relations determined by the pairs is the extension of the formula

$$\exists \mathtt{b} \in B.(\mathtt{a}\ \langle r \rangle\ \mathtt{b} \,\&\, \mathtt{b}\ \langle s \rangle\ \mathtt{c})$$

with respect to the variables $\mathtt{a}, \mathtt{c}$. (Indeed, the composite is the equivalence class of all the representation of the extension above.)

There is a standard notion of order between relations with the same domain and codomain given by factorization. In logical terms, for relations $[r], [r']\colon A \mathrel{\mapstochar\!\!\longrightarrow} B$, $[r] \leq [r']$ if and only if

$$\mathtt{a}\ \langle r \rangle\ \mathtt{b} \Rightarrow \mathtt{a}\ \langle r' \rangle\ \mathtt{b}.$$

One can now use the logic to prove that the definition above gives rise to a 2-category $\mathrm{Rel}(\mathbf{C})$ on the relations in $\mathbf{C}$. From now on, we shall indistinctly refer to a relation as a jointly monic pair r, dropping the square brackets, as done in [CS86].

As a last example, we check the well-known result that the global version of the Axiom of Unique Choice holds in $\mathbf{C}$. Given a relation $r\colon A \mathrel{\mapstochar\!\!\longrightarrow} B$ which satisfies

1. $\exists \mathtt{b} \in B.\mathtt{a}\ \langle r \rangle\ \mathtt{b}$,

2. $\mathtt{a}\ \langle r \rangle\ \mathtt{b} \,\&\, \mathtt{a}\ \langle r \rangle\ \mathtt{b}' \Rightarrow \mathtt{b} = \mathtt{b}'$,

there exists a (necessarily unique) map $f\colon A \longrightarrow B$ in $\mathbf{C}$ such that r is the graph of f, *i.e.* the extension $\llbracket f(\mathtt{a}) = \mathtt{b} \rrbracket_{\mathtt{a},\mathtt{b}}$.

Consider satisfaction of formula 1 at A with respect to one-element string (id_A). Thus, there are a regular epi $e\colon X \twoheadrightarrow A$ and a map $b\colon X \longrightarrow B$ such that $X \Vdash (\mathtt{a}\ \langle r \rangle\ \mathtt{b})[\mathrm{id}_A \circ e, b]$. By 1.1, there is a factor x as in the following diagram

$$\begin{tikzcd}
& X \arrow[ddl, "e"'] \arrow[d, "x"] \arrow[ddr, "b"] & \\
& \llbracket \mathtt{a}\ \langle r \rangle\ \mathtt{b} \rrbracket \arrow[dl, "r_1"] \arrow[dr, "r_2"'] & \\
A & & B.
\end{tikzcd} \tag{1}$$

Let $K \overset{k_1}{\underset{k_2}{\rightrightarrows}} X$ be the kernel pair of e, and consider satisfaction of formula 2 at K with respect to the string $(e \circ k_1, b \circ k_1, b \circ k_2)$. The conjunction in the antecedent holds at K because of (1) and $e \circ k_2 = e \circ k_1$. Hence the consequent holds at K too, yielding that $b \circ k_1 = b \circ k_2$. Since e is a coequalizer for its kernel pair, there is a unique factor $f\colon A \longrightarrow B$ such that

$$\begin{tikzcd}
& X \arrow[ddl, "e"'] \arrow[d, "x"] \arrow[ddr, "b"] & \\
& \llbracket \mathtt{a}\ \langle r \rangle\ \mathtt{b} \rrbracket \arrow[dl, "r_1"] \arrow[dr, "r_2"'] & \\
A \arrow[rr, dotted, "f"'] & & B.
\end{tikzcd}$$

Since the pair $A \xleftarrow{\mathrm{id}_A} A \xrightarrow{f}$ is trivially jointly monic, and the pair $(e, fe = b)$ factors through r, we obtain that the two pairs are isomorphic. □

Another example of interest for the present paper is that of *preorder*, called ordered object in [CS86]. It consists of a jointly monic fork $\mathbb{A} = (A_1 \overset{p_1}{\underset{p_2}{\rightrightarrows}} A)$ which is *reflexive* and *transitive*. Instead of drawing the relevant diagrams, one may operate as before to define the meaning of reflexivity and transitivity: they state respectively that in $\mathbf{C}$ the following hold

$$\begin{array}{l} \mathtt{a}\ \langle p \rangle\ \mathtt{a} \\ \mathtt{a}\ \langle p \rangle\ \mathtt{a}' \mathbin{\&} \mathtt{a}'\ \langle p \rangle\ \mathtt{a}'' \Rightarrow \mathtt{a}\ \langle p \rangle\ \mathtt{a}'' \end{array}$$

