Teoria delle Categorie
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Eugenio Moggi

moggi@disi.unige.it

DISI, Univ. of Genova

Introduzione e Motivazioni

- Encyclopedia on Scientific and Philosophical Thought [Geymonat75]

  volumes on 20th century most interesting, in particular
  - Physics: Relativity Theory, Quantum Mechanics
  - Logics: the failure of Hilbert's program, Computability, Category Theory
  - Philosophy of Science (Epistemology): Falsificationism (Karl Popper), . . .

  extremely valuable to provide an overview of parallel threads and their inter-dependencies, with some attempts to go into technical aspects
- Section on Category Theory[a]: very *enthusiastic* advertising of its potential

  main focus on toposes and interpretation of logic in them
- Categories for the Working Mathematician [MacLane71][b]: very *hard* to read without a good background in Mathematics

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[a] Universal properties vs concrete descriptions of mathematical constructions

[b] 1977 Italian translation by Betti, Carboni, Galuzzi, Meloni

Introduzione e Motivazioni

- Nowadays there several books, e.g. [AspertiLongo91], that do not require as much mathematical background as [MacLane71], but some background is needed in order to provide examples.
- Perhaps using “web technology” and platforms for “collaborative work” one could envisage an e-book that shares definitions and theorems form Category Theory, but examples and applications are customized on the readers background knowledge and interests.
- Links to relevant material for this course and to further readings can be found in the web page

  `http://www.disi.unige.it/person/MoggiE/AILA07/`

Introduzione e Motivazioni

[Goguen91]: why category theory is useful (in computer science, and more generally in a young subject, poorly organized, that needs all the help that it can get):

- Formulating definitions and theories (CT provides guidelines)
- Carrying out proofs
- Discovering and exploiting relations with other fields

  sufficiently abstract formulations can reveal surprising connections
- Dealing with abstraction and representation independence

  a copernican revolution w.r.t. set theory: CT looks at objects trought their relations with other objects
- Formulating conjectures and research directions mainly through relations with other fields
- Conceptual unification (by abstraction and use of few fundamental concepts)

CT useful also in a mature subject (e.g. to export ideas to other subjects):

- more general/abstract reformulations or cleaner/unified reformulations

There are also bad uses of CT, e.g. : specious generality, categorical overkill.

Part 1 - [AspertiLongo91, Ch 1]

Category, Graph and Diagram

A *category* $\mathcal{C}$ consists of

- a collection $\mathcal{C}_0$ of objects, notation $\boxed{a \in \mathcal{C}}$
- a collection $\mathcal{C}_1$ of morphisms (arrows, maps)
- operations $\mathrm{dom}, \mathrm{cod} \colon \mathcal{C}_1 \longrightarrow \mathcal{C}_0$ assigning to each arrow a domain and codomain

  we write $\boxed{f \in \mathcal{C}[a,b] \text{ or } a \xrightarrow{f} b \text{ or } f \colon a \longrightarrow b \text{ when } a = \mathrm{dom}(f) \text{ and } b = \mathrm{cod}(f)}$
- an operation $\mathrm{id} \colon \mathcal{C}_0 \longrightarrow \mathcal{C}_1$ assigning to each object a an identity $\boxed{\mathrm{id}_a \in \mathcal{C}[a,a]}$
- a *composition* operation $\circ$ assigning to each pair f and g of *composable* arrows

  $\boxed{a \xrightarrow{f} b \xrightarrow{g} c}$ a composite arrow $\boxed{g \circ f \in \mathcal{C}[a,c]}$

and identity and composition satisfy the following properties

(identity) $\boxed{\mathrm{id}_b \circ f = f = f \circ \mathrm{id}_a}$ for any $\boxed{a \xrightarrow{f} b}$

(associativity) $\boxed{h \circ (g \circ f) = (h \circ g) \circ f}$ for any $\boxed{a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d}$

Category, Graph and Diagram

- A *graph*[a] $\mathcal{G}$ consists of
  - a collection $\mathcal{G}_0$ of nodes (vertexes)
  - a collection $\mathcal{G}_1$ of arcs (edges, arrows)
  - operations $\mathrm{dom}, \mathrm{cod}\colon \mathcal{G}_1 \longrightarrow \mathcal{G}_0$ assigning to each arc a source and target

    we write $\boxed{a \xrightarrow{f} b \text{ when } a = \mathrm{dom}(f) \text{ and } b = \mathrm{cod}(f)}$

  Any category $\mathcal{C}$ has an underlying graph $\mathrm{dom}, \mathrm{cod}\colon \mathcal{C}_1 \longrightarrow \mathcal{C}_0$
- **Graph** is the category of *small* graphs (i.e. $\mathcal{G}_0$ and $\mathcal{G}_1$ are sets) with arrows $(g_0, g_1) \in$ **Graph**$[\mathcal{G}, \mathcal{G}'] \overset{\Delta}{\Longleftrightarrow} g_0\colon \mathcal{G}_0 \longrightarrow \mathcal{G}'_0$ and $g_1\colon \mathcal{G}_1 \longrightarrow \mathcal{G}'_1$ s.t.

  $\boxed{a \xrightarrow{f} b \text{ in } \mathcal{G} \text{ implies } g_0(a) \xrightarrow{g_1(f)} g_0(b) \text{ in } \mathcal{G}'}$

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[a] In Graph Theory what we call graph is called a *directed multi-graph*.

Category, Graph and Diagram

- Given a category $\mathcal{C}$ and a small graph $\mathcal{G}$ a *diagram* D of shape $\mathcal{G}$ in $\mathcal{C}$ is a graph morphism (d_0, d_1) from $\mathcal{G}$ to the underlying graph of $\mathcal{C}$, i.e. D corresponds to a consistent labeling of nodes and arcs of $\mathcal{G}$ with objects and arrows of $\mathcal{C}$

  given a *path* $p = \langle a_i \xrightarrow{f_i} a_{i+1} | i < n \rangle$ from a_0 to a_n in $\mathcal{G}$ we write $D[p]$ for the arrow in $\mathcal{C}[d_0(a_0), d_0(a_n)]$ obtained by composing the arrows $d_1(f_i)$ (when $n = 0$ then $D[p]$ is the identity on $d_0(a_0)$)

- A diagram D *commutes* $\stackrel{\Delta}{\Longleftrightarrow}$ for every pair of paths p and p' in $\mathcal{G}$ with the same source and target (say a to b) $D[p] = D[p']$ (as arrows in $\mathcal{C}[d_0(a), d_0(b)]$)

  commuting diagrams expressing the (identity) and (associativity) properties

$$\begin{tikzcd}
a \arrow[r, "\mathrm{id}_a"] \arrow[d, "f"] & a \arrow[d, "f"] \arrow[dl, "f"] \\
b \arrow[r, "\mathrm{id}_b"] & b
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, "g \circ f"] \arrow[d, "f"] & c \arrow[d, "h"] \\
b \arrow[r, "h \circ g"] \arrow[ur, "g"] & d
\end{tikzcd}$$

Examples

Dogma 1: to each species of mathematical structure, there corresponds a **category** whose objects have that structure, and whose morphisms preserve it.

$\mathcal{C}$	Objects a	Morphisms $f \in \mathcal{C}[a_1, a_2]$
Set	sets X	functions $f \in X_1 \longrightarrow X_2$
	to be precise morphisms are triples (X_1, f, X_2)	
pSet	sets X	partial maps $f \in X_1 \rightharpoonup X_2$
Rel	sets X	relations $R \subseteq X_1 \times X_2$
Mon	monoids $(X, \cdot, 1)$ $x\,1 = x = 1\,x \quad (x_1\,x_2)\,x_3 = x_1\,(x_2\,x_3)$	homomorphisms $f\colon X_1 \longrightarrow X_2$ $f(1_1) = 1_2 \quad f(x_1 \cdot_1 x_2) = f(x_1) \cdot_2 f(x_2)$
Grp	groups $(X, \cdot, 1, {}^{-1})$ monoid s.t. $x \cdot x^{-1} = 1 = x^{-1} \cdot x$	homomorphisms $f\colon X_1 \longrightarrow X_2$ monoid homomorphism: $f(x^{-1}) = f(x)^{-1}$
Vect	vector spaces	linear transformations
Top	topological spaces (X, τ) $\tau \subseteq \mathcal{P}(X)$ closed w.r.t. $\cup$ and finite $\cap$	continuous maps $f\colon X_1 \longrightarrow X_2$ $O \in \tau_2 \supset f^{-1}(O) \in \tau_1$
PO	partial orders $(X, \leq)$	monotone maps $f\colon X_1 \longrightarrow X_2$ $x_1 \leq_1 x_2 \supset f(x_1) \leq_2 f(x_2)$

Examples

- a collection C induces a *discrete* category $\mathcal{C}$ (i.e. every arrow is an identity): $\mathcal{C}_0 = \mathcal{C}_1 = C$ and $\mathrm{dom}(a) = a = \mathrm{cod}(a)$
- a *preorder* $(X, \leq)$, i.e. $\leq \subseteq X \times X$ is reflexive and transitive, induces a category $\mathcal{C}$ where every $\mathcal{C}[a, b]$ has at most one element: $\mathcal{C}_0 = X$, $\mathcal{C}_1 = \leq$, $\mathrm{dom}(a, b) = a$ and $\mathrm{cod}(a, b) = b$
  - $\subseteq$ is a preorder on sets (indeed a partial order)
  - $\in$ is not a preorder on sets (e.g. $X \in X$ fails in ZF set theory)
- a *monoid* $(X, \cdot, 1)$, induces a category $\mathcal{C}$ with exactly one object: $\mathcal{C}_0 = \{*\}$, $\mathcal{C}_1 = X$, $\mathrm{id}_* = 1$ and $x_1 \circ x_2 = x_1 \cdot x_2$

Categories from (your favorite) propositional logic

- entailment $A_1 \vdash A_2$ is a preorder on propositions, thus it induces a category **Ent**
- a more interesting category **Prf** is obtained by taking as $A_1 \xrightarrow{p} A_2$ *proofs* of the entailment $A_1 \vdash A_2$[a]

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[a] Intuitionistic proofs = typed λ-terms, see [AspertiLongo91, Ch 8]

Examples from Algebra

Let Ω be an *algebraic signature*, i.e. a family $\langle \Omega_n | n \rangle$ of sets (of operator symbols) indexed by natural numbers (considered as *arities*)

- $T_\Omega(X)$ denotes the set of Ω-terms with variables included in the set X

  T_Ω is the category of (finite) sets and *substitutions* $T_\Omega[X_1, X_2] \triangleq X_2 \longrightarrow T_\Omega(X_1)$

  given $\rho_1 \colon X_2 \longrightarrow T_\Omega(X_1)$ and $\rho_2 \colon X_3 \longrightarrow T_\Omega(X_2)$, the composite $\rho_2 \circ \rho_1$ is the $\rho \colon X_3 \longrightarrow T_\Omega(X_1)$ s.t. $\rho(x) \triangleq t[\rho_1]$ with $t = \rho_2(x) \in T_\Omega(X_2)$
- an Ω-algebra is a pair $(X, [\![-]\!])$, where X is a set and $[\![-]\!]$ is an interpretation of the operator symbols in X, i.e. $[\![op]\!] \colon X^n \longrightarrow X$ for $op \in \Omega_n$

