§.0 INTRODUCTION - (to Part I and II) -. By "data type" one usually intends a set of objects of the same "type" or "kind", suitable for manipulation by a computer program. Of course, computers actually manipulate formal representations of objects. The purpose of the mathematical semantics of programming languages, though, is to characterize data types (and functions on them) in a way which is independent of any specific representation mechanism. Thus the objects one deals with are mostly elements of domains borrowed from Set-Theory, Algebra, Category Theory ... , whose meaning is well understood within each framework and does not depend on the practice of programming. However, by doing so, what is lost is the notion of effective computability, which has an intrinsic operational character. This notion may be recovered by a suitable definition of "computable object" in abstract set-theoretic, algebraic, category-theoretic ... settings.

In particular, a more specific motivation for the study of effectiveness over semantic domains may be suggested by the following analogy.

The categories one needs for interpreting high level programming languages must possess strong completeness and closure properties so

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that the existence of objects, which are formally given by general definitional tools, is "a priori" assured: e.g. we want that cartesian products and morphism spaces still belong to the given category, for these constructs are commonly used in the design of high level languages.

Similarly, completeness and closure properties are the key idea for defining domains and categories in several areas of Mathematics, Banach, Hilbert or Sobolev spaces, say, may be considered as the (metric) completion of the possible solutions of a given set of equations. Once the solution of the problem studied is found in one such a space, it is then time to ask whether it is an "acceptable" solution from the intended viewpoint or whether it has been added by the completion technique. For example, for a given set of partial differential equations, one may (easily) find a solution in the related (Sobolev) space and then check whether it is an acceptable (regular) solution, i.e. whether it is differentiable in the ordinary sense.

Now, acceptable for a computer scientist means computable. It is then worth pursuing a general notion of effectiveness over abstract data types, since computable elements and maps provide the "regular" interpretation of programming constructs over semantic domains. Preliminary investigations on the effectiveness of the semantics of programs may be found in Scott (1976), Giannini & Longo (1983), Kanda (1984).

Unfortunately, the natural numbers, $\omega$, and the partial recursive functions, PR, are not sufficient for this investigation, since, in general, typed and type-free languages cannot be directly interpreted over PR or $\omega$. PR and $\omega$, though, may be used for defining effectiveness over more general data types. The methods are borrowed from higher type Recursion Theory or computability in abstract structures, nowadays strictly interrelated topics in view of the work done in the 70's by several authors (see references).

This paper is motivated by the study of completeness and closure properties of natural categories of effectively given data types. Countability, say, is a useful assumption for dealing with effectiveness over abstract data types.

Suppose one is given two countable sets $A$ and $B$, and two numberings (onto maps) $e_A : \omega \rightarrow A$, $e_B : \omega \rightarrow B$. There is then a natural de-
Definition of computable map between \( A \) and \( B \): call \( g : A \rightarrow B \) (effective) morphism iff there exists a recursive function \( f \) such that the following diagram commutes:

\[
\begin{array}{ccc}
\omega & \xrightarrow{f} & \omega \\
\downarrow{e_A} & & \downarrow{e_B} \\
A & \xrightarrow{\text{g}} & B
\end{array}
\]

The category of numbered sets \((EN)\) whose objects are pairs such as \( A = (A,e_A) \) and morphisms defined as in (1), has been studied in Ershov (1973, 1975). An introduction and some applications may be found in Visser (1980) and Bernardi & Sorbi (1983), mainly, or Barendregt & Longo (1982).

The first question one may ask about \( EN \) is whether there is a natural way to give, effectively, a numbering to the set \( EN(A,B) \), the set of morphisms from \( A \) to \( B = (B,e_B) \). In general, there is no such a "uniform" and "effective" coding of \( EN(A,B) \), given \( A \) and \( B \). As a matter of fact \( EN \) is far away from being Cartesian Closed.

Nonetheless \( EN \) has several nice properties. We recall a notion and a simple consequence of it, whose relevance should be clear.

\( R \) is the set of the (total) recursive functions; \( \omega \) is \((\omega, id)\).

0.1 Definition. \( A \) is a precomplete numbered set if

\[
\forall f \in R \exists f' \in R \forall n(f(n) + e_A f(n)) = e_A f'(n) \quad (\text{i.e. } f' \text{ extends } f \text{ w.r.t. } (A,e_A)).
\]

\( A \) is complete if

\[
\exists a \in A \forall n(f(n) + e_A f(n)) = a \quad (a \text{ is a special (bottom) element}).
\]

0.2 Generalized Recursion Theorem. Let \((A,e_A)\) be precomplete. Then

\[
\exists n(f(n) + e_A f(n)) = e_A (f(n)) \quad (3)
\]

The partial recursive functions suggest an obvious notion of partial morphism for numbered sets.

0.3 Definition. \( A \) and \( B \) be numbered sets. Then \( f \in EN(A,B) \) (f is
a partial morphism) if \( \exists f' \in PR \ f \circ e_A = e_B \circ f' \).

For the purposes of this paper, partial morphisms will be studied in a general category-theoretic setting, since partial maps come out naturally in computability theory. Note that (3) above is equivalent to
\[
\forall f \in EN \ (\forall \omega, A) \ \exists n(f(n) \downarrow) \Rightarrow f(n) = e_A(n).
\]
Completeness may be related to a Least Fixed Point Theorem (see later).

