

# Realizability Models

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MFPS Tutorial, New Orleans, 10 April 2007

# **Naïve** Realizability Models

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# Prerequisites

- partial functions  $\mathbb{N} \rightarrow \mathbb{N}$  computed by Turing machines
- $M(n)$  means that the function  $M$  computed by the Turing machine is defined on  $n$  and the output is  $M(n)$ .
- there are computable bijections

$$\begin{array}{ccc}
 (m, \ell) & \xrightarrow{\quad} & \langle m, \ell \rangle \\
 \mathbb{N} \times \mathbb{N} & \xleftrightarrow{\quad} & \mathbb{N} \\
 (n_0, n_1) & \xleftarrow{\quad} & n
 \end{array}$$

- “test on 0” is computable  $(i, n, m) \mapsto \text{case}(i, n, m) = \begin{cases} n & \text{if } i = 0 \\ m & \text{otherwise} \end{cases} : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$
- since Turing machines are strings in a fixed alphabet, they can be encoded by numbers: the number which encodes the machine  $M$  is  $\lfloor M \rfloor$ , the machine encoded by the number  $e$  is  $\lceil e \rceil$ .
- $U$  is a universal Turing machine so that

$$U(\langle e, n \rangle) = \lceil e \rceil(n) \quad \text{any } n.$$

and  $S$  satisfies

$$M(\langle m, \ell \rangle) = \lceil S(\lfloor M \rfloor, m) \rceil(\ell) = U(\langle S(\lfloor M \rfloor, m), \ell \rangle) \quad \text{any } M, m, \ell.$$

- the fundamental structure is that of **partial combinatory algebra** given on the set  $\mathbb{N}$  by the partial binary operation:

$$(t, n) \mapsto \lceil t \rceil(n) = \text{the output of the Turing machine encoded by } t \text{ on input } n$$

# Assemblies

An *assembly*  $X = (X_{el}, \Vdash_X)$  is

- a collection of elements  $X_{el}$  together with
- a relation  $\Vdash_X$  between numbers and elements of  $X_{el}$  which is surjective, *i.e.* for each  $x \in X_{el}$  there is a number such that  $n \Vdash_X x$

Write  $n \Vdash_X x$  as  $n \Vdash x \in X$ . Read it as “ $n$  realizes that  $x$  is in  $X$ ”.

Say that  $X \subseteq Y$  when there is a computable function  $M$  such that for every  $x \in X_{el}$  and every  $n \in \mathbb{N}$ , if  $n \Vdash x \in X$ , then  $M(n) \Vdash x \in Y$

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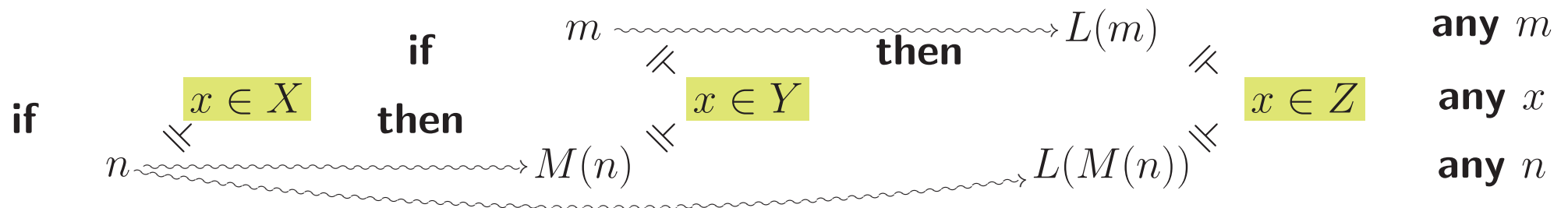
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**if**  $n \Vdash x \in X$  **then**  $M(n) \Vdash x \in Y$  **any**  $x$   
**any**  $n$   
**when**  $m \Vdash x \in Y$  **then**  $M'(m) \Vdash x \in X$  **any**  $m$

for suitable  $M$  and  $M'$



# Examples of assemblies

- $S$ : a set

$\nabla S \stackrel{\text{def}}{=} (S, \Vdash_{\nabla S})$  where  $n \Vdash_{\nabla S} s$  if  $s \in S$

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# Examples of assemblies

- $\Sigma = (\{0, 1\}, \Vdash_{\Sigma})$  where  $n \Vdash_{\Sigma} x$  if either  $\lceil n \rceil(n) \downarrow$  and  $x = 1$  or  $\lceil n \rceil(n) \uparrow$  and  $x = 0$ .

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 $n \rightsquigarrow$   $??$   $\rightsquigarrow$       any  $n$

- $\mathbf{1} = (\{0\}, \Vdash_{\mathbf{1}})$  where  $n \Vdash_{\mathbf{1}} x$  if  $n = 0 = x$ .

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 or  $\lceil n \rceil(n) \uparrow$  and  $x = 0$

Check  $\Sigma \subsetneq \nabla\{0, 1\}$

if  $n \Vdash x \in \nabla\{0, 1\}$  then  $n \Vdash x \in \Sigma$       any  $x$   
 any  $n$

??

- $\mathbf{1} = (\{0\}, \Vdash_{\mathbf{1}})$  where  $n \Vdash_{\mathbf{1}} x$  if  $n = 0 = x$ .

Check  $\mathbf{1} = \nabla\{0\}$

# Examples of assemblies

- $\Sigma = (\{0, 1\}, \Vdash_{\Sigma})$  where  $n \Vdash_{\Sigma} x$  if either  $\lceil n \rceil(n) \downarrow$  and  $x = 1$  or  $\lceil n \rceil(n) \uparrow$  and  $x = 0$ .

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if  $n \Vdash x \in \nabla\{0\}$  then  $n \Vdash x \in \mathbf{1}$       any  $x$   
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# Examples of assemblies

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if  $n \Vdash x \in \nabla\{0\}$  then  $n \Vdash x \in \mathbf{1}$       any  $x$   
any  $n$

# Elementary constructions

- Intersection:

$$X \cap Y \subseteq X$$

$$X \cap Y \subseteq Y$$

if  $Z \subseteq X$  and  $Z \subseteq Y$ , then  $Z \subseteq X \cap Y$

Define:  $X \cap Y \stackrel{\text{def}}{=} (X_{el} \cap Y_{el}, \Vdash_{X \cap Y})$

where  $n \Vdash_{X \cap Y} z$  if  $n_0 \Vdash_X z$  and  $n_1 \Vdash_Y z$

i.e.  $\langle m, l \rangle \Vdash z \in X \cap Y$  iff  $m \Vdash z \in X$  and  $l \Vdash z \in Y$

# Elementary constructions

- Intersection:

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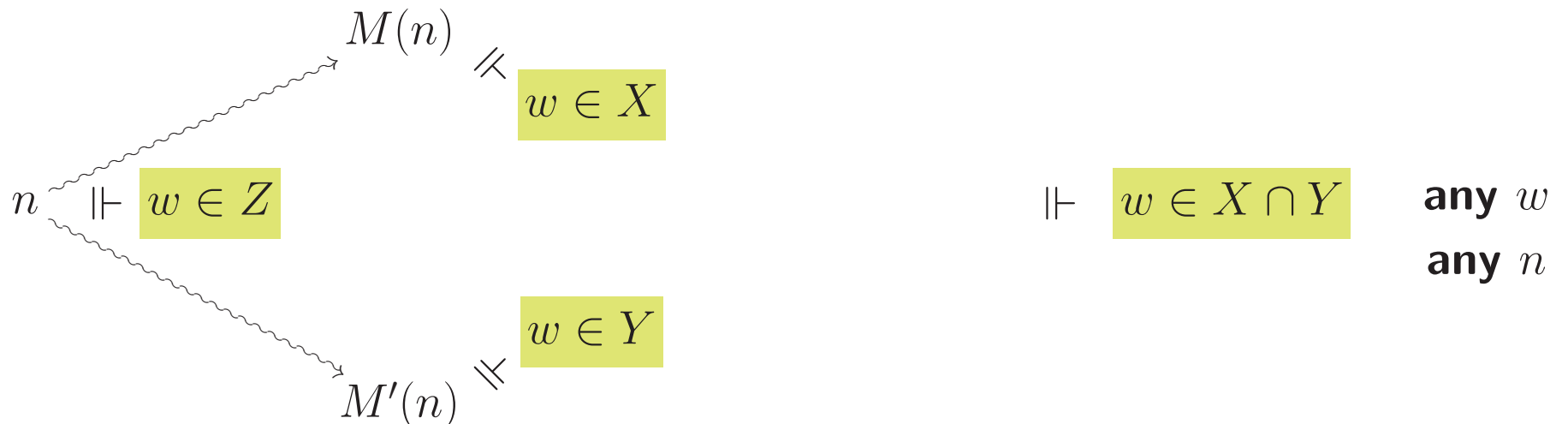
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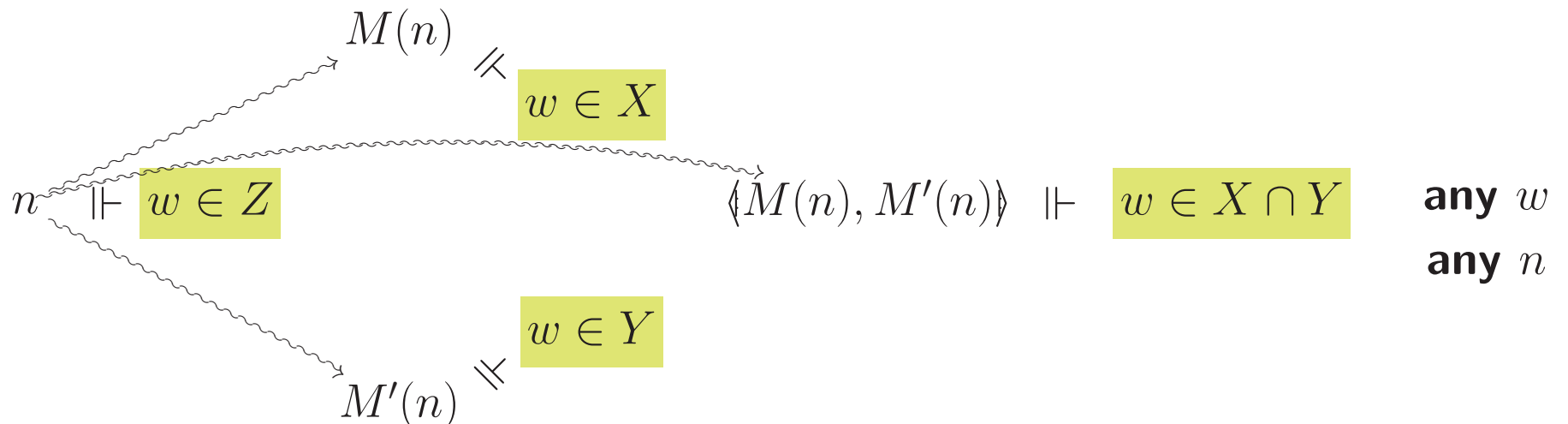
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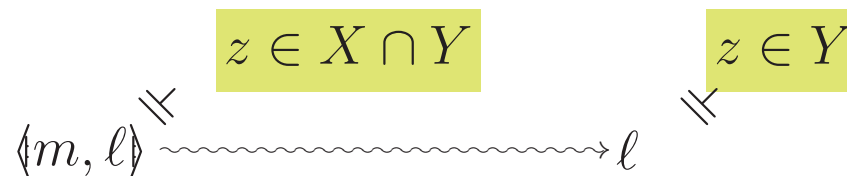
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$$X \cap Y \subseteq X$$



$$X \cap Y \subseteq Y$$



# Elementary constructions

- Union:

$$X \subseteq X \cup Y \quad Y \subseteq X \cup Y$$

$$\text{if } X \subseteq Z \text{ and } Y \subseteq Z \text{ then } X \cup Y \subseteq Z$$

Define:  $X \cup Y \stackrel{\text{def}}{=} (X_{el} \cup Y_{el}, \Vdash_{X \cup Y})$

where  $n \Vdash_{X \cup Y} z$  if either  $n_1 = 0, n_0 \Vdash_X z$  or  $n_1 = 1, n_0 \Vdash_Y z$

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# Elementary constructions

- Union:

$$X \subseteq X \cup Y$$

$$Y \subseteq X \cup Y$$

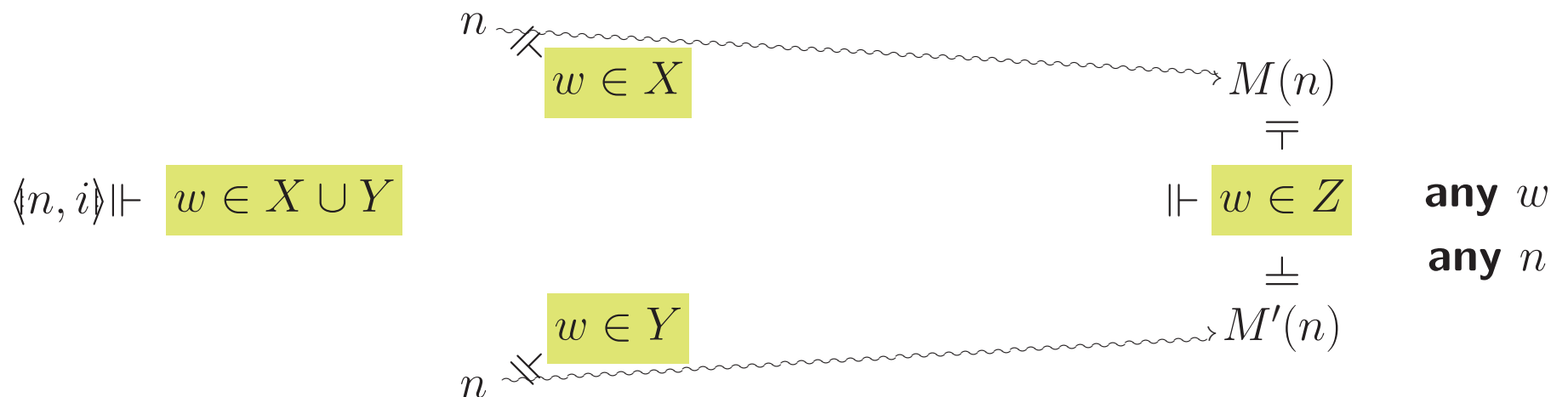
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# Elementary constructions

- Union:

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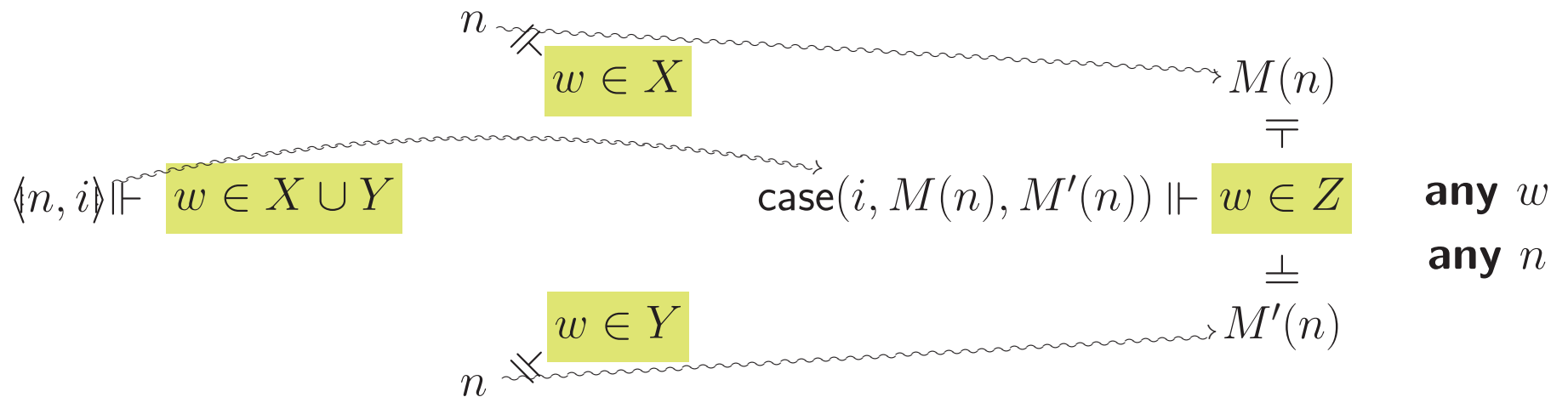
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if  $X \subseteq Z$  and  $Y \subseteq Z$  then  $X \cup Y \subseteq Z$



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$$X \subseteq X \cup Y$$

$$n \Vdash z \in X \iff n \Vdash z \in X \cup Y$$

# Elementary constructions

- Union:

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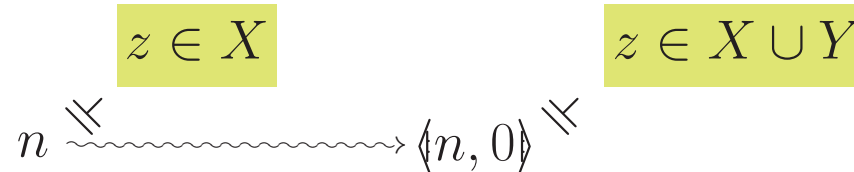
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$$X \subseteq X \cup Y$$

$$z \in X \quad z \in X \cup Y$$

$$n \Vdash \rightsquigarrow \langle n, 0 \rangle \Vdash$$

$$Y \subseteq X \cup Y$$

$$z \in Y \quad z \in X \cup Y$$

$$n \Vdash \rightsquigarrow \langle n, 1 \rangle \Vdash$$

# Elementary constructions

- (Pseudo)complement:

$$(X \implies_A Y) \cap X \subseteq Y$$

$$(X \implies_A Y) \subseteq A$$

if  $W \subseteq A$  and  $W \cap X \subseteq Y$  then  $W \subseteq (X \implies_A Y)$