  Alg$_\Omega$ is the category of Ω-algebras and Ω-*homomorphisms*[a]

$$\begin{tikzcd}
X_1^n \arrow[r, "f^n"] \arrow[d, "{[\![op]\!]_1}"'] & X_2^n \arrow[d, "{[\![op]\!]_2}"] \\
X_1 \arrow[r, "f"] & X_2
\end{tikzcd}$$

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[a] See [AspertiLongo91, Sec 4.1]

Addendum on pCL and pCAs

- *partial Combinatory Logic* (pCL) is a theory in *Logic of Partial Terms* ($M \downarrow$ means M defined, $M_1 = M_2$ means terms defined and equal, $M_1 \simeq M_2$ means $M_1 \downarrow \vee M_2 \downarrow \supset M_1 = M_2$)
  - Terms $M ::= x \mid K \mid S \mid M_1 M_2$ partial application (possibly other constants c)
  - Axioms $K\ x\ y = x$ and $S\ x\ y \downarrow$ and $S\ x\ y\ z \simeq x\ z\ (y\ z)$

  additional axioms are: (tot) $xy \downarrow$ (ext) $(\forall z.xz \simeq yz) \supset x = y$
- the *abstraction* $[x]M$ is a term defined by induction on M satisfying the following properties: $x \notin \mathrm{FV}([x]M)$, $([x]M) \downarrow$ and $([x]M)x \simeq M$

  $[x]x \triangleq I \triangleq SKK \quad [x]y \triangleq K\ y \quad [x]c \triangleq K\ c \quad [x]M_1M_2 \triangleq S([x]M_1)([x]M_2)$
- a model of pCL (called pCA) is *non trivial* $\overset{\Delta}{\iff} K \neq S$.

  Kleene's applicative structure $\omega = (N, \cdot)$, where $m \cdot n \simeq \{m\}(n)$, *is* a pCA
- There is an encoding $\underline{n}$ of $n \in N$ in pCL and a term M_U s.t. in any non-trivial pCA $M_U \underline{e}\ \underline{m} \simeq \underline{n} \iff \{e\}(m) \simeq n$ (when the pCA is non-total, then $M_U \underline{e}\ \underline{m} \downarrow \iff \{e\}(m) \downarrow$)

  i.e. in pCL every partial recursive function is *representable*

  Church's encoding $\underline{n} \triangleq [x][y]x^n y$, where $M^0 N \triangleq N$ and $M^{n+1} N \triangleq M(M^n N)$

Examples from Computability

- **EN** is the category of *numbered sets*

  (objects) $\underline{X} = (X, e)$ with $e\colon N \twoheadrightarrow X$ (i.e. e onto map)

  (arrows) $\underline{X}_1 \xrightarrow{f} \underline{X}_2 \overset{\Delta}{\iff}$ exists a recursive $f'\colon N \to N$ s.t.

$$\begin{tikzcd}
X_1 \arrow[r, "f"] & X_2 \\
N \arrow[u, two heads, "e_1"] \arrow[r, "f'"] & N \arrow[u, two heads, "e_2"]
\end{tikzcd}$$

- Let $\underline{A} = (A, \cdot)$ be a *partial Combinatory Algebra*, i.e. $\cdot$ is a partial application and
  - exist $K, S \in A$ s.t. $K\ a\ b = a$, $S\ a\ b \downarrow$ and $S\ a\ b\ c \simeq a\ c\ (b\ c)$ for any $a, b, c \in A$
- $\underline{A}$**-Set** is the category of sets with an $\underline{A}$-*realizability* relation

  (objects) $\underline{X} = (X, \Vdash)$ with $\Vdash\ \subseteq A \times X$ *onto* $\boxed{\forall x \in X . \exists a . a \Vdash x}$

  (arrows) $\underline{X}_1 \xrightarrow{f} \underline{X}_2 \overset{\Delta}{\iff} X_1 \xrightarrow{f} X_2$ has a *realizer* r $\boxed{a \Vdash_1 x \text{ implies } r\ a \Vdash_2 f(x)}$

Examples from Category Theory

- The category **Cat** whose objects are (small) categories (by Dogma 1)[a]
- the *dual*[b] category $\mathcal{C}^{op}$ of $\mathcal{C}$: $\mathcal{C}_0^{op} = \mathcal{C}_0$ and $\boxed{\mathcal{C}^{op}[a,b] = \mathcal{C}[b,a]}$
  $\mathrm{id}_a^{op} = \mathrm{id}_a$ and $g \circ^{op} f = f \circ g$
- the *product* category $\mathcal{C} \times \mathcal{D}$ of $\mathcal{C}$ and $\mathcal{D}$: $(\mathcal{C} \times \mathcal{D})_0 = \mathcal{C}_0 \times \mathcal{D}_0$ and $(\mathcal{C} \times \mathcal{D})_1 = \mathcal{C}_1 \times \mathcal{D}_1$
  $\mathrm{id}_{(a,a')} = (\mathrm{id}_a, \mathrm{id}_{a'})$ and $(g, g') \circ (f, f') = (g \circ f, g' \circ f')$
- the *slice*[c] category $\mathcal{C}/a$ of $\mathcal{C}$ over $a \in \mathcal{C}$: $(\mathcal{C}/a)_0 = \{f \in \mathcal{C}_1 | \mathrm{cod}(f) = a\}$
  $\mathcal{C}/a[f\colon b \to a, f'\colon b' \to a] = \{g \in \mathcal{C}[b, b'] | f' \circ g = f\}$ in fact $(f, g, f'))$
- A category $\mathcal{D}$ is a *subcategory* of $\mathcal{C} \overset{\Delta}{\iff} \mathcal{D}_0 \subseteq \mathcal{C}_0$ and $\mathcal{D}[a,b] \subseteq \mathcal{C}[a,b]$, and identities and composition in $\mathcal{D}$ coincide with those in $\mathcal{C}$

  $\mathcal{D}$ is a *full* subcategory when in addition $\mathcal{D}[a,b] = \mathcal{C}[a,b]$

  Set is a subcategory of **Rel** (but it is not full), since functions are relations (with certain properties)

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[a] Morphisms in **Cat** are functors, see [AspertiLongo91, Def 3.1.1]

[b] Duality is a powerful technique of Theory applicable to definitions and theorems.

[c] The objects of **Set**$/I$ *corresponds* to I-indexed families of sets.

Special Morphisms

Given a category $\mathcal{C}$ we say that

- $a \xrightarrow{e} b$ is *epic* $\overset{\Delta}{\iff}$ $\boxed{f \circ e = g \circ e \text{ implies } f = g}$ when $c \in \mathcal{C}$ and $f, g \in \mathcal{C}[b, c]$
- $a \xrightarrow{m} b$ is *monic* $\overset{\Delta}{\iff}$ $\boxed{m \circ f = m \circ g \text{ implies } f = g}$ when $c \in \mathcal{C}$ and $f, g \in \mathcal{C}[c, a]$

  monic and epic are dual properties, i.e. m is monic in $\mathcal{C} \iff m$ is epic in $\mathcal{C}^{op}$
- $a \xrightarrow{i} b$ is *iso* $\overset{\Delta}{\iff}$ $\boxed{j \circ i = \mathrm{id}_a \text{ and } i \circ j = \mathrm{id}_b}$ for some (unique) $j \in \mathcal{C}[b, a]$

  iso is a self-dual property, i.e. i is iso in $\mathcal{C} \iff i$ is iso in $\mathcal{C}^{op}$
- $a \xrightarrow{e} b$ is a *split epic* $\overset{\Delta}{\iff}$ $\boxed{e \circ m = \mathrm{id}_b}$ for some $m \in \mathcal{C}[b, a]$

  there is a dual property of *split monic*

The following statements and their dual hold (proofs are by *diagram chasing*):

- e split epic $\implies$ e epic
- m monic and split epic $\implies$ m iso

we write $\boxed{a \overset{m}{\rightarrowtail} b \text{ when } m \text{ is monic and } a \overset{e}{\twoheadrightarrow} b \text{ when } e \text{ is epic}}$

Special Morphisms

In **Set** one has the following concrete characterizations

- e epic $\iff$ e is surjective $\iff$ e split epic (by the axiom of choice)
- m monic $\iff$ m is injective ($m\colon a \longrightarrow b$ split monic $\iff$ m monic and $a \neq \emptyset$)
- i iso $\iff$ i is bijective

Give concrete characterizations in other sample categories, in particular consider

- $\mathcal{C}$ is a monoid, i.e. a category with exactly one object
- $\mathcal{C}$ is a preorder (every arrow is both monic and epic)

Part 2 - [AspertiLongo91, Ch 2]

Thinking Categorically (special objects)

- $0 \in \mathcal{C}$ is *initial* $\overset{\Delta}{\iff}$ $\boxed{\forall a \in \mathcal{C}. \exists ! f \in \mathcal{C}[0, a]}$
- $1 \in \mathcal{C}$ is *terminal* $\overset{\Delta}{\iff}$ $\boxed{\forall a \in \mathcal{C}. \exists ! f \in \mathcal{C}[a, 1]}$

  initial and terminal are dual properties, i.e. a is terminal in $\mathcal{C} \iff a$ is initial in $\mathcal{C}^{op}$

The following statements say that initial objects are determined up to unique iso

- if 0 is initial and $0 \xrightarrow{i} 0'$ is an iso, then $0'$ is initial
- if 0 and $0'$ are initial, then they are isomorphic $\boxed{\text{and the iso is unique}}$

  dual statements hold for terminal objects

  there are categories without initial/terminal objects (e.g. the empty category)

In **Set** one has the following concrete characterizations

- X is initial $\iff X = \emptyset$ ($\emptyset$ is both initial and terminal in **Rel** and **pSet**)
- X is terminal $\iff X$ has exactly one element

Give concrete characterizations in other sample categories.

Thinking Categorically (special objects)

- $0 \in \mathcal{C}$ is *initial* $\overset{\Delta}{\Longleftrightarrow}$ $\boxed{\forall a \in \mathcal{C}.\exists! f \in \mathcal{C}[0, a]}$
- $1 \in \mathcal{C}$ is *terminal* $\overset{\Delta}{\Longleftrightarrow}$ $\boxed{\forall a \in \mathcal{C}.\exists! f \in \mathcal{C}[a, 1]}$

  initial and terminal are dual properties, i.e. a is terminal in $\mathcal{C} \Longleftrightarrow a$ is initial in $\mathcal{C}^{op}$

The property of being initial/terminal is a simple form of *universal property*, i.e.