Of course, (3) in Theorem 0.2 is a very desirable property for handling abstract data types, in view of the recursive definitions. But exactly because of this, one may want more; namely the possibility of inheriting completeness and other properties at higher types, i.e. for the set of morphisms on numbered sets, since functions are among the typical data to be mostly defined recursively. This cannot be done in general, in view of the lack of the above mentioned closure properties for EN.

There are two reasonable ways to obtain the Cartesian Closure (CC) of a Category such as EN: one may restrict the attention to a subcategory or enlarge the Category itself. The point is that both ways should be "natural" and should give interesting categories.

In Part I we study a direct, elementary characterization of the "main" types of a well known sub-CCC of EN, Scott's effectively given domains (their computable sub-objects, to be precise). This will be done by a type structure over \( \omega \), based on two simple notions: acceptable pairing and relative (Gödel-)numberings (§.1).

§.2 and 3 presents CCC's with partial morphisms and partial objects and relates domains to EN also by using these notions.

Part II will introduce the CCC of Generalized Enumerations, whose definition is inspired by the notion of relative numbering and will relate it, as well as its computability properties, to EN.

§.1 An elementary approach to higher type computability

Let 0 be the type of \( \omega \). Then the integer types are defined by \( n+1 = n + n \) and the pure types by \( n+1 = n + 0 \). Partial computable functionals in the integer and pure types may be introduced by using only \( \omega \) and PR, with no mention of the category-theoretic and continuity

1.1 Definition. Let \( L^0 = \omega \) and fix \( L \subseteq \omega \rightarrow \omega \). Define then

\[
L^{n+1.5} = \{ \phi : L^n \rightarrow L^{n+1} / \lambda xy. \phi(x)(y) \in L^{n+1} \}
\]

\[
L^{n+2} = \{ f : L^{n+1} \times L^{n+1} / \forall \phi \in L^{n+1.5} f \circ \phi \in L^{n+1.5} \}.
\]

The key idea is that (some) functions in \( L^{n.5} \) Gödelize \( L^{n+1} \) by \( L^n \) (see the notion of relative numberings in 3.2.2).

There is another way to look at the HPEF, which makes explicit the role of the pairing function \(<,>\), implicitly used in the definition of \( L^{n+1.5} \).

1.2 Definition. Let \( U \) be a set and \( F \subseteq U \rightarrow U \). Then \( <,> : U \times U \rightarrow U \) is an acceptable pairing w.r.t. \( F \) if:

1) \( \exists p_1, p_2 \in F \forall x, y \in U \ p_1(<x_1, x_2>) = x_1 \), where \( p_1 \) and \( p_2 \) are total

2) \( \forall f, g \in F \lambda x. <f(x), g(x)> \in F \).

Following the polish tradition in constructive mathematics, an interesting class of (pure) type 2 total functionals on \( \mathbb{R} \) is defined in Rogers (1967; p. 364). Namely, \( f : \mathbb{R} \rightarrow \omega \) is Banach-Mazur if

\( \forall g \in \mathbb{R} \exists h \in \mathbb{R} f(\lambda y.g(<x, y>)) = h(x) \), where \( <,> : \omega \times \omega \rightarrow \omega \) is an effective pairing function (an acceptable pairing w.r.t. \( \mathbb{R} \), in our terminology).

This can be generalized and extended at higher types as follows.

1.3 Definition. (GBM) Let \( BM^0 = \omega \) and fix \( BM^1 \subseteq \omega \rightarrow \omega \). Define then

\[
BM^{n+2} \subseteq BM^{n+1} \rightarrow BM^{n+1}
\]

by

\[
f \in BM^{n+2} \text{ if } \forall g \in BM^{n+1} \exists h \in BM^{n+1} \forall x \in BM^n f(\lambda y.g(<x, g>)) = \lambda y.h(<x, y>),
\]

where \( <,> : BM^b \times BM^n \rightarrow BM^n \) is an acceptable pairing w.r.t. \( BM^{n+1} \).

What remains to be verified is that \( <,> \) actually exists in any type, for a suitable choice of \( BM^1 \). This will be done in §.3.

It is now easy to see that, if \( L^1 = BM^1 \), then \( \forall n \ L^n = BM^n \).

Just notice that

\[
(3) \ q \in L^{n+1.5} \text{ iff } \exists g' \in L^{n+1} \forall x, y \in L^n g(x)(y) = g'(<x, y>).
\]
Thus, for $f \in L^{n+2}$, $f \circ g(x) = f(\lambda y.g'(\langle x,y \rangle))$ and, for some $h \in L^{n+1}$, $f \circ g(x) = \lambda y.h(\langle x,y \rangle)$, by the definition of $L^{n+2}$ and (3) applied to $f \circ g$. The rest is obvious.

It is also a simple exercise to give a variant in the pure types of the GBM or the HPEF. Thus these functionals are an easy way to define partial computable functions in higher types, by taking $L^1 = PR$ or $BM^1 = PR$. Partial maps turned out to be essential in computability theory, mainly because they may be effectively numbered and possess universal functions. Moreover, the related type structures yield models of functional languages, namely of typed and type-free $\lambda$-calculus, as it will be mentioned below.