Define:  $(X \implies_A Y) \stackrel{\text{def}}{=} ((A_{el} \setminus X_{el}) \cup Y_{el}, \Vdash_{(X \implies_A Y)})$

where  $n \Vdash_{(X \implies_A Y)} z$  if  $n_0 \Vdash_A z$  and for all  $k \Vdash_X z$   $[n_1](k) \Vdash_Y z$

i.e.  $\langle m, e \rangle \Vdash z \in (X \implies_A Y)$  iff  $m \Vdash z \in A$  and  
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$$(X \implies_A Y) \cap X \subseteq Y$$

$$\langle \langle m, e \rangle, \ell \rangle \Vdash z \in (X \implies_A Y) \cap X \iff [e](\ell) \Vdash z \in Y$$

any  $z$   
any  $n, e, m$

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if  $W \subseteq A$  and  $W \cap X \subseteq Y$  then  $W \subseteq (X \implies_A Y)$

Suppose  $\cong \begin{array}{c} z \in W \\ n \rightsquigarrow M(n) \end{array} \cong z \in A$  and  $\cong \begin{array}{c} z \in W \cap X \\ \langle m, l \rangle \rightsquigarrow L(\langle m, l \rangle) \end{array} \cong z \in Y$



# Elementary constructions

• (Pseudo)complement:

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i.e.  $\langle m, e \rangle \Vdash z \in (X \implies_A Y)$  iff  $m \Vdash z \in A$  and  
for all  $k \Vdash z \in X$  one has  $[e](k) \Vdash z \in Y$

if  $W \subseteq A$  and  $W \cap X \subseteq Y$  then  $W \subseteq (X \implies_A Y)$

Suppose  $\Vdash \begin{array}{ccc} z \in W & z \in A & \\ n \rightsquigarrow M(n) & & \end{array}$  and  $\Vdash \begin{array}{ccc} z \in W \cap X & z \in Y & \\ \langle m, l \rangle \rightsquigarrow L(\langle m, l \rangle) & & \end{array}$

$\Vdash \begin{array}{ccc} z \in W & z \in (X \implies_A Y) & \\ n \rightsquigarrow & & \end{array}$

# Elementary constructions

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Suppose  $\begin{array}{c} z \in W \\ \Downarrow \\ n \rightsquigarrow M(n) \end{array}$  and  $\begin{array}{c} z \in A \\ \Downarrow \\ \langle m, \ell \rangle \rightsquigarrow L(\langle m, \ell \rangle) \end{array}$  and  $\begin{array}{c} z \in W \cap X \\ \Downarrow \\ \langle m, \ell \rangle \rightsquigarrow L(\langle m, \ell \rangle) \end{array}$  and  $\begin{array}{c} z \in Y \\ \Downarrow \\ \langle m, \ell \rangle \rightsquigarrow L(\langle m, \ell \rangle) \end{array}$

$\begin{array}{c} z \in W \\ \Downarrow \\ n \rightsquigarrow \langle M(n), S(\lfloor L \rfloor, n) \rangle \end{array}$   $\Downarrow$   $\begin{array}{c} z \in (X \implies_A Y) \end{array}$  where  $[S(\lfloor L \rfloor, m)](\ell) = L(\langle m, \ell \rangle)$ .

# Elementary constructions

• (Pseudo)complement:  $(X \implies_A Y) \cap X \subseteq Y$        $(X \implies_A Y) \subseteq A$

if  $W \subseteq A$  and  $W \cap X \subseteq Y$  then  $W \subseteq (X \implies_A Y)$

# Elementary constructions

- complement:  $(X \implies_A \nabla\emptyset) \cap X \subseteq \nabla\emptyset$        $(X \implies_A \nabla\emptyset) \subseteq A$   
if  $W \subseteq A$  and  $W \cap X \subseteq \nabla\emptyset$  then  $W \subseteq (X \implies_A \nabla\emptyset)$

# Elementary constructions

- complement:  $(X \implies_A \nabla\emptyset) \cap X \subseteq \nabla\emptyset$        $(X \implies_A \nabla\emptyset) \subseteq A$   
if  $W \subseteq A$  and  $W \cap X \subseteq \nabla\emptyset$  then  $W \subseteq (X \implies_A \nabla\emptyset)$

So  $\langle m, e \rangle \Vdash z \in (X \implies_A Y)$  iff  $m \Vdash z \in A$  and  
for all  $k \Vdash z \in X$  one has  $[e](k) \Vdash z \in Y$

# Elementary constructions

- complement:  $(X \implies_A \nabla\emptyset) \cap X \subseteq \nabla\emptyset$        $(X \implies_A \nabla\emptyset) \subseteq A$   
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So  $\langle m, e \rangle \Vdash z \in (X \implies_A \nabla\emptyset)$  iff  $m \Vdash z \in A$  and  
for all  $k \Vdash z \in X$  one has  $[e](k) \Vdash z \in \nabla\emptyset$

# Elementary constructions

• complement:  $(X \implies_A \nabla\emptyset) \cap X \subseteq \nabla\emptyset$        $(X \implies_A \nabla\emptyset) \subseteq A$

if  $W \subseteq A$  and  $W \cap X \subseteq \nabla\emptyset$  then  $W \subseteq (X \implies_A \nabla\emptyset)$

So  $\langle m, e \rangle \Vdash z \in (X \implies_A \nabla\emptyset)$  iff  $m \Vdash z \in A$  and  
for all  $k \Vdash z \in X$  one has  $[e](k) \Vdash z \in \nabla\emptyset$

iff  $m \Vdash z \in A$  and no  $k \Vdash z \in X$

# Elementary constructions

- complement:  $(X \implies_A \nabla\emptyset) \cap X \subseteq \nabla\emptyset$        $(X \implies_A \nabla\emptyset) \subseteq A$   
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iff  $m \Vdash z \in A$  and no  $k \Vdash z \in X$

Define:  $A \setminus X \stackrel{\text{def}}{=} (A_{el} \setminus X_{el}, \Vdash_{A \setminus X})$   
 where  $n \Vdash_{A \setminus X} z$  if  $n \Vdash_A z$



# Elementary constructions

- complement:  $(X \implies_A \nabla\emptyset) \cap X \subseteq \nabla\emptyset$        $(X \implies_A \nabla\emptyset) \subseteq A$   
 if  $W \subseteq A$  and  $W \cap X \subseteq \nabla\emptyset$  then  $W \subseteq (X \implies_A \nabla\emptyset)$

So  $\langle m, e \rangle \Vdash z \in (X \implies_A \nabla\emptyset)$  iff  $m \Vdash z \in A$  and  
 for all  $k \Vdash z \in X$  one has  $[e](k) \Vdash z \in \nabla\emptyset$   
 iff  $m \Vdash z \in A$  and no  $k \Vdash z \in X$

Define:  $A \setminus X \stackrel{\text{def}}{=} (A_{el} \setminus X_{el}, \Vdash_{A \setminus X})$   
 where  $n \Vdash_{A \setminus X} z$  if  $n \Vdash_A z$

**Exercise:** Check if  $(X \implies_A Y) = (A \setminus X) \cup Y$

# Elementary constructions

- Cartesian product:  $A \times B \stackrel{\text{def}}{=} (A_{el} \times B_{el}, \Vdash_{A \times B})$   
where  $n \Vdash_{A \times B} (z, w)$  if  $n_0 \Vdash_A z$  and  $n_1 \Vdash_B w$   
*i.e.*  $\langle m, \ell \rangle \Vdash (z, w) \in A \times B$  iff  $m \Vdash z \in A$  and  $\ell \Vdash w \in B$

# Elementary constructions

- Cartesian product:  $A \times B \stackrel{\text{def}}{=} (A_{el} \times B_{el}, \Vdash_{A \times B})$   
where  $n \Vdash_{A \times B} (z, w)$  if  $n_0 \Vdash_A z$  and  $n_1 \Vdash_B w$   
i.e.  $\langle m, \ell \rangle \Vdash (z, w) \in A \times B$  iff  $m \Vdash z \in A$  and  $\ell \Vdash w \in B$

If  $A \subseteq C$  and  $B \subseteq D$  then  $A \times B \subseteq C \times D$

# Elementary constructions

- Cartesian product:  $A \times B \stackrel{\text{def}}{=} (A_{el} \times B_{el}, \Vdash_{A \times B})$   
where  $n \Vdash_{A \times B} (z, w)$  if  $n_0 \Vdash_A z$  and  $n_1 \Vdash_B w$   
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If  $A \subseteq C$  and  $B \subseteq D$  then  $A \times B \subseteq C \times D$

- Diagonal:  $\Delta_A \stackrel{\text{def}}{=} (\{(z, z) \mid z \in A_{el}\}, \Vdash_{\Delta_A})$   
where  $n \Vdash_{\Delta_A} (z, w)$  if  $z = w$  and  $n \Vdash z \in A$