- a *property* $P(x)$ expressed in the language of Category Theory, s.t.
- the *structures* x satisfying the property are determined up to unique iso

$\boxed{\text{thus the structures on which } P(x) \text{ is defined are the objects of a category}}$

Thinking Categorically (universal properties I)

Given a category

- $a_1 \xleftarrow{\pi_1} a \xrightarrow{\pi_2} a_2$ is a *product* diagram in $\mathcal{C} \overset{\Delta}{\Longleftrightarrow}$

  for any $a_1 \xleftarrow{f_1} b \xrightarrow{f_2} a_2$ $\exists! f$ s.t.
$$\begin{tikzcd} a_1 & & a \arrow[ll, "\pi_1"'] \arrow[rr, "\pi_2"] & & a_2 \\ & & & & \\ & & b \arrow[uull, "f_1"] \arrow[uu, dashed, "f"] \arrow[uurr, "f_2"'] & & \end{tikzcd}$$
  commutes

  we write $\boxed{a_1 \times a_2 \text{ for } a \text{ and } \langle f_1, f_2\rangle \text{ for } f}$

- product diagrams for a_1 and a_2 are determined up to unique iso, i.e.

  if $a_1 \xleftarrow{\pi'_1} a' \xrightarrow{\pi'_2} a_2$ is another product diagram, then

  $\exists! i$ iso s.t.
$$\begin{tikzcd} a_1 & & a' \arrow[ll, "\pi'_1"'] \arrow[rr, "\pi'_2"] & & a_2 \\ & & & & \\ & & a \arrow[uull, "\pi_1"] \arrow[uu, dashed, "i"] \arrow[uurr, "\pi_2"'] & & \end{tikzcd}$$
  commutes

Thinking Categorically (universal properties I)

- a *coproduct* diagram $a_1 \xrightarrow{\iota_1} a \xleftarrow{\iota_2} a_2$ is the dual of a product diagram, i.e.

  for any $a_1 \xrightarrow{f_1} b \xleftarrow{f_2} a_2$ $\exists! f$ s.t.

$$\begin{tikzcd} a_1 \arrow[r, "\iota_1"] \arrow[dr, "f_1"'] & a \arrow[d, dashed, "f"] & a_2 \arrow[l, "\iota_2"'] \arrow[dl, "f_2"] \\ & b & \end{tikzcd}$$

  commutes

  we write $\boxed{a_1 + a_2 \text{ for } a \text{ and } [f_1, f_2] \text{ for } f}$

- coproduct diagrams for a_1 and a_2 are determined up to unique iso (by duality)
- the definitions of product and coproduct diagram generalize from the *binary* to *I-indexed* case (where I is a set)

  when $I = \emptyset$ the definitions coincide with that of terminal and initial object.

  The notation introduced for binary products and coproducts is modified as follows

  $\boxed{\prod_{i \in I} a_i \text{ and } \coprod_{i \in I} a_i}$ and $\boxed{\langle f_i | i \in I \rangle \text{ and } [f_i | i \in I]}$

Thinking Categorically (universal properties I)

In **Set** for any pair of object X_1 and X_2 we have that

- $X_1 \xleftarrow{\pi_1} X_1 \times X_2 \xrightarrow{\pi_2} X_2$ is a product diagram, where $X_1 \times X_2$ is the cartesian product and $\pi_i(x_1, x_2) = x_i$
- $X_1 \xrightarrow{\iota_1} X_1 \uplus X_2 \xleftarrow{\iota_2} X_2$ is a coproduct diagram, where $X_1 \uplus X_2$ is the disjoint union $\{(i, x) | x \in X_i\}$ and $\iota_i(x) = (i, x)$

When $\mathcal{C}$ is a preorder one has

- an initial object 0 is a least element $\perp$, and a terminal object 1 is a top element $\top$
- a product $a_1 \times a_2$ is a greatest lower bound $a_1 \wedge a_2$, and a coproduct $a_1 + a_2$ is a least upper bound $a_1 \vee a_2$

When the objects involved exist, there are *canonical* isomorphisms

- $a \times 1 \cong a \quad a_1 \times a_2 \cong a_2 \times a_1 \quad (a_1 \times a_2) \times a_3 \cong a_1 \times (a_2 \times a_3)$
- similar isomorphisms hold by replacing $\times$ with $+$ and 1 with 0

In **Set** (in biCCCs, but not in general) the *canonical* maps below are iso

- $0 \dashrightarrow a \times 0 \quad (a \times a_1) + (a \times a_2) \dashrightarrow a \times (a_1 + a_2)$

Thinking Categorically (universal properties II)

Given a category with a terminal object 1

- $1 \xrightarrow{z} a_N \xrightarrow{s} a_N$ is a *natural number object* (NNO for short) diagram in $\mathcal{C}$ $\overset{\Delta}{\iff}$ for any $1 \xrightarrow{f_z} a \xrightarrow{f_s} a$ $\exists! f$ s.t.
$$\begin{tikzcd} 1 \arrow[r, "z"] \arrow[dr, "f_z"'] & a_N \arrow[r, "s"] \arrow[d, dashed, "f"] & a_N \arrow[d, dashed, "f"] \\ & a \arrow[r, "f_s"] & a \end{tikzcd}$$
commutes
- NNO diagrams are determined up to unique iso
- In **Set** a NNO diagram is given by $1 \xrightarrow{z} N \xrightarrow{s} N$, where
  - N is the set of natural numbers,
  - $z(*) = 0$ (when 1 is the singleton $\{*\}$),
  - $s(n) = n + 1$ is the successor function,

  If $\mathcal{C}$ has N-indexed coproducts, then $\coprod_{n \in N} 1$ is a NNO.
- **EN** has a NNO (and finite coproducts), but does not have N-indexed coproducts.

Thinking Categorically (universal properties II)

Given a category with binary products

- $c \times a \xrightarrow{ev} b$ is an *exponential* diagram in $\mathcal{C} \overset{\Delta}{\iff}$ for any $c' \times a \xrightarrow{f} b$ $\exists! f' \colon c' \to c$ s.t.
$$\begin{tikzcd} c \times a \arrow[r, "ev"] & b \\ c' \times a \arrow[u, dashed, "f' \times \mathrm{id}_a"] \arrow[ur, "f"'] & \end{tikzcd}$$
commutes

  where $f' \times \mathrm{id}_a \triangleq \langle f' \circ \pi_1, \pi_2 \rangle$, we write $\boxed{b^a \text{ for } c \text{ and } \Lambda(f) \text{ for } f'}$
- exponential diagrams are determined up to unique iso
- In **Set** an exponential diagram is $Y^X \times X \xrightarrow{ev} Y$, where Y^X is the set of functions **Set**$[X, Y]$ and $ev(f, x) = f(x)$
- In $\underline{A}$**-Set** an exponential diagram is $\underline{Y}^{\underline{X}} \times \underline{X} \xrightarrow{ev} \underline{Y}$, where $\underline{Y}^{\underline{X}}$ is the set of realizable maps $\underline{A}$**-Set**$[\underline{X}, \underline{Y}]$ with an *obvious* realizability relation
- In **EN** the exponential object N^N does not exists (N is the NNO)
- in **Ent** (for propositional logic) B^A is implication $A \supset B$

Thinking Categorically (universal properties II)

- $\mathcal{C}$ has *enough points* $\overset{\Delta}{\Longleftrightarrow}$ it has a terminal object 1 and for any $f, g \in \mathcal{C}[a, b]$ $(\forall x\colon 1 \to a. f \circ x = g \circ x) \supset f = g$
- $\mathcal{C}$ is a *cartesian* category (has finite products) $\overset{\Delta}{\Longleftrightarrow}$ it has a terminal object 1 and binary products $a_1 \times a_2$ for any pair of objects
- $\mathcal{C}$ is a *cartesian closed* category (CCC for short) $\overset{\Delta}{\Longleftrightarrow}$ it is cartesian and it has exponentials b^a for any pair of objects
- $\mathcal{C}$ is a *bi-cartesian closed* category (biCCC for short) $\overset{\Delta}{\Longleftrightarrow}$ it is cartesian closed and it has *finite coproducts* In a biCCC $\coprod_{i \in n}(a \times a_i) \dashrightarrow a \times (\coprod_{i \in n} a_i)$ is an iso.

PO, $\underline{A}$**-Set**, **Cat** are biCCC. **Graph**, **Mon**, **Grp**, **Top** are not CCC.

Equational reformulation

- $\pi_i \circ \langle f_1, f_2 \rangle = f_i$ and $\langle \pi_1 \circ f, \pi_2 \circ f \rangle = f\colon b \longrightarrow a_1 \times a_2$
- $[f_1, f_2] \circ \iota_i = f_i$ and $[f \circ \iota_1, f \circ \iota_2] = f\colon a_1 + a_2 \longrightarrow b$
- $ev \circ (a \times \Lambda(f)) = f$ and $\Lambda(ev \circ (f' \times \mathrm{id}_a)) = f'\colon c' \longrightarrow b^a$

Addendum: Internal Languages

The *internal language* L of a category $\mathcal{C}$ consists of

- types $t ::= a \mid \ldots$ and contexts $\Gamma ::= x{:}\,t \mid \ldots$, with a object of $\mathcal{C}$ and x variable
- raw terms $M ::= x \mid f(M) \mid \ldots$ with f arrow of $\mathcal{C}$, and several judgments
  - $\Gamma \vdash M{:}\,t$ asserting well-formedness of term M

$$x\ \frac{}{x{:}\,t \vdash x{:}\,t} \qquad f\ \frac{\Gamma \vdash M{:}\,t}{\Gamma \vdash f(M){:}\,t'}\ [\![t]\!] \xrightarrow{f} [\![t']\!]$$

  - $\Gamma \vdash M_1 = M_2{:}\,t$ asserting equality of well-formed terms

The interpretation $[\![-]\!]$ of L in $\mathcal{C}$ goes a follows

- types t and contexts Γ are interpreted by objects of $\mathcal{C}$ $\boxed{[\![a]\!] = [\![x{:}\,a]\!] \triangleq a}$
- well-formed terms $\Gamma \vdash M{:}\,t$ are interpreted[a] by arrows $f{:}\,[\![\Gamma]\!] \longrightarrow [\![t]\!]$
  $\boxed{[\![x{:}\,t \vdash x{:}\,t]\!] \triangleq \mathrm{id}_a}$ with $a = [\![t]\!]$ and $\boxed{[\![\Gamma \vdash f(M){:}\,t']\!] \triangleq f \circ [\![\Gamma \vdash M{:}\,t]\!]}$
- equality judgments are interpreted by equality of arrows.

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[a] the interpretation is defined by induction on the *unique* derivation of $\Gamma \vdash M{:}\,t$.