Interestingly enough the proof that these hierarchies are well defined (i.e. that $\langle,\rangle$ exists in any type) goes together with the proof of their main properties, which heavily rely on category theoretic and continuity notions for EN and Scott's domains. One cannot avoid, then, some mathematics. Let's first discuss the issue of partiality in a category-theoretic frame.

§.2. Partial morphisms and partial objects

There are at least three different ways to introduce the notion of divergence in categories. By using partial morphisms, partial objects or both. In this section we consider concrete categories (with partial morphisms), i.e. subcategories of Set ($Set_p$), and see how these ways relate.

2.1 Definition. $Set_p$ is the category whose objects are sets and where $Set_p(x,y) = \{ f | f : X \to Y \text{ (partial)} \}$, for all objects $x,y$.

The following notion has been inspired by a talk given in Siena by A. Heller.

2.2 Definition. $C$ is a concrete category with partial morphisms (pC) if:

1) Every hom-set $C(x,y)$ contains an every-where divergent morphism $0_{x,y}$ s.t. for all objects $z,v$ and any $f \in C(z,x)$ and any $g \in C(y,v)$ one has
There exists a singleton object \( t \) s.t.
\[
C(t,t) = \{0_{t,t}, \text{id}_t\} \quad \text{and} \quad \forall x,y \forall f,g \in C(x,y) \ (f = g \iff \forall h \in C(t,x) \ f \circ h = g \circ h).
\]

Singleton objects clearly coincide to within isomorphism. Thus the category of total morphisms, defined as follows, does not depend on the choice of \( t \).

### 2.3 Definition
Let \( C \) be \( pC \) and \( t \) a singleton in \( C \). Define then \( C_T \) with objects in \( C \) and morphisms as follows:
\[
C_T(x,y) = \{f \in C(x,y) / \forall h \in C(t,x) (f \circ h = 0_{t,y} \iff h = 0_{t,x})\}.
\]

Clearly \( EN^p \) is \( pC \) and \( (EN^p)_T = EN \).

A \( pC \) may be embedded in \( Set \) in the same way as a concrete category may be embedded in \( Set \). Namely, the embedding functor
\[
I = C_T(t,-) : C \to Set \text{ is faithful and } I(C(t,x)) = Set(\text{It},Ix).
\]

As usual, one may also represent a partial \( f : x \to y \) by a total map \( f^\bot : x \to y^\bot \), where \( y^\bot \) is obtained from \( y \) by adding a fresh element to \( y \). Recall that, in a category \( C \), \( x \preceq y \) (\( x \) is a retract of \( y \)) if there exists a pair \( (\text{in}, \text{out}) \), with \( \text{in} \in C(x,y) \), \( \text{out} \in C(y,x) \) s.t.
\[
\text{out} \circ \text{in} = \text{id}_x.
\]
By this, we may give a notion of partial object, suitably related to partial morphisms.

### 2.4 Definition
Let \( C \) be a \( pC \). Define then

1) \( -^\bot : C \to C_T \) is a bottom functor if \( C(x,y) \preceq C_T(x,y^\bot) \).

2) \( x \) is a partial object if \( x \preceq x^\bot \) in \( C_T \).

\( (\text{Intuition: } \preceq \text{ upside down triangle} \) \).

### 2.5 Remark
Let \( t \) be a terminal object (a singleton) in \( C_T \). Then \( t \preceq x^\bot \); moreover, if \( x \) is a partial object, then \( t \preceq x \).

Partial morphisms and partial objects may be more fully related within Cartesian Closed Categories. These categories may be defined as in the classical case. One has to take care, though, of the behaviour of functors and natural transformations, which should be preserved
on partial morphisms. This may be done using an (implicit) notion of domain (see i, ii, iii in 2.6 below).

2.6 Definition. C is a pCCC if C is pC with the following adjunctions:

1) a terminal singleton object t for \( C_T \);

2) \( \Delta \cdot, x. \cdot \cdot \cdot : C_T \times C_T \to C_T \)

3) for any object a, \( \langle a \cdot, a \cdot \cdot \cdot : C_T \to C \),

where:

i) if \( f \in C(x, y) \) and \( g \in C(x, z) \), then

\[
\forall h \in C_T(t, x) \ (f \cdot g) \circ h = \begin{cases} 0 & \text{if } f \circ h = 0 \text{ or } g \circ h = 0 \\ (f \circ h)(g \circ h) & \text{otherwise} \end{cases}
\]

ii) if \( f \in C(x, y) \) and \( g \in C(x', y') \), then

\( f \cdot g = (f \circ p_1)(g \circ p_2) \);

iii) if \( f \in C(x, y \cdot a) \), then

\( \forall h \in C_T(t, x) \ (f^{-1}) \circ (h \cdot a) = \begin{cases} 0 & \text{if } f \circ h = 0 \\ f^{-1}(f \circ h) & \text{otherwise} \end{cases} \)

Observe that the extensions in the adjunctions in 2.6.2 and 3 are unique. As usual, \( x^y_p \) is an object and represents \( C(y, x) \).

2.7 Proposition. Let C be a pCCC, x and y objects in C and t a terminal object. Then

(i) \( x^t_p \approx x^t \), i.e. \( t \) is a bottom functor,

(ii) \( x^y_p \) is a partial object.