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Check  $\Delta_A \subseteq A \times A$

$$n \Vdash (z, w) \in \Delta_A \iff (z, w) \in A \times A \quad \text{any } z, w$$

# Elementary constructions

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Check  $\Delta_A \subseteq A \times A$

$$\begin{array}{ccc}
 \begin{array}{c} \Vdash \\ \parallel \\ n \end{array} & \begin{array}{c} (z, w) \in \Delta_A \\ \xrightarrow{\text{~~~~~}} \end{array} & \begin{array}{c} (z, w) \in A \times A \\ \parallel \\ \end{array} & \begin{array}{l} \text{any } z, w \\ \text{any } n \end{array}
 \end{array}$$

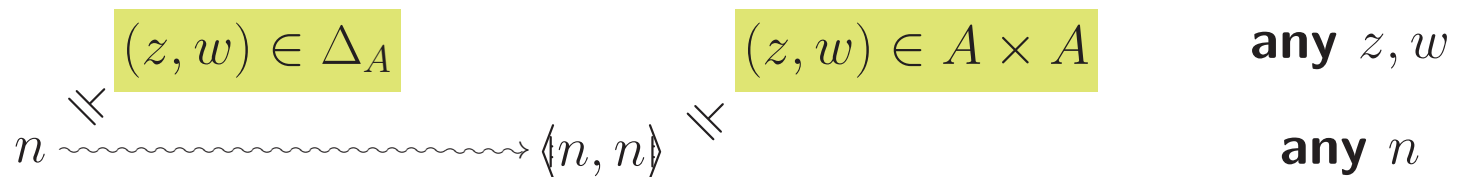
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Check  $\Delta_A \subseteq A \times A$



# Elementary constructions

- Projection: for  $X \subseteq A \times B$   $X \subseteq \pi_1^p(X) \times B$   
if  $X \subseteq Y \times B$  then  $\pi_1^p(X) \subseteq Y$

Define:  $\pi_1^p(X) \stackrel{\text{def}}{=} (\{z \in A_{el} \mid \exists w \in B_{el} (z, w) \in X_{el}\}, \Vdash_{\pi_1^p(X)})$   
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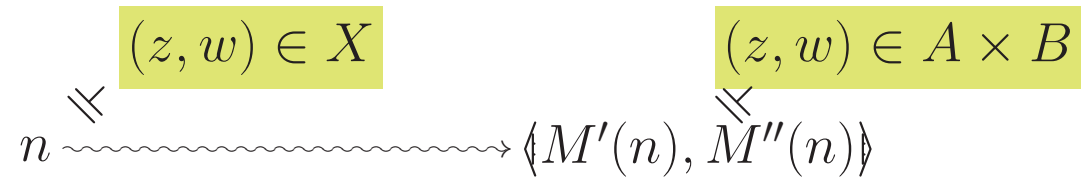


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Suppose  $X \subseteq A \times B$

$$n \stackrel{\cong}{\rightsquigarrow} \langle M'(n), M''(n) \rangle$$

$(z, w) \in X$   $(z, w) \in A \times B$

$$X \subseteq \pi_1^p(X) \times B$$

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$$X \subseteq \pi_1^p(X) \times B$$

$$n \stackrel{\Vdash}{\rightsquigarrow} \langle n, M''(n_1) \rangle$$

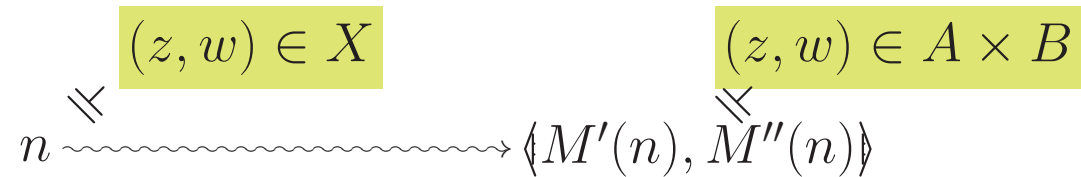
$(z, w) \in X$   $(z, w) \in \pi_1^p(X) \times B$

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$(z, w) \in X$   $(z, w) \in A \times B$

if  $X \subseteq Y \times B$  then  $\pi_1^p(X) \subseteq Y$

$$m \stackrel{\Vdash}{\rightsquigarrow} \langle L(m), L'(m) \rangle$$

$(z, w) \in X$   $(z, w) \in Y \times B$

# Elementary constructions

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$$m \stackrel{\Vdash}{\rightsquigarrow} \langle L(m), L'(m) \rangle \stackrel{\Vdash}{\rightsquigarrow} \langle L(m), L'(m) \rangle$$

$(z, w) \in X$   $(z, w) \in Y \times B$

$$m \stackrel{\Vdash}{\rightsquigarrow} L(m) \stackrel{\Vdash}{\rightsquigarrow} L(m)$$

$z \in \pi_1^p(X)$   $z \in Y$



# Elementary constructions

- Direct Image:

$$\begin{array}{l} \pi_1^d(X) \subseteq A \quad \pi_1^d(X) \times B \subseteq X \\ \text{for } Y \subseteq A \quad \text{if } Y \times B \subseteq X \text{ then } Y \subseteq \pi_1^d(X) \end{array}$$

Define:  $\pi_1^d(X) \stackrel{\text{def}}{=} (\{z \in A_{el} \mid \forall w \in B_{el} (z, w) \in X_{el}\}, \Vdash_{\pi_1^d(X)})$   
where  $n \Vdash_{\pi_1^d(X)} z$  if  $n_0 \Vdash_A z$  and for all  $k \Vdash_B w$   $[n_1](k) \Vdash_X (z, w)$

# Elementary constructions

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$$\pi_1^d(X) \times B \subseteq X$$

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$$\pi_1^d(X) \times B \subseteq X$$

$$\langle \langle m, e \rangle, \ell \rangle \stackrel{\Vdash}{\rightsquigarrow} (z, w) \in \pi_1^d(X) \times B \stackrel{\Vdash}{\rightsquigarrow} [e](\ell) \stackrel{\Vdash}{\rightsquigarrow} (z, w) \in X$$

# Elementary constructions

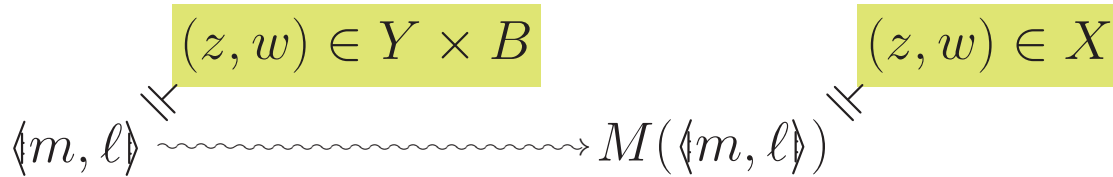
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# Elementary constructions

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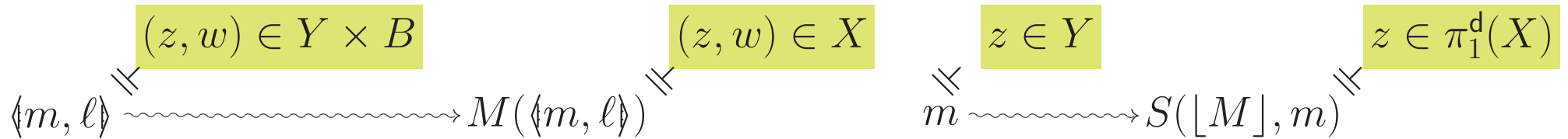
$$\pi_1^d(X) \subseteq A \quad \pi_1^d(X) \times B \subseteq X$$

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if  $Y \times B \subseteq X$

then  $Y \subseteq \pi_1^d(X)$



# Examples

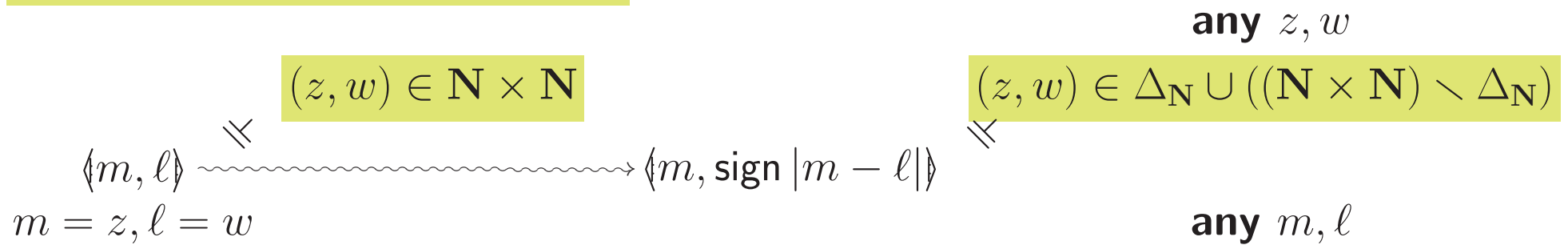
- $\mathbf{N} \times \mathbf{N} \subseteq \Delta_{\mathbf{N}} \cup ((\mathbf{N} \times \mathbf{N}) \setminus \Delta_{\mathbf{N}})$

$$\begin{array}{ccc} & (z, w) \in \mathbf{N} \times \mathbf{N} & \\ \langle m, l \rangle & \xrightarrow{\cong} & \langle m, \text{sign } |m - l| \rangle \\ m = z, l = w & & \end{array}$$

$$\begin{array}{ccc} \text{any } z, w & & \\ (z, w) \in \Delta_{\mathbf{N}} \cup ((\mathbf{N} \times \mathbf{N}) \setminus \Delta_{\mathbf{N}}) & & \\ \cong & & \\ \text{any } m, l & & \end{array}$$