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  - $\Gamma \vdash M{:}t$ asserting well-formedness of term M

$$x \; \frac{}{x{:}t \vdash x{:}t} \qquad f \; \frac{\Gamma \vdash M{:}t}{\Gamma \vdash f(M){:}t'} \; [\![t]\!] \xrightarrow{f} [\![t']\!]$$

  - $\Gamma \vdash M_1 = M_2{:}t$ asserting equality of well-formed terms

Substitution is Composition

- subst $\dfrac{\Gamma \vdash M{:}t \qquad x{:}t \vdash N{:}t'}{\Gamma \vdash [M/x]N{:}t'}$ is an admissible rule
- $[\![\Gamma \vdash [M/x]N{:}t']\!] = g \circ f$ if $[\![\Gamma \vdash M{:}t]\!] = c \xrightarrow{f} a$ and $[\![x{:}t \vdash N{:}t']\!] = a \xrightarrow{g} b$

Addendum: Internal Languages

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- raw terms $M ::= x \mid f(M) \mid \ldots$ with f arrow of $\mathcal{C}$, and several judgments
  - $\Gamma \vdash M{:}t$ asserting well-formedness of term M

$$x\ \frac{}{x{:}t \vdash x{:}t} \qquad f\ \frac{\Gamma \vdash M{:}t}{\Gamma \vdash f(M){:}t'}\ [\![t]\!] \xrightarrow{f} [\![t']\!]$$

  - $\Gamma \vdash M_1 = M_2{:}t$ asserting equality of well-formed terms

Equality of Terms

$$\frac{\Gamma \vdash M{:}t}{\Gamma \vdash M = M{:}t} \qquad \frac{\Gamma \vdash M_1 = M_2{:}t}{\Gamma \vdash M_2 = M_1{:}t} \qquad \frac{\Gamma \vdash M_1 = M_2{:}t \qquad \Gamma \vdash M_2 = M_3{:}t}{\Gamma \vdash M_1 = M_3{:}t}$$

$$\text{congr}\ \frac{\Gamma \vdash M_1 = M_2{:}t \qquad x{:}t \vdash M{:}t'}{\Gamma \vdash [M_1/x]M = [M_2/x]M{:}t'} \qquad \text{subst}\ \frac{\Gamma \vdash M{:}t \qquad x{:}t \vdash M_1 = M_2{:}t'}{\Gamma \vdash [M/x]M_1 = [M/x]M_2{:}t'}$$

$$\text{id}\ \frac{}{x{:}t \vdash x = \mathrm{id}_a(x){:}t}\ a = [\![t]\!] \qquad \text{comp}\ \frac{}{x{:}t \vdash h(x) = g(f(x)){:}t'}\ h = [\![t]\!] \xrightarrow{f} \xrightarrow{g} [\![t']\!]$$

Addendum: Internal Languages for Cartesian Categories

- types $t ::= a \mid 1 \mid t_1 \times t_2$ and contexts $\Gamma ::= x : t \mid 1 \mid \Gamma, x : t$
- raw terms $M ::= x \mid f(M) \mid () \mid (M_1, M_2) \mid \pi_1(M) \mid \pi_2(M)$
- additional rules for well-formedness of terms

$$\frac{}{\Gamma, x : t \vdash x : t}\ x \notin \Gamma \qquad \frac{\Gamma \vdash M : t}{\Gamma, x : s \vdash M : t}\ x \notin \Gamma$$

$$\frac{}{\Gamma \vdash () : 1} \qquad \frac{\Gamma \vdash M_1 : t_1 \qquad \Gamma \vdash M_2 : t_2}{\Gamma \vdash (M_1, M_2) : t_1 \times t_2} \qquad \frac{\Gamma \vdash M : t_1 \times t_2}{\Gamma \vdash \pi_i(M) : t_i}$$

Interpretation of types and terms require a choice of product diagrams.

There are $\Gamma \vdash M : t$ with multiple derivations, this can be avoided with a different choice of rules.

- additional rules for equality of terms

$$\frac{\Gamma \vdash M : 1}{\Gamma \vdash M = () : 1} \qquad \frac{\Gamma \vdash M_1 : t_1 \qquad \Gamma \vdash M_2 : t_2}{\Gamma \vdash \pi_i(M_1, M_2) = M_i : t_i} \qquad \frac{\Gamma \vdash M : t_1 \times t_2}{\Gamma \vdash M = (\pi_1(M), \pi_2(M)) : t_1 \times t_2}$$

Thinking Categorically (universal properties III)

- $a' \xrightarrow{m} a$ is an *equalizer* of $f_1, f_2 \colon a \longrightarrow b$ in $\mathcal{C}$ $\overset{\Delta}{\Longleftrightarrow}$ $f_1 \circ m = f_2 \circ m$ and for any $c \xrightarrow{g} a$ s.t. $f_1 \circ g = f_2 \circ g$ $\exists! g' \colon c \longrightarrow a'$ s.t. $g = m \circ g'$
- equalizers are determined up to unique iso
- a *coequalizer* $b \xrightarrow{e} b'$ of $f_1, f_2 \colon a \longrightarrow b$ is the dual of an equalizer, i.e. $e \circ f_1 = e \circ f_2$ and for any $b \xrightarrow{g} c$ s.t. $g \circ f_1 = g \circ f_2$ $\exists! g' \colon b' \longrightarrow c$ s.t. $g = g' \circ e$
- The following statements hold:
  - m equalizer $\Longrightarrow$ m monic
  - m split monic $\Longrightarrow$ m equalizer
  - m equalizer and epic $\Longrightarrow$ m iso
- In **Set** an equalizer of $f_1, f_2 \colon X \longrightarrow Y$ is $X' \xrightarrow{m} X$, where $X' = \{x | f_1(x) = f_2(x)\}$ and m is the inclusion of X' in X.

Thinking Categorically (universal properties III)

- $a_1 \xleftarrow{p_1} a' \xrightarrow{p_2} a_2$ is a *pullback* of $a_1 \xrightarrow{f_1} b \xleftarrow{f_2} a_2$ in $\mathcal{C}$ $\overset{\Delta}{\iff}$ $f_1 \circ p_1 = f_2 \circ p_2$ and for any $a_1 \xleftarrow{g_1} c \xrightarrow{g_2} a_2$ s.t. $f_1 \circ g_1 = f_2 \circ g_2$ $\exists! g' \colon c \longrightarrow a'$ s.t. $g_i = p_i \circ g'$

  the commuting diagram
  $$\begin{tikzcd} a' \arrow[r, "p_2"] \arrow[d, "p_1"'] & a_2 \arrow[d, "f_2"] \\ a_1 \arrow[r, "f_1"] & b \end{tikzcd}$$
  is called a pullback square

- a pullback *corresponds* to a product of f_1 and f_2 in the slice category $\mathcal{C}/b$, thus (with some abuse of notation) we write $\boxed{a_1 \times_b a_2 \text{ for } a' \text{ and } \langle g_1, g_2 \rangle_b \text{ for } g'}$

- pullbacks are determined up to unique iso

- a *pushout* $b_1 \xrightarrow{q_1} b' \xleftarrow{q_2} b_2$ of $b_1 \xleftarrow{f_1} a \xrightarrow{f_2} b_2$ is the dual of a pullback

- In **Set** a pullback of $X_1 \xrightarrow{f_1} Y \xleftarrow{f_2} X_2$ is $X_1 \xleftarrow{p_1} X' \xrightarrow{p_2} X_2$, where $X' = \{(x_1, x_2) | f_1(x_1) = f_2(x_2)\}$ and $p_i(x_1, x_2) = x_i$

Thinking Categorically (universal properties III)

Properties of Pullbacks

Given a commuting diagram

$$\begin{tikzcd}
b_3 \arrow[r, "h_2"] \arrow[d, "g_3"] \arrow[dr, phantom, "(2)"] & b_2 \arrow[r, "h_1"] \arrow[d, "g_2"] \arrow[dr, phantom, "(1)"] & b_1 \arrow[d, "g_1"] \\
a_3 \arrow[r, "f_2"] & a_2 \arrow[r, "f_1"] & a_1
\end{tikzcd}$$

- if (1) and (2) are pullback squares, then the outer rectangle is a pullback square
- if the outer rectangle and (1) are pullback squares, then (2) is a pullback square
- If

$$\begin{tikzcd}
b_2 \arrow[r, "g"] \arrow[d, "m_2"] & b_1 \arrow[d, "m_1"] \\
a_2 \arrow[r, "f"] & a_1
\end{tikzcd}$$

is a pullback square and m_1 is monic, then m_2 is monic

Subobjects and Toposes

In Set Theory the definition of subset exploits the membership relation $\in$, while in a category $\mathcal{C}$ *subobjects* must be defined in terms of arrows

- **Mono**(a) is the preorder whose elements are monic $a' \overset{m}{\rightarrowtail} a$ into a and $m_1 \leq m_2 \overset{\Delta}{\iff} \exists m.m_1 = m_2 \circ m$ (m is necessarily unique and monic)
- a *subobject* of a is the equivalence class $[m]$ of a monic into a w.r.t. the equivalence $m_1 \equiv m_2 \overset{\Delta}{\iff} m_1 \leq m_2 \wedge m_2 \leq m_1 \iff \exists! i$ iso s.t. $m_2 = m_1 \circ i$
- **Sub**(a) is the partial order whose elements are subobjects of a and $[m_1] \leq [m_2] \overset{\Delta}{\iff} m_1 \leq m_2$ (the choice of representatives is irrelevant)
- a *global element* of a is a map $1 \xrightarrow{x} a$ (with 1 terminal object of $\mathcal{C}$) global elements of a are necessarily monic and $x_1 \leq x_2 \iff x_1 \equiv x_2$

In **Set** the subobjects of X are in bijective correspondence with the subsets of X

$Y \in \mathcal{P}(X)$ corresponds to $[m_Y]$, where m_Y is the inclusion of Y into X, moreover

- the bijection is an isomorphism of partial orders between **Sub**(X) and $(\mathcal{P}(X), \subseteq)$
- singleton subsets correspond to (equivalence classes of) global elements

Subobjects and Toposes

SKIP

Other set-theoretic notions that have a category theoretic reformulation are

- a *relation* between a and b (in $\mathcal{C}$) is a subobject of $a \times b$

  the category **Rel**$(\mathcal{C})$, s.t. **Rel**$(\mathcal{C})[a, b]$ consists of the relations between a and b, exists only when $\mathcal{C}$ has certain properties (the difficulty is to define composition)

- a *partial map* from a to b (in $\mathcal{C}$) is the equivalence class of $a \overset{m}{\longleftarrow\!\!<} a' \xrightarrow{f} b$ w.r.t.

  the equivalence $(m_1, f_1) \equiv (m_2, f_2) \overset{\Delta}{\Longleftrightarrow} \exists! i$ iso s.t. $m_2 = m_1 \circ i \wedge f_2 = f_1 \circ i$

  the category **pMap**$(\mathcal{C})$, s.t. **pMap**$(\mathcal{C})[a, b]$ consists of the partial maps from a to b, exists only when $\mathcal{C}$ has certain properties (e.g. it suffices to have all pullbacks)

Subobjects and Toposes

- $1 \overset{t}{\rightarrowtail} \Omega$ (with 1 terminal object)[a] is a *subobject classifier* in $\mathcal{C} \overset{\Delta}{\iff}$ for any $a' \overset{m}{\rightarrowtail} a$ monic $\exists! f$ s.t.
$$\begin{tikzcd} a \arrow[r, "f"] & \Omega \\ a' \arrow[u, tail, "m"] \arrow[r] & 1 \arrow[u, tail, "t"'] \end{tikzcd}$$
is a pullback square
- subobject classifiers are determined up to unique iso
- $\mathcal{C}$ is a *topos* $\overset{\Delta}{\iff}$ it is a CCC with all pullbacks and a subobject classifier (there are other equivalent definitions).