The preorder is an *equivalence* relation if $A_1 \overset{p_1}{\underset{p_2}{\rightrightarrows}} A$ is *symmetric*:

$$\mathtt{a}\ \langle p \rangle\ \mathtt{a}' \Rightarrow \mathtt{a}'\ \langle p \rangle\ \mathtt{a}.$$

The preorder is an *order* if $A_1 \overset{p_1}{\underset{p_2}{\rightrightarrows}} A$ is *antisymmetric*:

$$\mathtt{a}\ \langle p \rangle\ \mathtt{a}' \mathbin{\&} \mathtt{a}'\ \langle p \rangle\ \mathtt{a} \Rightarrow \mathtt{a} = \mathtt{a}'.$$

When $\mathbf{C}$ is a topos, the preorders are precisely the Ω-enriched internal categories. But, for a general regular category $\mathbf{C}$, one cannot describe the preorders as examples of an enrichment. Nevertheless, one can develop a theory of functors and profunctors on these.

1.4 DEFINITION For preorders $\mathbb{A} = (A_1 \overset{p_1}{\underset{p_2}{\rightrightarrows}} A)$ and $\mathbb{B} = (B_1 \overset{q_1}{\underset{q_2}{\rightrightarrows}} B)$, a *monotone map* (or a *functor*) $f\colon \mathbb{A} \longrightarrow \mathbb{B}$ is a map $f\colon A \longrightarrow B$ in $\mathbf{C}$ which preserves the relation:

$$\mathtt{a}\ \langle p\rangle\ \mathtt{a}' \Rightarrow f(\mathtt{a})\ \langle q\rangle\ f(\mathtt{a}').$$

(These are said order-preserving in [CS86].) A monotone map *reflects* the preorder when the converse sequent holds:

$$f(\mathtt{a})\ \langle q\rangle\ f(\mathtt{a}') \Rightarrow \mathtt{a}\ \langle p\rangle\ \mathtt{a}'.$$

An *ideal* $r\colon \mathbb{A} \longrightarrow\!\!\!\!\!\!\mapsto \mathbb{B}$ is a relation $r\colon A \longrightarrow\!\!\!\!\!\!\mapsto B$ in $\mathbf{C}$ such that

$$\mathtt{a}\ \langle p\rangle\ \mathtt{a}' \mathbin{\&} \mathtt{a}'\ \langle r\rangle\ \mathtt{b}' \mathbin{\&} \mathtt{b}'\ \langle q\rangle\ \mathtt{b} \Rightarrow \mathtt{a}\ \langle r\rangle\ \mathtt{b}.$$

Preorders in $\mathbf{C}$ and ideals form a 2-category $\mathrm{Idl}(\mathbf{C})$ using relational composition of ideals. Note that the identity ideal on $\mathbb{A}$ is not its diagonal relation as that need not be an ideal. One must take its *idealization* so that $\mathrm{id}\colon \mathbb{A} \longrightarrow\!\!\!\!\!\!\mapsto \mathbb{A}$ is $[\![\mathtt{a}\ \langle p\rangle\ \mathtt{a}']\!]_{\mathtt{a},\mathtt{a}'}$—which comes to the same as the order itself. The 2-arrows are the same as for relations.

And preorders and monotone maps form another 2-category $\mathrm{Ord}(\mathbf{C})$. Composition in $\mathbf{C}$ of two monotone maps gives a monotone map, and the homsets are endowed with the "pointwise" order: in logical form,

$$f(\mathtt{a})\ \langle q\rangle\ f(\mathtt{a}'), \qquad \text{for } f, f'\colon \mathbb{A} \longrightarrow \mathbb{B}.$$

Since $\mathrm{Idl}(\mathbf{C})$ is a 2-category, all the 2-categorical concepts make sense in $\mathrm{Idl}(\mathbf{C})$. And, since the homsets are orders, all 2-diagrams are commutative, and many concepts simplify considerably. We are interested in the notion of adjoint pair in $\mathrm{Idl}(\mathbf{C})$. Because 2-commutativity is for free in $\mathrm{Idl}(\mathbf{C})$, the notion of adjoint pair boils down to the bare existence of unit and counit, and we shall rephrase here those requests in logical terms: an ideal $r\colon \mathbb{A} \longrightarrow\!\!\!\!\!\!\mapsto \mathbb{B}$ is *left adjoint to* $s\colon \mathbb{B} \longrightarrow\!\!\!\!\!\!\mapsto \mathbb{A}$ if

(a1) $\mathtt{a}\ \langle p\rangle\ \mathtt{a}' \Rightarrow \exists \mathtt{b} \in B.(\mathtt{a}\ \langle r\rangle\ \mathtt{b} \mathbin{\&} \mathtt{b}\ \langle s\rangle\ \mathtt{a}')$

(a2) $\exists \mathtt{a} \in A.(\mathtt{b}\ \langle s\rangle\ \mathtt{a} \mathbin{\&} \mathtt{a}\ \langle r\rangle\ \mathtt{b}') \Rightarrow \mathtt{b}\ \langle q\rangle\ \mathtt{b}'$

Note that the 2-notion of adjoint pair in $\mathrm{Ord}(\mathbf{C})$ gives rise to Galois connections.