# Examples

- $\mathbf{N} \times \mathbf{N} \subseteq \Delta_{\mathbf{N}} \cup ((\mathbf{N} \times \mathbf{N}) \setminus \Delta_{\mathbf{N}})$



- $(\nabla \mathbf{N} \setminus (\nabla \mathbf{N} \setminus \mathbf{N})) = \nabla \mathbf{N}$



# Recap on elementary constructions

$n \Vdash x \in \mathbf{N}$	$n = x \in \mathbf{N}$
$n \Vdash x \in \nabla S$	$z \in S$
$n \Vdash x \in \nabla \emptyset$	$\downarrow$
$n \Vdash x \in \mathbf{1}$	$n = 0 = x.$
$n \Vdash x \in \Sigma$	either $\lceil n \rceil(n) \downarrow$ and $x = 1$ or $\lceil n \rceil(n) \uparrow$ and $x = 0$
$\langle m, \ell \rangle \Vdash (z, w) \in A \times B$	$m \Vdash z \in A$ and $\ell \Vdash w \in B$
$n \Vdash (z, w) \in \Delta_X$	$z = w$ and $n \Vdash z \in X$
$\langle m, \ell \rangle \Vdash z \in X \cap Y$	$m \Vdash z \in X$ and $\ell \Vdash z \in Y$
$\langle n, i \rangle \Vdash z \in X \cup Y$	either $i = 0$ and $n \Vdash z \in X$ or $i = 1$ and $n \Vdash z \in Y$
$\langle m, e \rangle \Vdash z \in (X \implies_A Y)$	$m \Vdash z \in A$ and $\lceil e \rceil(k) \Vdash z \in Y$ for all $k \Vdash z \in X$
$\langle m, e \rangle \Vdash z \in A \setminus X$	$m \Vdash z \in A$ and no $k \Vdash z \in X$
$n \Vdash z \in \pi_1^p(X)$	$n \Vdash (z, w) \in X$ for some $k \Vdash w \in B$
$\langle m, e \rangle \Vdash z \in \pi_1^d(X)$	$m \Vdash z \in A$ and $\lceil n \rceil(k) \Vdash (z, w) \in X$ for all $k \Vdash w \in B$



# Functions

$f: A \rightarrow B$  must be  $f \subseteq A \times B$  such that

single-valued  $\forall a \in A \forall b \in B \forall b' \in B \quad [(a, b) \in f \wedge (a, b') \in f] \Rightarrow b = b'$

total  $\forall a \in A \exists b \in B \quad (a, b) \in f$

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$$[\pi_{12}^*(f) \cap \pi_{13}^*(f)] \subseteq \pi_{23}^* \Delta_B$$

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$f \subseteq A \times B$ :

$$n \stackrel{\cong}{\rightsquigarrow} (a, b) \in f \stackrel{\cong}{\rightsquigarrow} (a, b) \in A \times B$$

**any**  $a, b$

**any**  $n$

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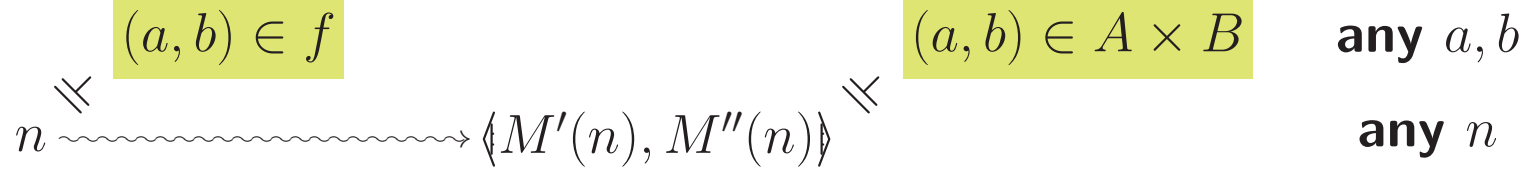
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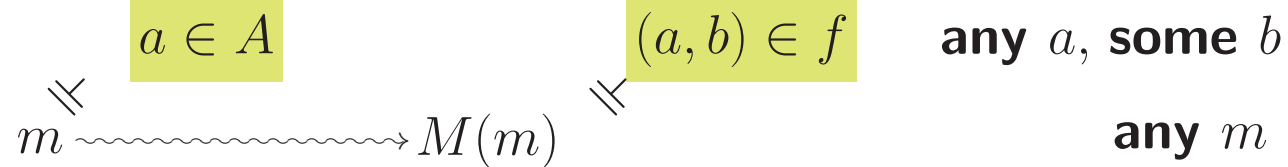
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$A \subseteq \pi_1^p(f)$ :



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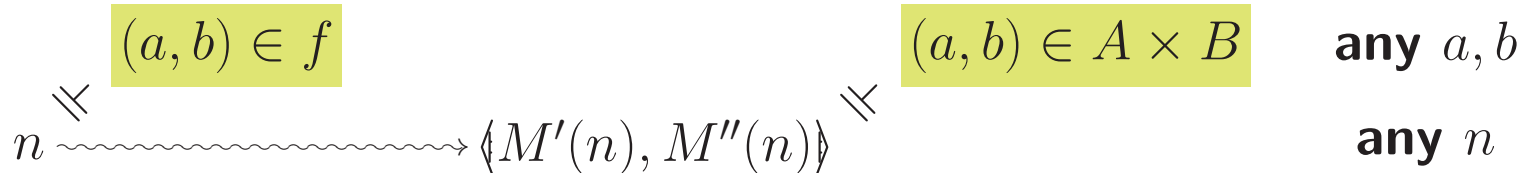
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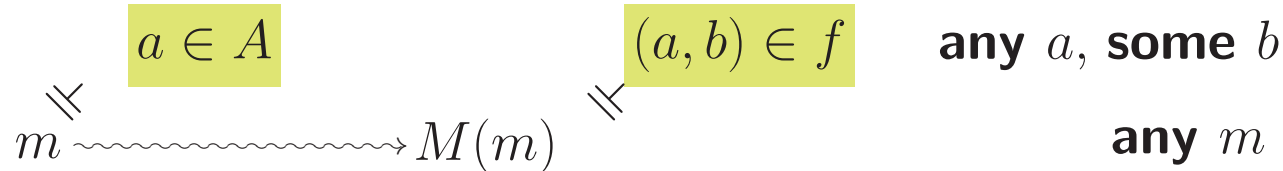
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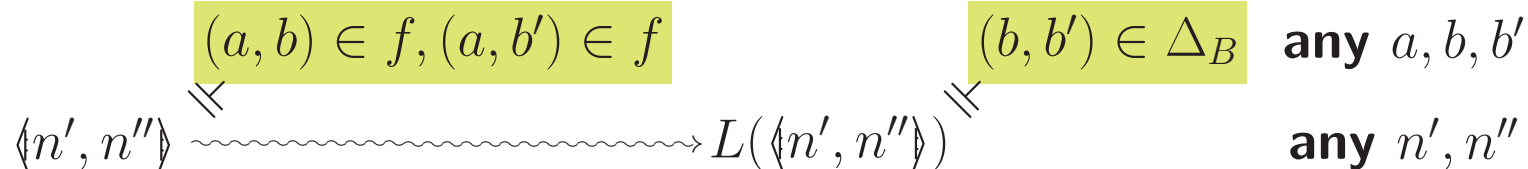
$f \subseteq A \times B$ :



$A \subseteq \pi_1^p(f)$ :



$[\pi_{12}^*(f) \cap \pi_{13}^*(f)] \subseteq \pi_{13}^*(\Delta_B)$ :