  Toposes are well-behaved categories, suitable to interpret intuitionistic HOL. They were introduced by Lawvere and Tierney (as a *substitute* for set theory). For more details see [BarrWells83].
- **Set** is a topos and a subobject classifier is given by a global element t of a two elements set, e.g. the set $\Omega = \{true, false\}$. Also the full subcategory **Fin** of finite sets is a topos (and the topos structure is inherited from **Set**).

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[a] This property of 1 is a consequence of the universal property of the monic t.

Addendum: Logic in a Topos

The interpretation of conjunction $\Omega\times\Omega \xrightarrow{\wedge} \Omega$, implication $\Omega\times\Omega \xrightarrow{\supset} \Omega$ and universal quantification $\Omega^a \xrightarrow{\forall_a} \Omega$ are the unique maps s.t. the following squares are pullbacks

$$\begin{tikzcd}
\Omega\times\Omega \arrow[r, "\wedge"] & \Omega & \Omega^a \arrow[r, "\forall_a"] & \Omega & \Omega\times\Omega \arrow[r, "\supset"] & \Omega \\
1\cong 1\times 1 \arrow[r] \arrow[u, tail, "t\times t"] & 1 \arrow[u, tail, "t"] & 1\cong 1^a \arrow[r] \arrow[u, tail, "t^a"] & 1 \arrow[u, tail, "t"] & \leq \arrow[r] \arrow[u, tail, "m"] & 1 \arrow[u, tail, "t"]
\end{tikzcd}$$

where m equalizer of $\Omega\times\Omega \underset{\pi_1}{\overset{\wedge}{\rightrightarrows}} \Omega$

$\boxed{\text{In intuitionistic HOL all logical constants are definable from } \supset \text{ and } \forall}$

$A\vee B \overset{\Delta}{\Longleftrightarrow} \forall x{:}\,\Omega.(A\supset x)\supset(B\supset x)\supset x$ and $A\wedge B \overset{\Delta}{\Longleftrightarrow} \forall x{:}\,\Omega.(A\supset B\supset x)\supset x$.

Part 3 - [AspertiLongo91, Ch 3]

Functors

A *functor* F from $\mathcal{C}$ to $\mathcal{D}$, notation $\boxed{F{:}\mathcal{C} \longrightarrow \mathcal{D}}$, consists of

- operations $F_0{:}\mathcal{C}_0 \longrightarrow \mathcal{D}_0$ and $F_1{:}\mathcal{C}_1 \longrightarrow \mathcal{D}_1$ $\boxed{\text{subscripts are usually omitted}}$ s.t.
- F preserves domain and codomain: $\boxed{a \xrightarrow{f} b \text{ in } \mathcal{C} \text{ implies } Fa \xrightarrow{Ff} Fb \text{ in } \mathcal{D}}$
- F preserves identity and composition: $\boxed{F(\mathrm{id}_a) = \mathrm{id}_{Fa} \text{ and } F(g \circ f) = Fg \circ Ff}$

Cat is the category of (small) categories and functors (the definition of identity functors and functor composition are obvious).

- $F{:}\mathcal{C} \longrightarrow \mathcal{D}$ is *faithful* $\overset{\Delta}{\iff}$ $\boxed{\forall a, b \in \mathcal{C}. \forall f, g \in \mathcal{C}[a, b].\ Ff = Fg \text{ implies } f = g}$
- $F{:}\mathcal{C} \longrightarrow \mathcal{D}$ is *full* $\overset{\Delta}{\iff}$ $\boxed{\forall a, b \in \mathcal{C}. \forall g \in \mathcal{D}[Fa, Fb]. \exists f \in \mathcal{C}[a, b] \text{ s.t. } g = Ff}$
- $F{:}\mathcal{C} \longrightarrow \mathcal{D}$ *equivalence*[a] $\overset{\Delta}{\iff}$ full, faithful and $\boxed{\forall b \in \mathcal{D}. \exists a \in \mathcal{C} \text{ s.t. } b \cong Fa \text{ in } \mathcal{D}}$

In **Cat** “monic $\implies$ faithful” and “iso $\implies$ equivalence”, but “epic $\implies$ full” fails.

Functors

Dogma 2: to any *natural* construction on structures of one species, yielding structures of another species, there corresponds a **functor** from the category of the first species to the category of the second.

- Functors between discrete categories correspond to functions between the underlying collections of objects
- Functors between preorders correspond to monotonic maps
- Functors between monoids correspond to monoid homomorphisms
- If $\mathcal{C}$ is a subcategory of $\mathcal{D}$, then there is a *inclusion functor* $In: \mathcal{C} \longrightarrow \mathcal{D}$, i.e. $\boxed{In(a) = a \text{ and } In(f) = f}$. In is monic. When $\mathcal{C}$ is full, then also In is full.
- Given $\mathcal{C}$ whose objects are sets with *additional structure* (and arrows are functions *respecting* the structure), there is a *forgetful functor* $U: \mathcal{C} \longrightarrow$ **Set**, which maps an object to the underlying set and is the identity on arrows (thus U is faithful). Examples are: **Mon**, **Grp**, **Top**, **PO**, $\mathbf{Alg}_\Omega$, **EN**, $\underline{A}$**-Set**. Similarly one can define
- U: **Grp** $\longrightarrow$ **Mon** mapping a group to the underlying monoid (this U is also full)
- U_0, U_1: **Graph** $\longrightarrow$ **Set** mapping a graph to the underlying set of nodes/arcs.
- U: **Cat** $\longrightarrow$ **Graph** mapping a category to the underlying graph.

Functors

Dogma 2: to any *natural* construction on structures of one species, yielding structures of another species, there corresponds a **functor** from the category of the first species to the category of the second.

- *diagonal functor* $\Delta : \mathcal{C} \longrightarrow \mathcal{C} \times \mathcal{C}$ is given by $\Delta(a) = (a, a)$ and $\Delta(f) = (f, f)$
- *projection functor* $\pi_i : \mathcal{C}_1 \times \mathcal{C}_2 \longrightarrow \mathcal{C}_i$ is given by $\pi_i(a_1, a_2) = a_i$ and $\pi_i(f_1, f_2) = f_i$
- Given a biCCC $\mathcal{C}$, we define the following functors (using choice)
  - $\times : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ mapping (a_1, a_2) to $a_1 \times a_2$, where $a_1 \xleftarrow{\pi_1} a_1 \times a_2 \xrightarrow{\pi_2} a_2$ is a *chosen* product diagram
  - $+ : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ mapping (a_1, a_2) to $a_1 + a_2$, where $a_1 \xrightarrow{\iota_1} a_1 + a_2 \xleftarrow{\iota_2} a_2$ is a *chosen* coproduct diagram
  - $-^a : \mathcal{C} \longrightarrow \mathcal{C}$ (for each $a \in \mathcal{C}$) mapping b to b^a, where $b^a \times a \xrightarrow{ev} b$ is a *chosen* exponential diagram. b^a is *contravariant* in a, i.e. we have a binary functor $\mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathcal{C}$ mapping (a, b) to b^a.

The definition of $f_1 \times f_2$, $f_1 + f_2$ and g^a (and the proof that they preserve identities and composition) exploit the universal properties of products, coproducts and exponentials.

Functors

Dogma 2: to any *natural* construction on structures of one species, yielding structures of another species, there corresponds a **functor** from the category of the first species to the category of the second.

- Given a category $\mathcal{C}$ with pullbacks, for each $a \xrightarrow{f} b$ we define (using choice) the *pullback functor* $f^*\colon \mathcal{C}/b \longrightarrow \mathcal{C}/a$ mapping $c \xrightarrow{g} b$ to $c' \xrightarrow{g'} a$, where

$$\begin{tikzcd}
c' \arrow[r, "f'"] \arrow[d, "g'"] & c \arrow[d, "g"] \\
a \arrow[r, "f"] & b
\end{tikzcd}$$

  is a *chosen* pullback square.

- the pullback functor induces a monotonic map $f^*\colon \textbf{Sub}(b) \longrightarrow \textbf{Sub}(a)$, called *inverse image*

- $\mathcal{C}$ is *locally small* $\overset{\Delta}{\iff} \forall a, b \in \mathcal{C}$ the collection $\mathcal{C}[a, b]$ is a set.

  Given a locally small $\mathcal{C}$, the *hom-functor* $\mathcal{C}[-,-]\colon \mathcal{C}^{op} \times \mathcal{C} \longrightarrow$ **Set** maps (a, b) to $\mathcal{C}[a, b]$, while $\mathcal{C}[f, g]$ is the function $h \mapsto g \circ h \circ f$ (with the appropriate domain).

Functors

Dogma 2: to any *natural* construction on structures of one species, yielding structures of another species, there corresponds a **functor** from the category of the first species to the category of the second.

- There are two functors extending the powerset $\mathcal{P}(X)$ construction on sets
  - the contravariant powerset $P\colon \mathbf{Set}^{op} \longrightarrow \mathbf{Set}$ mapping $f\colon Y \longrightarrow X$ into $Pf\colon \mathcal{P}(X) \longrightarrow \mathcal{P}(Y)$, where $Pf(X') = f^{-1}(X') = \{y|f(y) \in X'\}$
  - the covariant powerset $\exists\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ mapping $f\colon X \longrightarrow Y$ to $\exists f\colon \mathcal{P}(X) \longrightarrow \mathcal{P}(Y)$, where $\exists f(X') = f(X') = \{f(x)|x \in X'\}$
- Give examples of construction on sets that do not extend to a functor on **Set**

  e.g. case analysis on the cardinality of X set, or on whether X is a member of a given set.

- Give examples of functors between some of the *concrete* categories defined so far

  A faithful functor from **EN** into $\underline{A}$**-Set**, exploiting the encoding of N in any non-trivial pCA

  Full and faithful functors from **Set** into **Top**, **PO** and $\underline{A}$**-Set**

  Functors from **Top** to **PO** and conversely

Natural Transformations

Given two functors $F, G\colon \mathcal{C} \longrightarrow \mathcal{D}$, a *natural transformation* $\boxed{\tau\colon F \longrightarrow G}$ consists of an operation $\tau\colon \mathcal{C}_0 \longrightarrow \mathcal{D}_1$, we may write $\boxed{\tau_a \text{ for } \tau(a)}$, s.t.

$\boxed{\forall a \in \mathcal{C}. \tau_a \in \mathcal{D}[Fa, Ga]}$ and $\boxed{\forall a, b \in \mathcal{C}. \forall f \in \mathcal{C}[a, b].\ \tau_b \circ Ff = Gf \circ \tau_a}$

or equivalently, for all $a \xrightarrow{f} b$ in $\mathcal{C}$ the square[a]

$$\begin{tikzcd} Fa \arrow[r, "\tau_a"] \arrow[d, "Ff"'] & Ga \arrow[d, "Gf"] \\ Ga \arrow[r, "\tau_b"] & Gb \end{tikzcd}$$

commutes in $\mathcal{D}$

To make explicit also the categories involved we write

$$\boxed{\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow \tau"] & \mathcal{D} \end{tikzcd}}$$

---

[a] They are called *naturality squares*.