Proof. (i) obvious (ii). We have to prove that \( x^y_p \cong (x^y_p)^\perp \) in \( \mathcal{C}_T \).

Let us identify \( x \times t \) with \( x \) and \( x^i \) with \( x^t_p \), by (i). Note then that the following diagrams commute:

\[
\begin{array}{ccc}
(x^y_p)^\perp \times t & \xrightarrow{\text{eval}} & x^y_p \\
\downarrow^{(\Lambda \text{id})} & & \downarrow^\text{id} \\
x^y_p \times t & \xrightarrow{\text{eval} \cdot \text{id}} & x^y_p \times t
\end{array}
\]

\[
\begin{array}{ccc}
(x^y_p) \times y & \xrightarrow{\text{eval} \cdot \text{id}} & x \\
\downarrow^{(x^y_p)^\perp \times y} & & \downarrow^{\Lambda \cdot \text{eval} \cdot \text{id}} \\
x^y_p \times y & \xrightarrow{(\Lambda(\Lambda \cdot \text{eval}) \cdot \text{id})} & x^y_p \times y
\end{array}
\]
Finally set \( \text{in} = \text{Id:} \ x^Y \rightarrow (x^Y)^i \) and \( \text{out} = \Lambda(\Lambda \text{eval}): (x^Y)^i \rightarrow x^Y \).

2.8 Proposition. Let \( C \) be a pCCC, \( C_T \) CCC and \( x \) a partial object. Then, for any object \( y \), one has:

(i) \( x^Y \triangleleft x^Y \)

(ii) \( x^Y \triangleleft x^Y \times t^Y \)

Proof. (i) \( x^Y \triangleleft (x^Y) \sim (x^Y) \sim x^Y \sim x^Y \), by 2.6 (i). (ii) \( \text{in}_1 = \text{out}: x^Y \rightarrow x^Y \), by (i); \( \text{in}_2 = \text{out}: x^Y \rightarrow t^Y \), by \( t \triangleleft x \) (see 2.5). Moreover, \( \text{out} = \sim: x^Y \times t^Y \rightarrow x^Y \), by the extended adjunction as defined in 2.6.

By 2.7 and 2.8, total and partial morphisms, as well as partial objects, are nicely related. In particular, when the target object is partial, partial morphisms do not change the higher type structure in an essential way. In contrast to this, when the target \( x \) is not partial, we only know that \( x^Y \) is a subobject of \( x^Y \), while nothing can be said about higher types.

We conclude this section by returning to the categories we are interested in for the purposes of computability in abstract data types: domains and numbered sets.

A presentation of the CCC's of domains and effectively given domains, with continuous (and computable) maps as morphisms, may be found in Scott (1982) (see also Giannini & Longo (1983)). A constructive domain is (isomorphic to) the collection of all computable elements in an effectively given domain.

2.9 Generalized Myhill-Shepherdson Theorem (Ershov (1976)). The category of constructive domains is a full sub-CCC of \( \text{EN} \).

Proof. (see Giannini & Longo (1983), say).

We are now in the position to reword a simple result in Ershov (1973/5). A pCC is a partial Cartesian Category in the obvious way.

2.10 Proposition. \( \text{EN}_p \) is a pCC with a bottom functor.

\( \text{EN} \) is clearly not a full sub-category of \( \text{EN}_p \). However, one may
still naturally relate domains to $\mathbb{D}$ by the following simple variant of 2.9. Note also that all now empty domains are partial objects.

2.11 **Theorem.** The category of constructive domains with strict maps is a full sub-$\mathbb{D}$ of $\mathbb{D}$.

§.3. **Relative numberings and principal morphisms in $\mathbb{D}$**

3.1 **Definition.** Let $A, B$ be objects in $\mathbb{D}$ and $f, g : A \rightarrow B$. Define then

$$f \preceq_A g \text{ if } \exists h \in \mathbb{D}(A, A) f = g \circ h$$

Note that, if $A = B = \omega$, this is a classical notion of recursion-theoretic reducibility. Acceptable Gödel-numberings inspired 3.2.1.

3.2 **Definition.** Let $A$ and $B$ be in $\mathbb{D}$. Define then

1) $f \in \mathbb{D}(\omega, A)$ is an **acceptable numbering** of $A = (A, e_A)$ if $e_A < f$.

2) $f \in \mathbb{D}(A, B)$ is a **relative numbering** of $B$ w.r.t. $A$ if $e_B < f \circ e_A$ (i.e. $f \circ e_A$) is an acceptable numbering of $B$.

3) $f \in \mathbb{D}(A, B)$ is a **principal morphism** if $\forall h \in \mathbb{D}(A, B) h < f$.

3.3 **Remark.** $f \in \mathbb{D}(\omega, A)$ and $f \in \mathbb{D}(A, B)$ in 3.2.1-2 are equivalent to $f < e_A$ and $f \circ e_A < e_B$. Principal morphisms may be easily generalized to arbitrary categories. In CCC's, principal morphisms characterize models of Combinatory Logic, see Longo & Moggi (1983b).

3.4 **Remark.** It is easy to prove that, if $f \in \mathbb{D}(A, B)$ is a relative numbering, then

$$A \text{ (pre-)complete } \Rightarrow B \text{ (pre-)complete},$$

see Longo & Moggi (1983).