# Functions

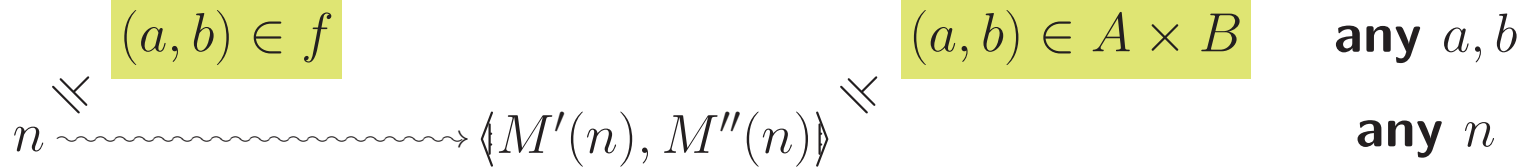
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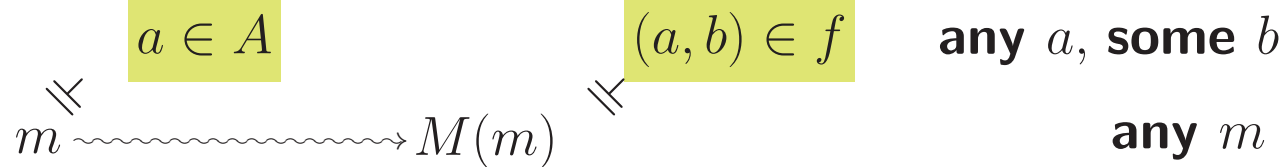
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 $A \subseteq \pi_1^p(f)$

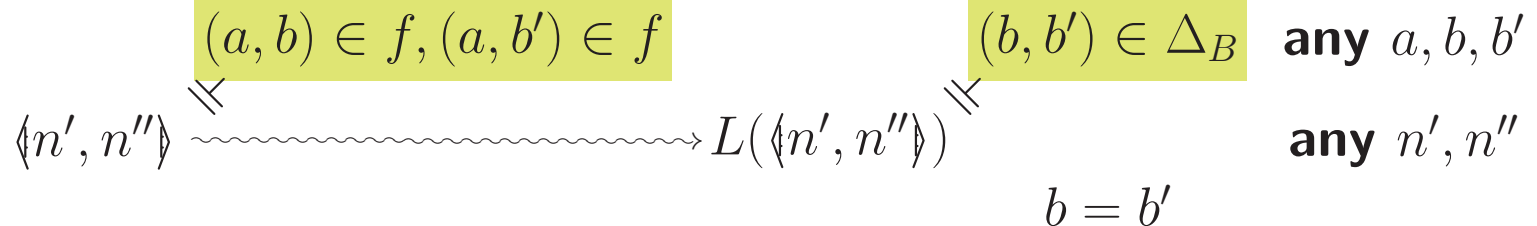
$f \subseteq A \times B$ :



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# Functions

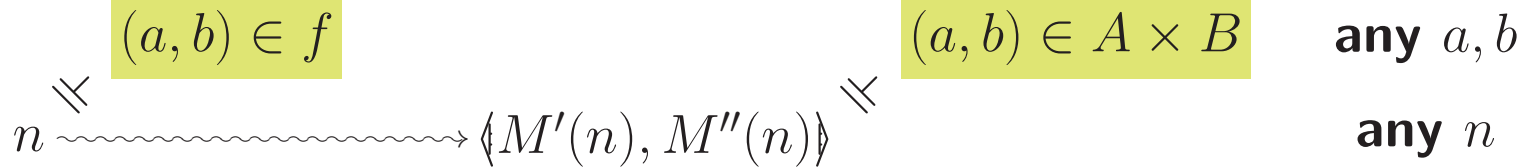
$f: A \rightarrow B$  must be  $f \subseteq A \times B$  such that

single-valued  $\forall a \in A \forall b \in B \forall b' \in B \quad [(a, b) \in f \wedge (a, b') \in f] \Rightarrow b = b'$

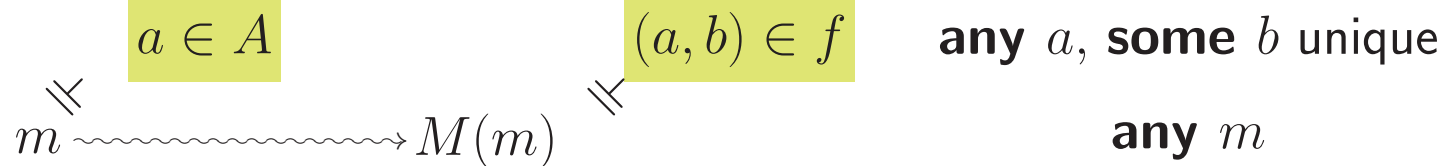
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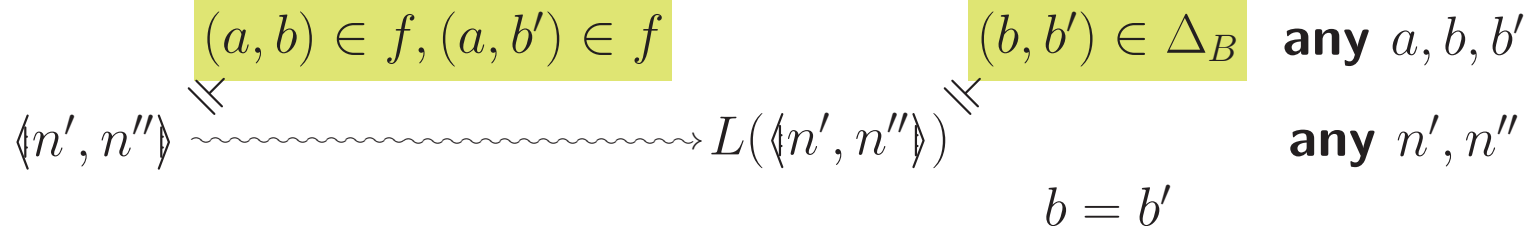
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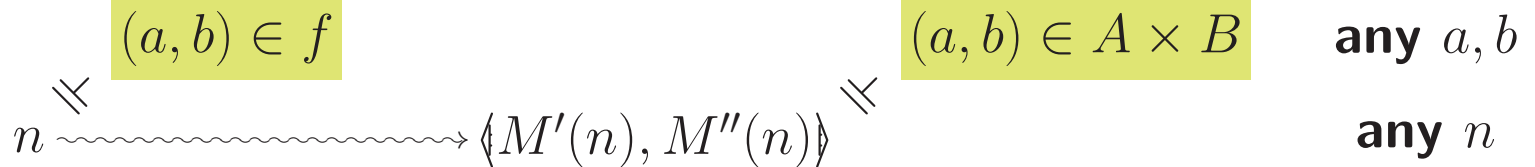
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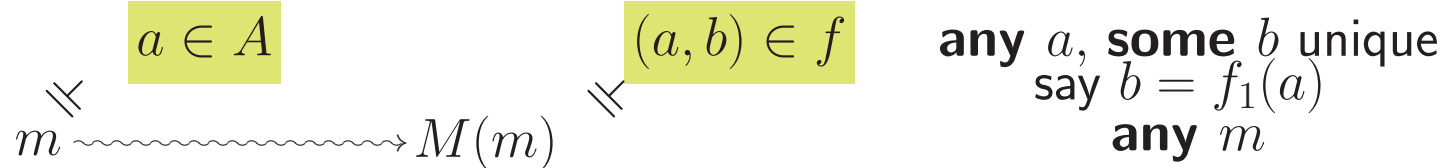
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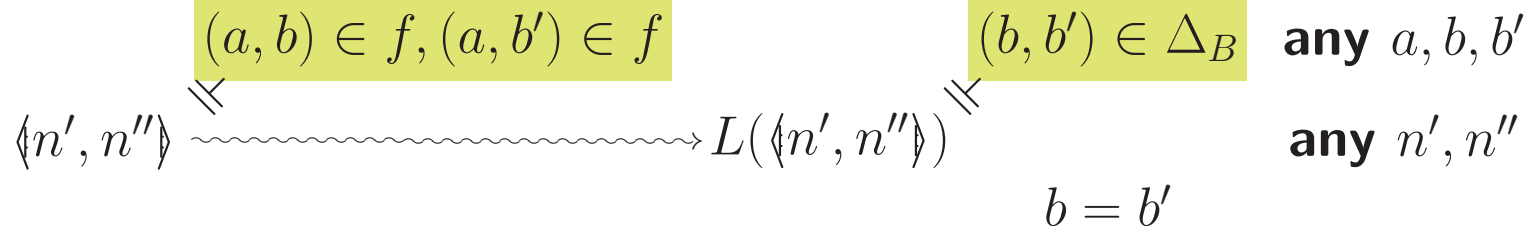
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There is also a computable function  $H$  which

for every  $a \in A_{el}$  and  $n \in \mathbb{N}$ , if  $n \Vdash a \in A$  then  $H(n) \Vdash f_1(a) \in B$

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Moreover, define:  $[f_1] \stackrel{\text{def}}{=} (\{(a, f_1(a)) \mid a \in A_{el}, \Vdash_{[f_1]}\})$   
 where  $n \Vdash_{[f_1]} (a, f_1(a))$  if  $n \Vdash_A a$ .