Natural Transformations

Dogma 3: to each *natural translation* from a construction $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ to a construction $G\colon \mathcal{A} \longrightarrow \mathcal{B}$ there corresponds a **natural transformation** $F \longrightarrow G$.

- the *identity natural transformation* $$\begin{tikzcd} \mathcal{A} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "F"'] \arrow[r, phantom, "\Downarrow \mathrm{id}_F"] & \mathcal{B} \end{tikzcd}$$ is $\mathrm{id}_F(a) = \mathrm{id}_{Fa}$

- if $$\begin{tikzcd} \mathcal{A} \arrow[r, bend left=70, "F_1"] \arrow[r, "F_2" description] \arrow[r, bend right=70, "F_3"'] \arrow[r, phantom, shift left=3ex, "\Downarrow \tau_1"] \arrow[r, phantom, shift right=3ex, "\Downarrow \tau_2"] & \mathcal{B} \end{tikzcd}$$ the *vertical composite* $$\begin{tikzcd} \mathcal{A} \arrow[r, bend left=50, "F_1"] \arrow[r, bend right=50, "F_3"'] \arrow[r, phantom, "\Downarrow \tau_2 \circ \tau_1"] & \mathcal{B} \end{tikzcd}$$ is $(\tau_2 \circ \tau_1)_a = \tau_2(a) \circ \tau_1(a)$

In fact, (when $\mathcal{A}$ is small) there is a *functor category* $\mathcal{B}^{\mathcal{A}}$ of functors $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ and natural transformations, we write $\boxed{\textbf{Nat}[F,G] \text{ for } \mathcal{B}^{\mathcal{A}}[F,G]}$. Moreover, there is a functor $ev\colon \mathcal{B}^{\mathcal{A}} \times \mathcal{A} \longrightarrow \mathcal{B}$ s.t. $ev(F,a) = Fa$, which is an exponential diagram in **Cat**.

Natural Transformations

Dogma 3: to each *natural translation* from a construction $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ to a construction $G\colon \mathcal{A} \longrightarrow \mathcal{B}$ there corresponds a **natural transformation** $F \longrightarrow G$.

- if $\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F_1"{name=U}] \arrow[r, bend right, "G_1"'{name=D}] \arrow[from=U, to=D, Rightarrow, "\tau_1"] & \mathcal{B} \arrow[r, bend left, "F_2"{name=U2}] \arrow[r, bend right, "G_2"'{name=D2}] \arrow[from=U2, to=D2, Rightarrow, "\tau_2"] & \mathcal{C} \end{tikzcd}$ the *horizontal composite* $\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F_2 \circ F_1"{name=U}] \arrow[r, bend right, "G_2 \circ G_1"'{name=D}] \arrow[from=U, to=D, Rightarrow, "\tau_2\tau_1"] & \mathcal{C} \end{tikzcd}$ is

$$\begin{tikzcd}
F_2(F_1a) \arrow[r, "F_2(\tau_1(a))"] \arrow[d, "\tau_2(F_1a)"'] \arrow[dr, dashed, "(\tau_2\tau_1)_a"] & F_2(G_1a) \arrow[d, "\tau_2(G_1a)"] \\
G_2(F_1a) \arrow[r, "G_2(\tau_1(a))"'] & G_2(G_1a)
\end{tikzcd}$$

the square commute by naturality of τ_2

Natural Transformations

Dogma 3: to each *natural translation* from a construction $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ to a construction $G\colon \mathcal{A} \longrightarrow \mathcal{B}$ there corresponds a **natural transformation** $F \longrightarrow G$.

- $$\begin{tikzcd} \mathcal{A} \arrow[r, shift left=2ex, "F"] \arrow[r, shift right=2ex, "G"'] \arrow[r, phantom, "\Downarrow\tau"] & \mathcal{B} \end{tikzcd}$$ is a *natural iso* $\overset{\Delta}{\iff} \exists\tau'\colon G \longrightarrow F$ s.t. $\tau \circ \tau' = \mathrm{id}_G$ and $\tau' \circ \tau = \mathrm{id}_F$
- τ is a natural iso $\iff$
  τ natural and $\forall a \in \mathcal{A}.\ \tau_a$ iso in $\mathcal{B}$ $\iff$
  τ is an iso in $\mathcal{B}^{\mathcal{A}}$ (provided $\mathcal{A}$ is small)
- "$F\colon \mathcal{A} \longrightarrow \mathcal{B}$ is an equivalence" can be rephrased as follow (using choice):
  exists $G\colon \mathcal{B} \longrightarrow \mathcal{A}$ and natural isos $G \circ F \longrightarrow \mathrm{id}_{\mathcal{A}}$ and $F \circ G \longrightarrow \mathrm{id}_{\mathcal{B}}$
- universal properties induce both functors and natural transformations, e.g. if $\mathcal{C}$ is a biCCC, then in addition to the functors $- \times -$, $- + -$ and $-^a$ we have

$$\begin{tikzcd} \mathcal{C} \times \mathcal{C} \arrow[r, shift left=2ex, "\times"] \arrow[r, shift right=2ex, "\pi_i"'] \arrow[r, phantom, "\Downarrow\pi_i"] & \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \times \mathcal{C} \arrow[r, shift left=2ex, "\pi_i"] \arrow[r, shift right=2ex, "+"'] \arrow[r, phantom, "\Downarrow\iota_i"] & \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \arrow[r, shift left=2ex, "-^a \times a"] \arrow[r, shift right=2ex, "\mathrm{id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow ev"] & \mathcal{C} \end{tikzcd}$$ where

$\pi_i(a_1, a_2)$ is $a_1 \times a_2 \xrightarrow{\pi_i} a_i$, $\iota_i(a_1, a_2)$ is $a_i \xrightarrow{\iota_i} a_1 + a_2$, $ev(b)$ is $b^a \times a \xrightarrow{ev} b$

Yoneda

- $F\colon\mathcal{C}\longrightarrow$ **Set** is *representable* $\overset{\Delta}{\Longleftrightarrow}$ exists $a\in\mathcal{C}$ and a natural iso $\phi\colon\mathcal{C}[a,-]\longrightarrow F$
- one can recast universal properties in terms of representable functors, e.g.
  - a product diagram $a_1 \xleftarrow{\pi_1} a \xrightarrow{\pi_2} a_2$ corresponds to a natural iso from $\mathcal{C}[-,a]\colon\mathcal{C}^{op}\longrightarrow$ **Set** to $(\mathcal{C}\times\mathcal{C})[-,(a_1,a_2)]\circ\Delta$
  - a coproduct diagram $a_1 \xrightarrow{\iota_1} a \xleftarrow{\iota_2} a_2$ corresponds to a natural iso from $\mathcal{C}[a,-]\colon\mathcal{C}\longrightarrow$ **Set** to $(\mathcal{C}\times\mathcal{C})[(a_1,a_2),-]\circ\Delta$
  - an exponential diagram $c\times a \xrightarrow{ev} b$ corresponds to a natural iso from $\mathcal{C}[-,c]\colon\mathcal{C}^{op}\longrightarrow$ **Set** to $\mathcal{C}[-\times a,b]$
  - a subobject classifier $1\xrightarrow{t}\Omega$ corresponds to a natural iso from $\mathcal{C}[-,\Omega]$ to a suitable contravariant functor **Sub**$(-)$
- Yoneda lemma: given $F\colon\mathcal{C}\longrightarrow$ **Set** and $a\in\mathcal{C}$ the following mapping is a bijection

  $\psi\colon$**Nat**$[\mathcal{C}[a,-],F]\longrightarrow F(a)$ s.t. $\psi\colon\phi\longmapsto\phi_a(\mathrm{id}_a)$ since $\phi_b(f\colon a\to b)=Ff(\phi_a(\mathrm{id}_a))$
- Yoneda embedding: given a small $\mathcal{C}$, the *category of presheaves* **Set**$^{\mathcal{C}^{op}}$ is a topos, and the functor $Y\colon\mathcal{C}\longrightarrow$ **Set**$^{\mathcal{C}^{op}}$ s.t. $Ya=\mathcal{C}[-,a]$ is full and faithful

Addendum SKIP: Properties of Presheaves

A product diagram[a] $\pi_i\colon F \longrightarrow F_i$ in **Set**$^{\mathcal{C}^{op}}$[b] for the I-indexed family $\langle F_i | i \in I\rangle$ is

$$F(a) \triangleq \prod_{i\in I} F_i(a) \quad F(f) \triangleq \prod_{i\in I} F_i(f) \quad \pi_i(a) \triangleq \pi_i\colon \prod_{i\in I} F_i(a) \longrightarrow F_i(a)$$

A subobject classifier $1 \xrightarrow{t} \Omega$ is (where $a, b, c \in \mathcal{C}$ and $f \in \mathcal{C}[b, a]$ and $g \in \mathcal{C}[c, b]$)

$$\Omega(a) \triangleq \{X \in \prod_{b\in\mathcal{C}} \mathcal{P}(\mathcal{C}[b,a]) | \forall g. X_b \circ g \subseteq X_c\} \quad (\Omega f X)_c \triangleq \{g | f \circ g \in X_c\} \quad (t_a)_b \triangleq \mathcal{C}[b,a]$$

By Yoneda we must have $\Omega(a) \cong \mathbf{Nat}[Ya, \Omega] \cong \mathbf{Sub}(Ya)$

An exponential diagram $H \times F \xrightarrow{ev} G$ in **Set**$^{\mathcal{C}^{op}}$ is

$$H(a) \triangleq \{s \in \prod_{f\in\mathcal{C}/a} (Fb \to Gb) | \forall f, g. \begin{tikzcd} Fb \arrow[r, "s_f"] \arrow[d, "Fg"] & Gb \arrow[d, "Gg"] \\ Fc \arrow[r, "s_{f\circ g}"] & Gc \end{tikzcd} \} \qquad (Hfs)_g \triangleq s_{f\circ g}$$

$$ev_a(s, x) \triangleq s_{\mathrm{id}_a}(x)$$

---

[a] Similarly coproducts, equalizers and pullbacks diagram are definable *pointwise*.

[b] When $\mathcal{C}$ is a preorder, the objects of **Set**$^{\mathcal{C}^{op}}$ are *Kripke sets*.