1.5 THEOREM *There is a 2-embedding* $\mathrm{Ord}(\mathbf{C}) \hookrightarrow \mathrm{Idl}(\mathbf{C})$ *which is the identity on the objects and takes a monotone map to a left adjoint in* $\mathrm{Idl}(\mathbf{C})$.

Proof: Is straightforward. The definition of the embedding takes $f\colon \mathbb{A} \longrightarrow \mathbb{B}$ to the relation f^* defined as the extension $[\![f(\mathtt{a})\ \langle q\rangle\ \mathtt{b}]\!]_{\mathtt{a},\mathtt{b}}$ whose right adjoint is f_*, extension of the connected formula $\mathtt{b}\ \langle q\rangle\ f(\mathtt{a})$. □

Relations of the form f^* are called principal in [CS86]. We can now recall the fundamental notion of Cauchy-complete object for the special case of the 2-category $\mathrm{Idl}(\mathbf{C})$ with respect to the embedding of $\mathrm{Ord}(\mathbf{C})$.

1.6 DEFINITION A preorder $\mathbb{A}$ is *Cauchy-complete* if every left adjoint into it is of the form f^* for some monotone map f in $\mathrm{Ord}(\mathbf{C})$.

Note that every object is Cauchy-complete in the category $\mathrm{Rel}(\mathbf{C})$ with respect to the embedding $C \hookrightarrow \mathrm{Rel}(\mathbf{C})$.

2 Characterization of the Cauchy-complete preorders

2.1 PROPOSITION *In a regular category, every order is Cauchy-complete.*

Proof: Suppose $\mathbb{B}$ is an order and $r\colon \mathbb{A} \longrightarrow\!\!\!\!\!\!\!+\ \mathbb{B}$ is an ideal with a right adjoint $s\colon \mathbb{B} \longrightarrow\!\!\!\!\!\!\!+\ \mathbb{A}$. Then, by condition (a2) of adjoint pairs,

$$\exists \mathtt{b} \in B.(\mathtt{a}\ \langle r\rangle\ \mathtt{b}\ \&\ \mathtt{b}\ \langle s\rangle\ \mathtt{a})$$

and, by (a1), such a $\mathtt{b}$ is unique:

$$\begin{array}{rcl} \mathtt{a}\ \langle r\rangle\ \mathtt{b}\ \&\ \mathtt{b}\ \langle s\rangle\ \mathtt{a}\ \&\ \mathtt{a}\ \langle r\rangle\ \mathtt{b}'\ \&\ \mathtt{b}'\ \langle s\rangle\ \mathtt{a} & \Rightarrow & [\mathtt{b}\ \langle s\rangle\ \mathtt{a}\ \&\ \mathtt{a}\ \langle r\rangle\ \mathtt{b}']\ \&[\mathtt{b}'\ \langle s\rangle\ \mathtt{a}\ \&\ \mathtt{a}\ \langle r\rangle\ \mathtt{b}] \\ & \Rightarrow & \exists \mathtt{a} \in A.[\mathtt{b}\ \langle s\rangle\ \mathtt{a}\ \&\ \mathtt{a}\ \langle r\rangle\ \mathtt{b}']\ \& \\ & & \qquad \exists \mathtt{a} \in A.[\mathtt{b}'\ \langle s\rangle\ \mathtt{a}\ \&\ \mathtt{a}\ \langle r\rangle\ \mathtt{b}] \\ & \Rightarrow & \mathtt{b}\ \langle q\rangle\ \mathtt{b}'\ \&\ \mathtt{b}'\ \langle q\rangle\ \mathtt{b} \\ & \Rightarrow & \mathtt{b} = \mathtt{b}' \end{array}$$

where the conclusion invokes antisymmetry in $\mathbb{B}$. By (AUC), there is a map $f\colon A \longrightarrow B$ such that

$$\mathtt{a}\ \langle r\rangle\ \mathtt{b}\ \&\ \mathtt{b}\ \langle s\rangle\ \mathtt{a} \Leftrightarrow f(\mathtt{a}) = \mathtt{b}. \tag{2}$$