Then

$f = [f_1]$

# Examples

•  $[f_1]: A \rightarrow \nabla S$      **if**  $a \in A$      **then**  $f_1(a) \in \nabla S$      **any**  $a$   
 $n \rightsquigarrow$       $n$

To give a function from an assembly  $A$  to  $\nabla S$  is to give a set-function from  $A_{el}$  to  $S$ .





# Examples

- $[f_1]: A \rightarrow \nabla S$   
**if**  $n \stackrel{\cong}{\parallel} \boxed{a \in A}$  **then**  $\boxed{f_1(a) \in \nabla S}$  **any**  $a$   
 $n \rightsquigarrow 0$  **any**  $n$

To give a function from an assembly  $A$  to  $\nabla S$  is to give a set-function from  $A_{el}$  to  $S$ .

- $[f_1]: \mathbf{N} \rightarrow \mathbf{N}$  for some computable function  $H$

**if**  $n \stackrel{\cong}{\parallel} \boxed{a \in \mathbf{N}}$  **then**  $\boxed{f_1(a) \in \mathbf{N}}$  **any**  $a$   
 $n \rightsquigarrow H(n)$  **any**  $n$



# Examples

- $[f_1]: A \rightarrow \nabla S$ 

<b>if</b>	$n \equiv$	$a \in A$	<b>then</b>	$n \equiv$	$f_1(a) \in \nabla S$	<b>any</b> $a$
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To give an endofunction on  $\mathbb{N}$  is to give a computable set-function on  $\mathbb{N}$ .

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# Examples

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 $0 \rightsquigarrow H(0)$

# Examples

- $[f_1]: A \rightarrow \nabla S$     if  $\begin{array}{c} a \in A \\ \parallel \\ n \end{array}$  then  $\begin{array}{c} f_1(a) \in \nabla S \\ \parallel \\ 0 \end{array}$     any  $a$   
any  $n$

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$$\text{if } \begin{array}{c} a \in \nabla S \\ \parallel \\ 0 \end{array} \text{ then } \begin{array}{c} f_1(a) \in \mathbb{N} \\ \parallel \\ H(0) \end{array} \quad \text{any } a$$





# Some category structure on assemblies

- natural number object:  $\mathbf{N}$

- finite products:  $A \times B$      $\mathbf{1} \stackrel{\text{def}}{=} \nabla\{0\}$

- coproducts:  $A + B \stackrel{\text{def}}{=} (A \times \nabla\{0\}) \cup (B \times \nabla\{1\})$

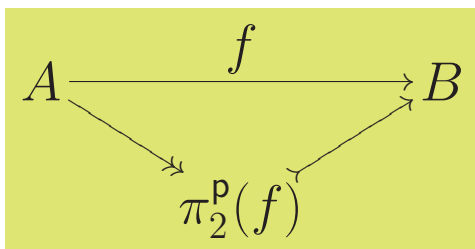
- equalizers: for  $[f_1], [g_1]: A \rightarrow B$  an equalizer is  $A \cap \nabla\{a \in A_{el} \mid f_1(a) = g_1(a)\} \subseteq A$

- exponentials:  $B^A \stackrel{\text{def}}{=} (\{f_1 \mid [f_1]: A \rightarrow B\}, \Vdash_{B^A})$

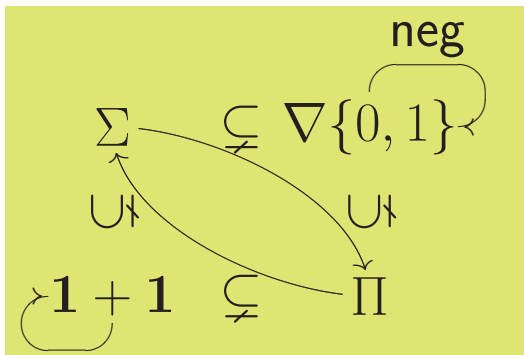
where  $e \Vdash_{B^A} f_1$  means that

if  $a \in A$  then  $f_1(a) \in B$  any  $a$   
 $\Vdash_n \rightsquigarrow [e](n) \Vdash$  any  $n$

- factorization:



# Some assemblies of truth values



$n \Vdash x \in \Sigma$  if either  $\lceil n \rceil(n) \downarrow$  and  $x = 1$   
or  $\lceil n \rceil(n) \uparrow$  and  $x = 0$

$n \Vdash x \in \Pi$  if either  $\lceil n \rceil(n) \uparrow$  and  $x = 1$   
or  $\lceil n \rceil(n) \downarrow$  and  $x = 0$

neg swaps 0 and 1.

- $f: A \rightarrow \nabla\{0, 1\}$  is completely determined by the subset  $f_1^{-1}\{1\} \subseteq A_{el}$
- for  $f: A \rightarrow \Sigma$  the subset  $\{n \mid n \Vdash_A x \text{ for some } x \in f_1^{-1}\{1\}\} \subseteq \{n \mid n \Vdash_A x \text{ for some } x \in A_{el}\}$  is relatively r.e. *i.e.* it is of the form  $\{n \mid n \Vdash_A x \text{ for some } x \in A_{el}\} \cap W$  for some  $W \subseteq_{\text{r.e.}} \mathbb{N}$
- for  $f: A \rightarrow \mathbf{1} + \mathbf{1}$  the subset  $\{n \mid n \Vdash_A x \text{ for some } x \in f_1^{-1}\{1\}\} \subseteq \{n \mid n \Vdash_A x \text{ for some } x \in A_{el}\}$  is relatively recursive *i.e.* both it and its complement are relatively r.e.

$$\Sigma = \{p \mid \exists f \in \mathbf{N}^{\mathbf{N}} \quad p \Leftrightarrow \exists n \in \mathbf{N} \ f(n) = 0\}$$

Take the relatively r.e. subset

$$\{e \mid [e](n) = 0 \text{ for some } n \in \mathbf{N}\} \cap (\mathbf{N}^{\mathbf{N}})_{el} \subseteq (\mathbf{N}^{\mathbf{N}})_{el}$$

and let  $s: \mathbf{N}^{\mathbf{N}} \rightarrow \Sigma$  be the induced function.

- $\forall p \in \Sigma \exists f \in \mathbf{N}^{\mathbf{N}} \quad (f, p) \in s$

$$\Sigma \subseteq \pi_2^p(s):$$

$$\begin{array}{ccc}
 & p \in \Sigma & (f, p) \in s \\
 \cong & & \cong \\
 n & \rightsquigarrow & C(n)
 \end{array}$$

take **any**  $p$ , **some**  $f$   
 $f: k \mapsto [C(n)](k)$   
**any**  $n$

$[C(n)](k)$  computes  $k$  steps of the computation  $[n](n)$  and returns 0 if it stops, 1 otherwise.

- $\forall f, g \in \mathbf{N}^{\mathbf{N}} \quad s(f) = s(g) \Leftrightarrow [\exists n \ f(n) = 0 \Leftrightarrow \exists m \ g(m) = 0]$

$$\{(f, g) \in \mathbf{N}^{\mathbf{N}} \times \mathbf{N}^{\mathbf{N}} \mid s(f) = s(g)\} = \{(f, g) \in \mathbf{N}^{\mathbf{N}} \times \mathbf{N}^{\mathbf{N}} \mid \exists n \ f(n) = 0 \Leftrightarrow \exists m \ g(m) = 0\}$$

# There is an enumeration $\mathcal{W}: \mathbb{N} \twoheadrightarrow \Sigma^{\mathbb{N}}$

- $\Sigma^{\mathbb{N}}$  is the assembly  $(\Sigma^{\mathbb{N}})_{el}$  consists of the r.e. subsets of  $\mathbb{N}$   
 $e \Vdash V \in \Sigma^{\mathbb{N}}$  iff  $V = \{n \in \mathbb{N} \mid [e](n) \downarrow\}$

$$\mathcal{W}_1: \mathbb{N} \rightarrow \text{RE}: e \mapsto \{n \in \mathbb{N} \mid [e](n) \downarrow\}$$

$$\begin{array}{ccc}
 \begin{array}{c} \parallel \\ n \end{array} & \begin{array}{c} \boxed{e \in \mathbb{N}} \\ \rightsquigarrow \\ n \end{array} & \begin{array}{c} \parallel \\ \mathcal{W}_1(e) \in \Sigma^{\mathbb{N}} \end{array} & \begin{array}{l} \text{any } e \\ \text{any } n \end{array}
 \end{array}$$

$$\begin{array}{ccc}
 \begin{array}{c} \parallel \\ e \end{array} & \begin{array}{c} \boxed{W \in \Sigma^{\mathbb{N}}} \\ \rightsquigarrow \\ e \end{array} & \begin{array}{c} \parallel \\ n \in \mathbb{N} \end{array} & \begin{array}{l} \text{any } W, \text{ some } n \text{ with } \mathcal{W}_1(n) = W \\ \text{take } n = e \\ \text{any } e \end{array}
 \end{array}$$

# Markov's Rule

Suppose  $X \subseteq \mathbf{N} \times A$  is such that  $\forall a \in A \forall n \in \mathbf{N} [(n, a) \in X \vee (n, a) \notin X]$ .

$$\mathbf{N} \times A = X \cup [(\mathbf{N} \times A) \setminus X]$$

If  $A \setminus \pi_1^p(X) = \nabla \emptyset$  then  $A \subseteq \pi_1^p(X)$ . If  $A \setminus \pi_1^p(X) = \nabla \emptyset$  then  $A \subseteq \pi_1^p(X)$ .