3.5 **Proposition.** Let $f \in \mathbb{D}(A, B)$ be a relative numbering. Then one has

(i) for $h : B \rightarrow C$, $h \in \mathbb{D}(B, C)$ iff $h \circ f \in \mathbb{D}(A, C)$,

(ii) if $f \preceq_A g \in \mathbb{D}(A, B)$, then also $g$ is relative.

(Thus, in presence of a relative numbering, any principal morphism is a relative numbering too).
Proof. (i) \( \Rightarrow \) : obvious. \( \Leftarrow \) : for \( f \) is relative.
Thus \( \exists h' R h e B = e_c h' \), by the assumption.

(ii) \( e_B = f e_A f' \), for some \( f' e R \) since \( f \) is relative,
\[ = e^g e_A f', \] for some \( l \in \text{EN}(A, A) \) since \( f \leq A \),
\[ = e g e_A f' \), for some \( l' e R \).

In view of the strict limit on the number of pages imposed by the Publisher, from now on we are forced to skip the proofs. An elementary proof (i.e. with no category theory) of 3.11 may be found in Longo & Moggi (1983). The authors plan an expanded version of the present paper.

Write \( A \triangleleft B \) (or \( A \triangleleft B \)) for \( A \) is a retraction of \( B \) in \( \text{EN}_p \) (or in \( \text{EN} \)).

3.6 Theorem. Let \( \omega \triangleleft A \) and \( B \triangleleft B \). Then one has

(i) \( \exists f e \text{EN}(A, B) \) relative numbering,

(ii) if one also has \( A \times A \triangleleft A \) and \( B \) exists, then \( \exists g e \text{EN}(A, B) \)

The following Lemma shows how retractions are inherited at higher types.

3.7 Lemma. Assume that, for \( A \) and \( B \) in \( \text{EN} \), \( B_A \) exists. Then

(i) \( \omega \triangleleft B( \triangleleft B ) \Rightarrow \omega \triangleleft B \triangleleft (B_A) \),

(ii) \( B \triangleleft B \Rightarrow B \triangleleft (B_A) \),

(iii) \( B \times B \triangleleft B \Rightarrow B \triangleleft B \triangleleft (B_B) \).

The type structure of domains over \( \omega \) in \( \text{EN} \) may be defined as follows.

3.8 Definition. Let \( T \) be the smallest set of finite types symbols containing \( 1 \) (i.e. \( 1 \in T; \sigma, \tau \in T \Rightarrow \sigma x \tau, \sigma \rightarrow \tau \in T \)). Define then

\[ E^{\sigma x \tau}_C = E^{\sigma}_C \times E^{\tau}_C \text{ and } E^{\sigma \rightarrow \tau}_C = (E^{\tau}_C)^{E^{\sigma}_C} \]

Of course, \( \{E^\sigma_C/\sigma \in T\} \) is the sub-CCC generated by \( \omega^p = \text{PR} \) in \( \text{EN} \).

The subscript \( c \) recalls that each \( E^\sigma_C \) is actually a constructive
domain, by 2.8. Thus all the numbered sets in the type structure, are actually partial objects. By 2.4 and the results following it, the total maps in each $E^o_C$ may be rightfully considered as partial computable functionals.

3.9 Lemma. $\forall \sigma, \tau \in T \quad \exists \phi, \psi \in E^o_C \quad E^o_C \not\subset (E^o_C)^I$ and $E^o_C \times E^o_C \not\subset E^o_C$, for all $n > 0$.

3.10 Theorem. $\forall \sigma, \tau \in T \quad \exists \phi, \psi \in EN(E^o_C, E^o_C)$ principal morphism and relative numbering.

Proof. By 3.6 (ii), 3.9 and 2.8, $EN(E^o_C, E^o_C)$ contains a principal morphism. Moreover, by 3.6 (i) and 3.9, it also contains a relative numbering. By 3.5 (ii) we are done.

3.11 Remark. By this and by results in Longo & Moggi (1983b), each $E_C$ yields a type-free Combinatory Algebra; actually a model of $\lambda$-calculus.

In view of all the information we have on numbered sets, we are now in the position to give the main theorem in Longo & Moggi (1983) as a simple corollary. This proves that the $BM^n$'s and the HPEF (see §.1) give the integer types in the type structure over $\omega$. The pure type variant of 3.12 is easily given.

3.12 Corollary. Let $L^1 = PR$. Then, for all $n > 0$,

$$L^n = E^n_C$$

$$L^{n+1.5} = EN(E^n_C, E^{n+1}_C).$$

Proof. (By induction) 0), 1) by definition

1.5) by a simple argument (show that $L^{1.5}$ contains all acceptable gödel-numberings of $PR$).

$n+2) EN(E^{n+1}_C, E^{n+1}_C) \not\subset L^{n+2}$, by definition and $n+1.5$.

Conversely, let $n+1, \tau \in EN(E^n_C, E^{n+1}_C)$ as in 3.10. Then

$n+2) EN(E^{n+1}_C, E^{n+1}_C) \not\subset L^{n+2}$, by definition and $n+1.5$.

Conversely, let $n+1, \tau \in EN(E^n_C, E^{n+1}_C)$ as in 3.10. Then

By 3.5 (i).