To finish the proof, we show that r is f^*. Since r is an ideal, it is obvious that

$$f(\mathtt{a})\ \langle q\rangle\ \mathtt{b} \Rightarrow \mathtt{a}\ \langle r\rangle\ \mathtt{b}$$

by (2). Conversely, using (2) again,

$$\begin{array}{rcl} \mathtt{a}\ \langle r\rangle\ \mathtt{b} & \Rightarrow & f(\mathtt{a})\ \langle s\rangle\ \mathtt{a}\ \&\ \mathtt{a}\ \langle r\rangle\ \mathtt{b} \\ & \Rightarrow & \exists \mathtt{a} \in A.[f(\mathtt{a})\ \langle s\rangle\ \mathtt{a}\ \&\ \mathtt{a}\ \langle r\rangle\ \mathtt{b}] \\ & \Rightarrow & f(\mathtt{a})\ \langle q\rangle\ \mathtt{b} \end{array}$$

where condition (a2) for adjoint pairs yields the last consequence. $\square$

The previous result hints directly at the solution for how to construct a Cauchy-completion of a preorder: force on the preorder the largest possible order. By this we mean the following: Given a preorder $\mathbb{A}$, consider the relation

$$\mathtt{a}\ \langle p\rangle\ \mathtt{a}'\ \&\ \mathtt{a}'\ \langle p\rangle\ \mathtt{a} \tag{3}$$

which is clearly an equivalence relation. Suppose there exists a regular epi $q\colon A \longrightarrow\!\!\!\!\triangleright\ \widehat{A}$ in $\mathbf{C}$ such that the relation in (3) is its kernel pair. We have then the following proposition, where (i) is folklore, and only (ii) shows a new property.

2.2 PROPOSITION *With the notation above,*

(i) *there is a poset structure on $\widehat{A}$ such that q is monotone and reflects the order relation,*

(ii) *the ideal $q^*\colon \mathbb{A} \longrightarrow\!\!\!\!\!\!\!+\ \widehat{\mathbb{A}}$ is an isomorphism in* Ord($\mathbf{C}$).

Proof: (i) It is a remake, in a general astract setting, of the standard construction of the poset reflection of a preorder. First of all, note that the following holds in $\mathbf{C}$

$$\exists \mathtt{a} \in A.q(\mathtt{a}) = \mathtt{c}$$

for $\mathtt{c}$ of tye $\widehat{A}$, by the hypothesis that q is regular epi, and

$$\mathtt{a}\ \langle p\rangle\ \mathtt{a}'\ \&\ \mathtt{a}'\ \langle p\rangle\ \mathtt{a} \Leftrightarrow q(\mathtt{a}) = q(\mathtt{a}')$$

by the hypothesis that the relation (3) is the kernel pair of q. Now, let $\widehat{P} \underset{\widehat{p_2}}{\overset{\widehat{p_1}}{\rightrightarrows}} \widehat{A}$ be the relation defined as the extension of

$$\exists \mathtt{a} \in A.\exists \mathtt{a}' \in A.[q(\mathtt{a}) = \mathtt{c}\ \&\ q(\mathtt{a}') = \mathtt{c}'\ \&\ \mathtt{a}\ \langle p\rangle\ \mathtt{a}']$$

The logic makes it straightforward to check that that relation satisfies the requirements in (i).
(ii) Since the homsets in Idl($\mathbf{C}$) are orders, and q^* is a left adjoint, it is enough to show that it is also a right adjoint. To that, one must show the following properties hold in $\mathbf{C}$

(a) $\mathtt{c}\ \langle\widehat{p}\rangle\ \mathtt{c}' \Rightarrow \exists \mathtt{a} \in A.[c\ \langle\widehat{p}\rangle\ q(\mathtt{a})\ \&\ q(\mathtt{a})\ \langle\widehat{p}\rangle\ \mathtt{c}']$,

(b) $\exists \mathtt{c} \in \widehat{A}.[q(\mathtt{a})\ \langle\widehat{p}\rangle\ \mathtt{c}\ \&\ \mathtt{c}\ \langle\widehat{p}\rangle\ q(\mathtt{a}')] \Rightarrow \mathtt{a}\ \langle p\rangle\ \mathtt{a}'$.

Property (a) is proved by picking any $\mathtt{a}$ such that $q(\mathtt{a}) = \mathtt{c}$ as given by regularity of q. Property (b) follows directly from the fact that $q\colon \mathbb{A} \longrightarrow \widehat{\mathbb{A}}$ reflects the relation. $\square$

So it appears clearly that Cauchy-completions can be achieved when definitional power is assumed on the regular category to ensure coequalizers of equivalence relations: this is precisely the notion of (Barr-)exactness. A regular category is *exact* if every equivalence relation has a coequalizer—which is usually named a *quotient*.

So Theorem 14 of [CS86] can be strengthened as follows

2.3 COROLLARY *In an exact category, the Cauchy completion of a preorder exists and is its order reflection.*

This seems to appear in contrast with Proposition 5 of [BD86], but there the notion of poset is taken to mean a set endowed with a reflexive and transitive relation: yet, care is advised as there is a preliminary comment to that proposition which seemed to imply that also antisymmetry were assumed.

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