Addendum: Internal Categories [AspertiLongo91, Sec 7.3]

Many mathematical notions can be recast within an ambient category $\mathcal{E}$, so that one recovers the original notion when $\mathcal{E} = \textbf{Set}$. For instance:

- When $\mathcal{E}$ has finite products, an *internal monoid* in a $\mathcal{E}$ consists of an object $M \in \mathcal{E}$ two arrows $1 \xrightarrow{e} M \xleftarrow{m} M \times M$ s.t. certain diagrams commute

$$\begin{tikzcd}
1 \times M \arrow[r, "e \times \mathrm{id}"] \arrow[dr, "\pi_2"'] & M \times M \arrow[d, "m"] & M \times 1 \arrow[l, "\mathrm{id} \times e"'] \arrow[dl, "\pi_1"] \\
 & M &
\end{tikzcd}
\qquad
\begin{tikzcd}
M \times M \times M \arrow[r, "\mathrm{id} \times m"] \arrow[d, "m \times \mathrm{id}"] & M \times M \arrow[d, "m"] \\
M \times M \arrow[r, "m"] & M
\end{tikzcd}$$

  Mon$(\mathcal{E})$ is (by dogma 1) the category whose objects are monoids in $\mathcal{E}$.

- When $\mathcal{E}$ has finite limits, one can recast basic notions (and results) of Category Theory within $\mathcal{E}$, e.g. an *internal category* consists of two objects $C_0, C_1 \in \mathcal{E}$ and arrows $\begin{tikzcd} C_0 \arrow[r, "i" description] & C_1 \arrow[l, shift right=2ex, "d_1"'] \arrow[l, shift left=2ex, "d_0"] & C_1 \times_0 C_1 \arrow[l, "c"'] \end{tikzcd}$ [a] s.t. certain diagrams commute.

  Cat$(\mathcal{E})$ is (by dogma 1) the category whose objects are categories in $\mathcal{E}$.

---

[a] $C_1 \times_0 C_1$ is the pullback of $C_1 \xrightarrow{d_1} C_0 \xleftarrow{d_0} C_1$.

Addendum: Internal Categories [AspertiLongo91, Sec 7.3]

Many mathematical notions can be recast within an ambient category $\mathcal{E}$, so that one recovers the original notion when $\mathcal{E} = $ **Set**.

Moreover, an ambient category $\mathcal{E}$ can serve as a *non-standard* universe, where properties (expressed in the internal language and) inconsistent with classical Set Theory (thus not valid in **Set**) become true $\boxed{\text{SEMANTIC FREEDOM}}$. For instance

- there are biCCC (even toposes) with a NNO N s.t. every map $N \longrightarrow N$ is Turing-computable (in **Set** this is false for cardinality reasons)
- there are CCC $\mathcal{E}$ (even toposes) with nontrivial *reflexive objects*, i.e. a U s.t. $U^U \lhd U$ or $U^U \cong U$ (in **Set** only the terminal object 1 is reflexive)
- there are CCC $\mathcal{E}$ (even toposes) with nontrivial objects U with *fix-point operators*, i.e. a map $\mathrm{fix}\colon U^U \longrightarrow U$ s.t. $f\colon U^U \vdash f(\mathrm{fix}\, f) = \mathrm{fix}\, f\colon U$ (in **Set** only 1 has fix)

Addendum: Indexed Categories [AspertiLongo91, Sec 7.1]

Given a set I and a category $\mathcal{A}$ of structures of a certain species, one can define a category $\mathcal{A}^I$ whose objects are I-indexed families of objects of $\mathcal{A}$. Given a *base category* $\mathcal{B}$, then one can take $\mathcal{A}^{\mathcal{B}^{op}}$ as the category of $\mathcal{B}$*-indexed objects of* $\mathcal{A}$.

- If $\mathcal{B}$ is the discrete category corresponding to a set I, then functors $A : \mathcal{B}^{op} \longrightarrow \mathcal{A}$ correspond to I-indexed families $\langle a_i | i \in I \rangle$.
- An *internal set* in $\mathcal{B}$, i.e. a $b \in \mathcal{B}$, induces a $\mathcal{B}$-indexed set $\mathcal{B}[-, b] : \mathcal{B}^{op} \longrightarrow$ **Set**, via the full and faithful Yoneda embedding $Y : \mathcal{B} \longrightarrow \mathbf{Set}^{\mathcal{B}^{op}}$.

For many species of mathematical structures (that can be recast within $\mathcal{B}$), one has a Yoneda-like embedding $Y : \mathcal{A}(\mathcal{B}) \longrightarrow \mathcal{A}^{\mathcal{B}^{op}}$. Yoneda-like embeddings exist for the following categories $\mathcal{A}$ (of mathematical structures): **Mon**, **Grp**, **PO**, **Graph**, **Cat**.

$\mathcal{B}$-indexed notions *generalize* internal notions within $\mathcal{B}$

In particular, $\mathcal{B}$-indexed notions do not rely on additional properties of $\mathcal{B}$.

Addendum: Indexed Categories [AspertiLongo91, Sec 7.1]

- A $\mathcal{B}$-*indexed category* is a functor $C \colon \mathcal{B}^{op} \longrightarrow$ **CAT**[a] (**CAT** includes *large* categories), we may write $\boxed{C_b \text{ for } C(b)}$ and $\boxed{f^* \text{ for } C(f)}$
- A $\mathcal{B}$-*indexed functor* $F \colon C \longrightarrow D$ is a family of functors $F_b \colon C_b \longrightarrow D_b$ s.t.

$$\begin{tikzcd} C_a \arrow[r, "F_a"] \arrow[d, "f^*"] & D_a \arrow[d, "f^*"] \\ Cb \arrow[r, "F_b"] & D_b \end{tikzcd}$$

commutes for each

$$\begin{tikzcd} a \\ b \arrow[u, "f"] \end{tikzcd}$$

in $\mathcal{B}$

- A $\mathcal{B}$-*indexed natural transformation* $\tau \colon F \longrightarrow G$ is a family of natural transformations $\tau_b \colon F_b \longrightarrow G_b$ s.t.

$$\begin{tikzcd} C_a \arrow[r, bend left, "F_a"{name=U}] \arrow[r, bend right, "G_a"'{name=L}] \arrow[d, "f^*"] & D_a \arrow[d, "f^*"] \\ C_b \arrow[r, bend left, "F_b"{name=U2}] \arrow[r, bend right, "G_b"'{name=L2}] & D_b \arrow[from=U, to=L, Rightarrow, "\tau_a"] \arrow[from=U2, to=L2, Rightarrow, "\tau_b"] \end{tikzcd}$$

commutes for each

$$\begin{tikzcd} a \\ b \arrow[u, "f"] \end{tikzcd}$$

in $\mathcal{B}$

---

[a] We adopt the *strict* notion, see also the notion of *fibration*.

Addendum: Indexed Categories [AspertiLongo91, Sec 7.1]

Given $\mathcal{B}$ with a choice of finite products, then B is the $\mathcal{B}$-indexed category s.t.

- the *fiber* B_b has the same objects of $\mathcal{B}$ and arrows $B_b[x,y]=\mathcal{B}[b\times x,y]$

  the identity for x and the composite of $g_2\in B_b[y,z]$ and $g_1\in B_b[x,y]$ are

  $b\times x\xrightarrow{\pi_2} x$ and $b\times x\xrightarrow{\langle\pi_1,g_1\rangle} b\times y\xrightarrow[2]{g} z$
- given $f\colon b\longrightarrow a$ the *re-indexing* functor $f^*\colon B_a\longrightarrow B_b$ is s.t.

  $f^*(x)=x$ and $f^*(g)=b\times x\xrightarrow{f\times\mathrm{id}} a\times x\xrightarrow{g} y$ when $g\in B_a[x,y]$

Given $\mathcal{B}$ with a coherent choice of pullbacks, then $B/$ is the $\mathcal{B}$-indexed category s.t.

- the *fiber* $B/_b$ is the slice category $\mathcal{B}/b$
- given $f\colon b\longrightarrow a$ the *re-indexing* functor along f is $f^*\colon\mathcal{B}/a\longrightarrow\mathcal{B}/b$[a]

Objects of $B/_b$ are b-*indexed family* $f\colon a\to b$. An $x\in B_b$ can be identified with the *constant* b-indexed family $\pi_1\colon b\times x\to b$. Indeed, there is a full and faithful $\mathcal{B}$-indexed functor $In\colon B\longrightarrow B/$ s.t. $In_b(x)=b\times x\xrightarrow{\pi_1} b$ and $In_b(g)=b\times x\xrightarrow{\langle\pi_1,g\rangle} b\times y$.

---

[a] A coherent choice of pullbacks ensures that $\mathrm{id}^*=\mathrm{id}$ and $(f_1\circ f_2)^*=f_2^*\circ f_1^*$.

Addendum: Hyperdoctrines [Lawvere69]

An $\mathcal{B}$-*hyperdoctrine* P is a $\mathcal{B}$-indexed category s.t. each fiber P_b is a preorder[a].

Given $\mathcal{B}$ with pullbacks[b], then **Sub** is the $\mathcal{B}$-hyperdoctrine s.t.

- the *fiber* **Sub**$_b$ is the partial order **Sub**(b) of subobjects of b
- given $f\colon b \longrightarrow a$ the *re-indexing* along f is inverse image $f^*\colon$ **Sub**$(a) \longrightarrow$ **Sub**(b).

Predicate Logic can be interpreted in an hyperdoctrine[c] as follows:

- types and contexts are interpreted by objects in $\mathcal{B}$
- well-formed terms are interpreted by arrows in $\mathcal{B}$, and composition is substitution
- well-formed formula are interpreted by objects in P_b, entailment is interpreted by the preorder on P_b, and re-indexing is substitution.

---

[a] In some cases one may require the fibers to be partial orders.

[b] With a coherent choice of pullbacks, one can take **Mono** (full $\mathcal{B}$-indexed subcategory of $B/$)

[c] Usually there are additional requirements, e.g. $\mathcal{B}$ is cartesian and each P_b is biCCC.