$n+2) EN(E^{n+1}_C, E^{n+1}_C) \not\subset L^{n+2}$, via $(<,>, (P_0, P_1))$, by 3.9. The pairing is
clearly acceptable w.r.t. $E^c_{n+2}$, in the sense of definition 1.2; hence
\[
\phi \in L_{n+2.5} \Rightarrow g = \lambda x. \phi(p_0(x))(p_1(x)) \in E^c_{n+2}
\]
\[
\Rightarrow \phi = \lambda x. (\lambda y. g(<x,y>)) \in EN(E^c_{n+1}, E^c_{n+2})
\]

Conversely,
\[
\phi \in EN(E^c_{n+1}, E^c_{n+2}) \Rightarrow f = \lambda xy. \phi(x)(y) \in EN(E^c_{n+1} \times E^c_{n+1}, E^c_{n+1})
\]
\[
\Rightarrow \lambda x. \phi(p_0(x))(p_1(x)) = f^o(p_0 \circ p_1) \in EN(E^c_{n+1}, E^c_{n+1}) = L_{n+1}.
\]

3.13 Remark. The key issue in this part has been the study of partial morphisms and objects in $EN$ and the related sub-CCC's. Note that, in precomplete numbered sets, partial morphisms may be always extended to total ones. As for complete numbered sets one can say more, in view of 2.4: with some work, it may be actually shown that complete numbered sets and partial objects coincide in $EN$.

(For references, see end of part II).
§1. Introduction. The type structure \( \{L^n\}_{n \in \omega} \) studied in §.3 of part I (i.e. with \( L^1 = \text{PR} \)) actually gives the partial computable functionals (see Ershov (1975)) in the integer types. The key fact was the possibility of enumerating each type \( n+1 \) by type \( n \), via a principal relative numbering. This generalizes the fact that \( \text{PR} \), i.e. \( L^1 \), can be effectively numbered by \( \omega \), i.e. \( L^0 \).

If one takes \( L^1 = \mathbb{R} \) (the total recursive maps) this is no longer possible, i.e. there is no effective numbering of \( \mathbb{R} \) by \( \omega \), therefore \( \{L^n\}_{n \in \omega} \) (with \( L^1 = \mathbb{R} \)) is not representable in \( \text{EN} \).

As pointed out in §.1 of part I, the definition of HPEF is rather general, and still works if we take as \( L^1 \) a set \( L \) of partial maps from \( \omega \) to \( \omega \) (instead of \( \text{PR} \) or \( \mathbb{R} \)).

We give a characterization of \( \{L^n\}_{n \in \omega} \), for \( L^1 \) enumeration-acceptable (see 1.1 below), in terms of a concrete CCC (and pCCC) based on the notion of numbering:

1.1 Definition. Let \( L : \omega \rightarrow \omega \), then \( L \) is enumeration-acceptable iff:
1) \( L \circ L \in L \), \( \text{id} \in L \);
1') \( 0 \in L \) (0 is the everywhere divergent function);
2) \( \forall n \in \omega \ \forall x \in L \ n \in L \);
3) there is an acceptable pairing of \( \omega \) w.r.t. \( L \) (see 1.2 of part I);
4) equality in \( \omega \) is decidable w.r.t. \( L \), i.e.
   \[ \forall f, g \in L \ \exists h \in L \ h(<x,y,z>) = \begin{cases} fz & \text{if } (x = y) \\ gz & \text{else} \end{cases} \]
1.2 Definition. The category of numbered sets on $L$ ($\text{EN}_L^P$) is defined by:

i) $A = (A, e_A) \in \text{EN}_L^P$ iff $e_A : \omega \to A$ is onto;

ii) $f \in \text{EN}_L^P(A, B)$ iff $f : A \to B$ and $\exists g \in L$ s.t. $f \circ e_A = e_B \circ g$.

Let $L$ be enumeration-acceptable and 1)-4) be the assumptions on $L$ in (1.1), then one easily has:

1) implies that $\text{EN}_L^P$ is a category;

1') implies that $\text{EN}_L^P$ has null morphisms;

2) implies that $\text{EN}_L^P$ has a singleton object;

2) and 3) imply that $\text{EN}_L^P$ is cartesian.

Remark. Note that the notion of enumeration-acceptable class of function is also a sound recursion-theoretic generalization of basic properties of PR. As a matter of fact, if $(\omega, \cdot)$ is a Uniformely Reflexive Structure, then $(\omega \to \omega) = \{f : \omega \to \omega \mid \exists a \in \omega \forall b \in \omega f(b) = a \cdot b\}$ is enumeration acceptable $\text{EN}_L^P$ is not a pCCC, hence the type structure generated from $\omega$ does not need to exist in it.