Since  $\mathbf{N} \times A = X \cup [(\mathbf{N} \times A) \setminus X]$ , the relation

$(X \times \nabla \{0\}) \cup [((\mathbf{N} \times A) \setminus X) \times \nabla \{1\}] \subseteq (\mathbf{N} \times A) \times (1 + 1)$  is total and single-valued.

Hence there are two r.e.

# Markov's Rule

Let  $A$  and  $P \subseteq \mathbf{N} \times A$  be assemblies such that  $\forall z \in \mathbf{N} \forall x \in A ((z, x) \in P \vee (z, x) \notin P)$

$$\text{i.e. } \mathbf{N} \times A = P \cup [(\mathbf{N} \times A) \setminus P]$$

Then  $\forall x \in A (\neg \neg \exists z \in \mathbf{N} (z, x) \in P \Rightarrow \exists z (z, x) \in P)$

$$\text{i.e. } (A \setminus \pi_1^P(P)) \setminus \pi_1^P(P) = \pi_1^P(P)$$

Suppose

$$(n, a) \in \mathbf{N} \times A$$

$$(n, a) \in P \cup [(\mathbf{N} \times A) \setminus P]$$

$$\langle n, m \rangle \stackrel{\cong}{\rightsquigarrow} \langle M'(\langle n, m \rangle), M''(\langle n, m \rangle) \rangle \stackrel{\cong}{\rightsquigarrow}$$

Note that, for any  $n, m$ , the value  $M''(\langle n, m \rangle)$  is either 0 or 1.

Consider

$$a \in (A \setminus \pi_1^P(P)) \setminus \pi_1^P(P)$$

$$a \in \pi_1^P(P)$$

$$m \stackrel{\cong}{\rightsquigarrow} \langle M'(\langle \min n. [M''(\langle n, m \rangle) = 0], m \rangle) \rangle \stackrel{\cong}{\rightsquigarrow}$$

# Caucuses

An assembly  $A$  is a *caucus* if the relation  $a \dashv\vdash n$  is single-valued.

Recall that, by definition,  $\Vdash_A$  is surjective. Hence, on a caucus,  $\dashv\vdash_A$  is a function  $r: A_{el} \rightarrow \mathbb{N}$ .

**The rule of choice holds for every caucus.** [Otherwise said: every caucus is projective.]

Take any assembly  $B$  and any  $R \subseteq A \times B$  such that  $\forall a \in A \exists b \in B (a, b) \in R$  i.e.  $A \subseteq \pi_1^p(R)$

$$\begin{array}{c}
 \begin{array}{c} a \in A \\ \cong \\ m \rightsquigarrow M(m) \end{array} \quad \begin{array}{c} (a, b) \in R \\ \cong \\ \text{any } a, z, w, \\ \text{some } b \\ \text{any } m, n \end{array} \quad \begin{array}{c} (z, w) \in R \\ \cong \\ n \rightsquigarrow \langle M'(n), M''(n) \rangle \end{array} \quad \begin{array}{c} (z, w) \in A \times B \\ \cong \end{array}
 \end{array}$$

By the standard Axiom of Choice, get  $g: A_{el} \times \mathbb{N} \rightarrow B_{el}$  such that

$$\begin{array}{c}
 \begin{array}{c} a \in A \\ \cong \\ m \rightsquigarrow M(m) \end{array} \quad \begin{array}{c} (a, g(a, m)) \in R \\ \cong \\ \text{any } a \\ \text{any } m \end{array}
 \end{array}$$

Define  $f_1: A_{el} \rightarrow B_{el}: a \mapsto g(a, r(a))$  and check that  $\forall a \in A (a, [f_1](a)) \in R$  i.e.  $[f_1] \subseteq R$

$$\begin{array}{c}
 \begin{array}{c} a \in A \\ \cong \\ m \rightsquigarrow M''(M(m)) \\ m = r(a) \end{array} \quad \begin{array}{c} (a, g(a, r(a))) \in R \\ \cong \\ \text{any } a \\ \text{any } m \end{array}
 \end{array}$$



# Projective Assemblies/Caucuses

- $\mathbb{N}$  is projective
- Every sub-assembly of  $\mathbb{N}$  of the form  $\mathbb{N} \cap \nabla Q$  for  $Q \subseteq \mathbb{N}$  is projective
- $1$  is projective
- Every assembly  $A$  where a single number  $k$  may realize, *i.e.* if  $n \Vdash a \in A$  then  $n = k$ .
- Every assembly of the form  $\nabla S$  is projective.

# Projective Covers

Given any assembly  $A$ , consider the assembly

$$P_A \stackrel{\text{def}}{=} (A_{el} \times \mathbb{N}, \Vdash_{P_A})$$

where  $n \Vdash_{P_A} (a, m)$  if  $n = m \Vdash_A a$

$P_A$  is projective.

Moreover, the first projection  $p_1: P_{A_{el}} \xrightarrow{\cong} A_{el} \times \mathbb{N} \rightarrow A_{el}$  defines a surjection  $[p_1]: P_A \twoheadrightarrow A$

since

$$\begin{array}{ccc}
 & (a, m) \in P_A & a \in A \\
 n \cong & \xrightarrow{\text{wavy}} & n \cong \\
 n = m \Vdash_A a & & 
 \end{array}
 \quad
 \begin{array}{l}
 \text{any } a, m \\
 \text{any } n
 \end{array}$$

$$\begin{array}{ccc}
 a \in A & (a, m) \in P_A & \text{any } a, \text{ some } m \text{ with } p_1((a, m)) = a \\
 n \cong & \xrightarrow{\text{wavy}} & n \cong \\
 & & \text{take } m = n \\
 & & \text{any } n
 \end{array}$$

# Discrete/modest assemblies

Call an assembly  $D$  *discrete* (or *modest*) if every function  $\nabla\{0,1\} \rightarrow D$  is constant.

- $\mathbb{N}$  is discrete
- a sub-assembly of  $\mathbb{N}$  is discrete
- a surjective image of a discrete assembly is discrete
- a discrete assembly  $D$  is a surjective image of a sub-assembly of  $\mathbb{N}$ ,  
*i.e.* is totally determined by a partial equivalence relation on  $\mathbb{N}$ .

Exercise:  $D$  is discrete if and only if the map  $\text{const}: D \rightarrow D^{\nabla\{0,1\}}$  is a bijection.

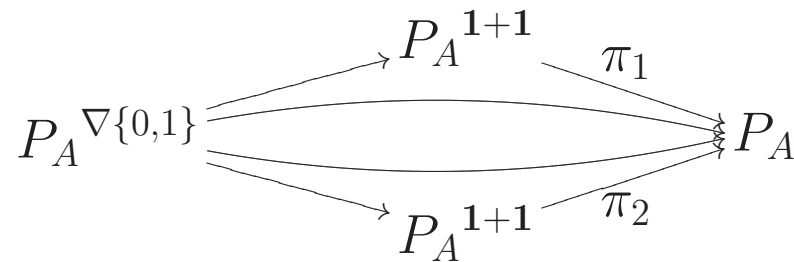
# Characterization of discrete/modest assemblies

General properties of the projective cover  $P_A$ :

- the second projection  $p_2: P_{A_{el}} \xrightarrow{\cong} A_{el} \times \mathbb{N} \rightarrow \mathbb{N}$  defines a function  $[p_2]: P_A \rightarrow \mathbb{N}$

- the pairing  $([p_1], [p_2]): P_A \rightarrow A \times \mathbb{N}$  is one-one

- the parallel maps



form the kernel pair of the map  $[p_2]: P_A \rightarrow \mathbb{N}$ .

For  $D$  discrete, prove that  $[p_2]: P_D \rightarrow \mathbb{N}$  is one-one.

# The effective topos

Assemblies are not enough.

The reason is “lack of quotient of equivalence relations”.

Define composition of relations as

$$R; S \stackrel{\text{def}}{=} \pi_{13}^p(\pi_{12}^*(R) \cap \pi_{23}^*(S))$$

An equivalence relation is  $R \subseteq A \times A$   $\Delta_A \subseteq R$   $R^o \subseteq R$   $R; R \subseteq R$

Formally add quotients by taking the equivalence relations themselves as objects and define a *map*  $F: R \rightarrow S$  to be  $F \subseteq A \times B$  such that

$$R; F; S = F \quad R \subseteq F; F^o \quad F^o; F \subseteq R$$

Every object is a surjective image of a projective assembly.

There is a surjection  $[p_1]: \nabla(P(\mathbb{N})) \twoheadrightarrow \Omega$ .

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