However every category $C$ may be embedded in the category of presheaves on $C$, $\text{Set}^{\text{op}}$ (which is a CCC), by a full and faithful functor, which preserves products and representations of morphisms (see Scott (1980), McLane (1971)). We will define a full sub-CCC ($\text{GEN}_L^P$) of $\text{Set}^{\text{op}}$ with the following property:

the embedding functor of $\text{EN}_L^P$ in $\text{Set}^{\text{op}}$ ($I_{\text{EN}}$) factorizes through that of $\text{GEN}_L^P$ in $\text{Set}^{\text{op}}$ ($I_{\text{GEN}}$)

i.e.: $\exists I$

\[ \begin{diagram} 
\text{EN}_L^P \rto \ertop \dlto & \text{GEN}_L^P \ertop \dlto \\
I_{\text{EN}} \drtwoar[2] & I_{\text{GEN}} \drtwoar[2] \\
\text{Set}^{\text{op}}(\text{EN}_L^P) \drtwoar[3] & \text{Set}^{\text{op}}(\text{GEN}_L^P) \drtwoar[3] 
\end{diagram} \]

1.3 Definition. Let $L$ be enumeration-acceptable, define then the category of generalized numbered sets on $L$ by:

i) $X = (X, e_X) \in \text{GEN}_L^P$ iff
1) $E_x \subseteq \omega \rightarrow X$,

2) $U\{\text{img } e | e \in E_x\} = X$,

3) $\forall e_0, e_1 \in E_x \exists e \in E_x \exists f_0, f_1 \in L$ s.t.

(\text{i.e. } E_x \text{ is a directed sets w.r.t. } L\text{-reducibility});

ii) $f \in \text{GEN}_p^L(X, Y)$ iff $f : X \rightarrow Y$ and

$\forall e \in E_x \exists g \in L \exists e' \in E_y \ f \circ e = e' \circ g$, i.e.

(Intuition: one cannot gödelize all of $R$, but one can effectively enumerate it piecewise).

\textbf{Notation. } $\text{EN}_L^L$ and $\text{GEN}_L^L$ are the categories of total morphisms.

\textbf{Lemma. } $\text{GEN}_L^L$ has coproducts.

\textbf{Hint: } $X \sqcup Y = (X \sqcup Y, \{e \sqcup e' | e \in E_x \land e' \in E_y\})$,

where $(e \sqcup e')(x, y, z) = \text{if } (x = y) \text{ then } ez \text{ else } e'z$, i.e. it is

the sup of $e, e'$ w.r.t. $L$-reducibility.

\textbf{1.4 Theorem. } i) $\text{GEN}_L^L$ is a CCC and

\hspace{1cm} \text{ii) } $\text{GEN}_p^L$ is a pCCC.

\textbf{Hint: } $X \times Y = (X \times Y, \{e \times e' | e \in E_x \land e' \in E_y\})$;

\begin{tikzpicture}
  \node (X) at (0,0) {$X$};
  \node (Y) at (2,0) {$Y$};
  \node (P) at (1,1) {$P$};
  \node (Q) at (1,-1) {$Q$};
  \node (R) at (0.5,0.5) {$R$};
  \node (S) at (1.5,0.5) {$S$};
  \node (T) at (1,0) {$T$};

  \draw[->] (X) to node[above] {$e$} (Y);
  \draw[->] (X) to node[left] {$e'$} (Y);
  \draw[->] (P) to node[above] {$e\times e'$} (Q);
  \draw[->] (P) to node[left] {$e\times e'$} (Q);
  \draw[->] (R) to node[left] {$e\times e'$} (S);
  \draw[->] (R) to node[right] {$e\times e'$} (S);
  \draw[->] (T) to node[below] {$e\times e'$} (T);
  \draw[->] (T) to node[above] {$e\times e'$} (T);
\end{tikzpicture}
use 3) and 4) (in Definition 1.1) for proving that $E_{X \times Y}$ is directed; let $\omega = (\omega, \{id\}) \in \text{GEN}_{L}^{L}$, then

i) $\frac{X Y}{p} = (\text{GEN}_{L}^{L}(X, Y), A(\text{GEN}_{L}^{L}(\omega \times X, Y)))$;

ii) $\frac{Y X}{p} = (\text{GEN}_{L}^{L}(X, Y), A(\text{GEN}_{L}^{L}(\omega \times X, Y)))$

where $A$ is the curry operator on maps, use 3) and 4) (in 1.1) for proving that $\omega \cup \omega \cup \omega$ and $(\omega \times X) \cup (\omega \times X) \simeq (\omega \cup \omega) \times X$, then it follows easily that $E_{X \times Y}$ is directed.

Remark. In general, if $C$ is $pCCC$, it does not follow that $C$ is $CCC$ (the problem are objects s.t. $Y \not\in Y^*$).

1.5 Definition. The embedding functor, $I$, of $\text{EN}_{L}^{L}$ into $\text{GEN}_{L}^{L}$ is defined by:

i) $I(A, e_{A}) = (A, \{e_{A}\})$ on objects and

ii) $I$ is the identity on maps

The properties of $I$ are summarized by theorem 1.7 below.

1.6 Lemma. Let $f: X \rightarrow Y$, then

$f \in \text{GEN}_{L}^{L}(X, Y) \iff f \circ \text{GEN}_{L}^{L}(\omega, X) \subseteq \text{GEN}_{L}^{L}(\omega, Y),$

$f \in \text{EN}_{L}^{L}(X, Y) \iff f \circ \text{EN}_{L}^{L}(\omega, X) \subseteq \text{EN}_{L}^{L}(\omega, Y)$

1.7 Theorem. i) $I$ is full and faithful,

ii) $I$ preserves products and representations of total and partial morphisms.

The main reason, for using generalized numbered sets instead of presheaves, is that the former are more similar to numbered sets than the latter, thus we can easily extend meaningful concepts from $\text{EN}_{L}^{L}$ to $\text{GEN}_{L}^{L}$ (such as the notion of partial morphism and relative numbering), whereas this seems impossible for presheaves.

§.2 HPEF and generalized numbered sets.

This section is devoted to the characterization of the generalized
HPEF \((L^n)_{n<\omega}\), in the integer (or pure) types (see §.I of part I), with
the corresponding type structure in \(\text{GEN}_p^L\). For \(L\) is an arbitrary
enumeration-acceptable function set, the full generality of \(\text{GEN}_p^L\) is
required.

The main step is to find the right counterpart to the notion of
relative numbering given in \(\text{EN}\) (see 3.2 part I).

2.1 Definition. \(X\) factorizes \(Y\) iff
\[\text{GEN}_p^L(\omega, Y) = \text{GEN}_p^L(X, Y) \circ \text{GEN}_p^L(\omega, X)\]
or equivalently \(\forall e' \in E_Y \exists e \in E_X \exists f \in \text{GEN}_p^L(X, Y)\) s.t.
\[e \rightarrow e' \downarrow \xrightarrow{f} \rightarrow Y\]

Remark. Let \(A, B \in \text{EN}_L\), then
\(\exists f \in \text{EN}_L(A, B)\) relative numbering \(\iff A\) factorizes \(B\).

2.2 Proposition. If \(X\) factorizes \(Y\) and \(f: Y \rightarrow Z\), then
\(f \in \text{GEN}_p^L(Y, Z) \iff f \circ \text{GEN}_p^L(X, Y) \subseteq \text{GEN}_p^L(X, Z)\). (see 3.5 part I)

2.3 Theorem. Let \(\omega \upharpoonright_p X\) and \(Y \upharpoonright Y\), the \(X\) factorizes \(Y\).
(see 3.6 part I)

The integer type structure in \(\text{GEN}_p^L\) is defined as in (3.8 part I):

2.4 Definition. \(E_0^L = \omega, E_1^L = \omega \upharpoonright_p E_0^L = E_0^E_1^L\)
We have that:

2.5 Lemma. \(\omega \upharpoonright_p E_n^L, E_{n+1}^L \downarrow (E_n^E_1^L), E_n^L \times E_n^L \downarrow E_n^L\).
Hint: use \(\omega \upharpoonright_1 \omega \upharpoonright \omega\) for proving that \(E_1^L \times E_1^L \downarrow E_1^L\), where

\[\begin{array}{c}
\text{in}_0^1 \\
\omega \\
\{f, g\} \\
\downarrow \omega
\end{array}\]

then it follows by induction that \(\forall n > 0 \ E_n^L \times E_n^L \downarrow E_n^L\)
From (2.5) and (2.3) it follows

2.6 Theorem. $E_n^L$ factorizes $E_{n+1}^L$ for $n > 0$

2.7 Theorem. Let $L$ be enumeration-acceptable, then $\{L_n^\} \_n<\omega$ is defined and for all $n > 0$:

\[ L_n = E_n^L, \quad n+1.5 \quad L_{n+1.5} = \text{GEN}(E_n^L, E_{n+1}^L). \]

hint: (see also (3.12) in part I)

0), 1) by definition,

n+1.5) follows from n), n+1) and $E_n^L \times E_n^L \subseteq E_n^L$,

n+2) follows from n+1.5, (2.6) and (2.2)

The existence of principal morphisms in $L_{n+1.5}^n$ does not follow from (2.6) (compare to 3.10 in part I), in fact it requires stronger hypotheses:

2.8 Theorem. \( \forall n \geq 0 \) there is a principal morphism in $L_{n+1.5}^n \implies \forall n \geq 0$

\( E_n^L \) is representable in $\text{EN}_L$ (i.e. $E_n^L$ is the image (w.r.t. I) of a numbered set).

§.3 Generalized numbered sets and presheaves.

At last we return of the relations between $\text{GEN}^L$ and $\text{Set}^{(\text{EN}^L)^{\text{OP}}}$. First let us define the embedding functors $I_{\text{EN}}$ and $I_{\text{GEN}}$.

3.1 Definition. i) $I_{\text{EN}} = \lambda A, \lambda B. \text{EN}^L(B,A)$ is the usual Yoneda embedding of $\text{EN}^L$ in $\text{Set}^{(\text{EN}^L)^{\text{OP}}}$,

ii) $I_{\text{GEN}} = \lambda B. \text{GEN}^L(\text{IB},X):\text{GEN}^L \to \text{Set}^{(\text{EN}^L)^{\text{OP}}}$

3.2 Theorem. i) $I_{\text{EN}}$ and $I_{\text{GEN}}$ are full and faithful,

ii) preserve products and representations of morphisms,

iii) $I_{\text{EN}} = I_{\text{GEN}} \circ I$

(2.7) may be restated, using presheaves only, as follows.

3.3 Theorem. Let $L$ be enumeration-acceptable, then there exist two presheaves $F$ and $G$ such that:

n) $L_n = E_n^L, \quad n+1.5 \quad L_{n+1.5} = E_{n+1}^L, \quad \text{where} \quad E_0^L = F, \quad E_1^L = G^F$ and

\[ E_{n+1}^L = E_n^L \]
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