Addendum: Hyperdoctrines [Lawvere69]

The *internal language* L of a $\mathcal{B}$-hyperdoctrine P extends the language of $\mathcal{B}$ with

- raw formulas $A ::= p(M) \mid \ldots$ with p object of some fiber P_b, and the judgments
  - $\Gamma \vdash A$ asserting well-formedness of formula A $\quad p\ \dfrac{\Gamma \vdash M{:}t}{\Gamma \vdash p(M)}\ p \in P_{[\![t]\!]}$
  - $\Gamma \vdash A_1 \implies A_2$ asserting that A_1 entails A_2
- well-formed formula $\Gamma \vdash A$ are interpreted by objects in $P_{[\![\Gamma]\!]}$

$\boxed{[\![\Gamma \vdash p(M)]\!] \stackrel{\Delta}{=} f^*(p)}$ with $f = [\![\Gamma \vdash M{:}t]\!]$

Substitution is Re-indexing

- subst $\dfrac{\Gamma \vdash M{:}t \qquad x{:}t \vdash A}{\Gamma \vdash [M/x]A}$ is an admissible rule
- $[\![\Gamma \vdash [M/x]A]\!] = f^*(p)$ if $[\![\Gamma \vdash M{:}t]\!] = c \xrightarrow{f} a$ and $[\![x{:}t \vdash A]\!] = p \in P_a$

Addendum SKIP: Enriched Categories [Kelly82]

Given a cartesian category $\mathcal{V}$[a], a $\mathcal{V}$-*enriched category* $\mathcal{C}$ consists of

- a collection $\mathcal{C}_0$ of objects
- a family of objects $\mathcal{C}[a,b] \in \mathcal{V}$ with $a, b \in \mathcal{C}_0$
- two families of arrows $i_a\colon 1 \longrightarrow \mathcal{C}[a,a]$ and $c_{a,b,c}\colon \mathcal{C}[b,c] \times \mathcal{C}[a,b] \longrightarrow \mathcal{C}[a,c]$ s.t.

$$\begin{tikzcd}
1 \times \mathcal{C}[a,b] \arrow[r] \arrow[d, "i_b \times \mathrm{id}"] & \mathcal{C}[a,b] \arrow[d, equal] & \mathcal{C}[a,b] \times 1 \arrow[l] \arrow[d, "\mathrm{id} \times i_a"] \\
\mathcal{C}[b,b] \times \mathcal{C}[a,b] \arrow[r, "c"] & \mathcal{C}[a,b] & \mathcal{C}[a,b] \times \mathcal{C}[a,a] \arrow[l, "c"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}[c,d] \times \mathcal{C}[b,c] \times \mathcal{C}[a,b] \arrow[r, "c \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times c"] & \mathcal{C}[b,d] \times \mathcal{C}[a,b] \arrow[d, "c"] \\
\mathcal{C}[c,d] \times \mathcal{C}[a,c] \arrow[r, "c"] & \mathcal{C}[a,d]
\end{tikzcd}$$

$\mathcal{V}$-enriched functors and natural transformations are defined in the obvious way.

- Given a CCC $\mathcal{V}$, then V is the $\mathcal{V}$-enriched category s.t. $V[a,b] \triangleq b^a$ for any $a, b \in \mathcal{V}$
- An ultra-metric space (X, d), i.e. $d(x,z) \leq \max\{d(x,y), d(y,z)\}$, is a $\mathcal{V}$-enriched category, where $\mathcal{V}$ is the poset of real numbers ≥ 0 with the reverse order (thus 0 is terminal and $\max(x,y)$ is the product of x and y).

---

[a] The notion does not extend to other structures (it suffices to take $\mathcal{V}$ *monoidal*) .

Part 4 - [AspertiLongo91, Ch 5] give examples of hyperdoctrines go back to Yoneda

Universal Arrows

Given a functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ and an object $d \in \mathcal{D}$

- a *universal arrow* from d to F consists of a pair $\langle u, c\rangle$ with $c \in \mathcal{C}$ and $d \xrightarrow{u} Fc$ s.t.

  $\forall a \in \mathcal{C}. \forall f\colon d \longrightarrow Fa.\ \exists! f'\colon c \longrightarrow a$ s.t.

$$\begin{tikzcd}
d \arrow[r, "u"] \arrow[dr, "f"'] & Fc \arrow[d, "Ff'"] \\
 & Fa
\end{tikzcd}$$

- a universal arrow $\langle u, c\rangle$ from d to F is determined up to unique iso, i.e.
  - if $c \xrightarrow{i} c'$ is an iso in $\mathcal{C}$, then $\langle (Fi) \circ u, c'\rangle$ is a universal arrow from d to F
  - if $\langle u', c'\rangle$ is a universal arrow from d to F, then

    $\exists! i\colon c \longrightarrow c'$ iso s.t.

$$\begin{tikzcd}
d \arrow[r, "u"] \arrow[dr, "u'"'] & Fc \arrow[d, "Fi"] \\
 & Fc'
\end{tikzcd}$$

Universal Arrows

Given a functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ and an object $d \in \mathcal{D}$

- a *universal arrow* from d to F consists of a pair $\langle u, c\rangle$ with $c \in \mathcal{C}$ and $d \xrightarrow{u} Fc$ s.t.

  $\forall a \in \mathcal{C}.\forall f\colon d \longrightarrow Fa.\ \exists! f'\colon c \longrightarrow a$ s.t.

$$\begin{tikzcd}
d \arrow[r, "u"] \arrow[dr, "f"'] & Fc \arrow[d, "Ff'"] \\
 & Fa
\end{tikzcd}$$

- a *universal arrow* from F to d consists of a pair $\langle c, u\rangle$ with $c \in \mathcal{C}$ and $Fc \xrightarrow{u} d$ s.t.

  $\forall a \in \mathcal{C}.\forall f\colon Fa \longrightarrow d.\ \exists! f'\colon a \longrightarrow c$ s.t.

$$\begin{tikzcd}
d & Fc \arrow[l, "u"'] \\
 & Fa \arrow[ul, "f"] \arrow[u, "Ff'"']
\end{tikzcd}$$

- $\langle c, u\rangle$ universal from F to $d \iff \langle u, c\rangle$ universal from d to $F^{op}\colon \mathcal{C}^{op} \longrightarrow \mathcal{D}^{op}$.

Reformulations and Examples

- A universal arrow $\langle u, c\rangle$ from $d \in \mathcal{D}$ to $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ corresponds to an initial object in the category $d \uparrow F$ given by

  (objects) $\langle f, a\rangle$ with $a \in \mathcal{C}$ and $f \in \mathcal{D}[d, Fa]$

  (arrows) $\langle f_1, a_1\rangle \xrightarrow{g} \langle f_2, a_2\rangle$ with $g \in \mathcal{C}[a_1, a_2]$ s.t.
$$\begin{tikzcd} d \arrow[r, "f_1"] \arrow[dr, "f_2"'] & Fa_1 \arrow[d, "Fg"] \\ & Fa_2 \end{tikzcd}$$
  commutes

- When $\mathcal{C}$ and $\mathcal{D}$ are locally small, then exists a universal arrow from d to $F \iff$ the functor $\mathcal{D}[d, F-]\colon \mathcal{C} \longrightarrow$ **Set** is representable
  - if $\langle u, c\rangle$ is a universal arrow, then $\phi\colon \mathcal{C}[c, -] \longrightarrow \mathcal{D}[d, F-]$ s.t. $\phi_a(f) = (Ff) \circ u$ is a natural iso
  - if $\phi\colon \mathcal{C}[c, -] \longrightarrow \mathcal{D}[d, F-]$ is a natural iso, then $\langle \phi_c(\mathrm{id}_c), c\rangle$ is a universal arrow.

Reformulations and Examples

Any universal property (for $\mathcal{C}$) introduced so far can be recast in terms of universal arrows to/from a functor $F\colon\mathcal{C} \longrightarrow \mathcal{D}$ by a suitable choice of $\mathcal{D}$ and F.

- an I-indexed coproduct diagram $\iota_i\colon c_i \longrightarrow c$ corresponds to a universal arrow $\langle\langle \iota_i | i \in I\rangle, c\rangle$ from $\langle c_i | i \in I\rangle$ to $\Delta\colon\mathcal{C} \longrightarrow \mathcal{C}^I$, where $\Delta(c) \triangleq \langle c | i \in I\rangle$ and $\mathcal{C}^I$ is

  (objects) I-indexed families $a = \langle a_i | i \in I\rangle$ of objects of $\mathcal{C}$

  (arrows) $\langle f_i | i \in I\rangle\colon a \longrightarrow b$ provided $\forall i \in I. f_i \in \mathcal{C}[a_i, b_i]$

  dually, I-indexed product diagrams corresponds to universal arrows from Δ

- an equalizer $a \xrightarrow{m} a_1$ of $f_1, f_2\colon a_1 \longrightarrow a_2$ corresponds to a universal arrow $\langle a, (m, f_i \circ n)\rangle$ from $\Delta\colon\mathcal{C} \longrightarrow \mathcal{C}^{\rightrightarrows}$ to (f_1, f_2), where $\Delta(c) \triangleq (\mathrm{id}_c, \mathrm{id}_c)$ and $\mathcal{C}^{\rightrightarrows}$ is

  (objects) pairs $f = (f_1, f_2\colon a_1 \longrightarrow a_2)$ of parallel arrows in $\mathcal{C}$

  (arrows) $(h_1, h_2)\colon f \longrightarrow g$ provided $$\begin{tikzcd} a_1 \arrow[r, "h_1"] \arrow[d, "f_1"] & b_1 \arrow[d, "g_1"] \\ a_2 \arrow[r, "h_2"] & b_2 \end{tikzcd}$$ and $$\begin{tikzcd} a_1 \arrow[r, "h_1"] \arrow[d, "f_2"] & b_1 \arrow[d, "g_2"] \\ a_2 \arrow[r, "h_2"] & b_2 \end{tikzcd}$$

Reformulations and Examples

Any universal property (for $\mathcal{C}$) introduced so far can be recast in terms of universal arrows to/from a functor $F\colon\mathcal{C} \longrightarrow \mathcal{D}$ by a suitable choice of $\mathcal{D}$ and F.

- an exponential diagram $ev\colon c \times a \longrightarrow b$ corresponds to a universal arrow $\langle c, ev\rangle$ from $- \times a\colon\mathcal{C} \longrightarrow \mathcal{C}$ to b
- a subobject classifier $t \in$ **Sub**(Ω) corresponds to a universal arrow from $1 \in$ **Set** to **Sub**$\colon\mathcal{C}^{op} \longrightarrow$ **Set**, where **Sub**(a) is the *set* of subobjects of a in $\mathcal{C}$ and

$$\mathbf{Sub}(f\colon b \longrightarrow a)[m\colon a' \rightarrowtail a] \triangleq [m'] \text{ with } \begin{tikzcd} b \arrow[r, "f"] & a \\ b' \arrow[u, tail, "m'"] \arrow[r] & a' \arrow[u, tail, "m"'] \end{tikzcd} \text{ is a pullback}$$

When $F\colon\mathcal{C} \longrightarrow \mathcal{D}$ is a monotonic maps between preorders, a universal arrow from d to F amounts to *the least* c s.t. $d \leq Fc$.

Reformulations and Examples

Any universal property (for $\mathcal{C}$) introduced so far can be recast in terms of universal arrows to/from a functor $F\colon\mathcal{C} \longrightarrow \mathcal{D}$ by a suitable choice of $\mathcal{D}$ and F.

More Examples

We have given several examples of functors, do the universal arrows to/from these functor exists? For instance, consider the forgetful functors $U\colon\mathcal{C} \longrightarrow$ **Set**

$\mathcal{C}$	a_X s.t. $u\colon X \longrightarrow U(a_X)$ univ.	b_X s.t. $u\colon U(b_X) \longrightarrow X$ univ.
Mon	$a_X =$ free monoid X^* on X	when $\lvert X\rvert = 1$: $b_X = 1$
Grp	$a_X =$ free group on X	when $\lvert X\rvert = 1$: $b_X = 1$
Top	$a_X = (X, \mathcal{P}(X))$ discrete top. on X	$b_X = (X, \{\emptyset, X\})$ chaotic top. on X
PO	$a_X = (X, =)$ discrete p.o. on X	when $\lvert X\rvert \leq 1$: $b_X = (X, =)$
Alg$_\Omega$	$a_X =$ free Ω-algebra $T_\Omega(X)$ on X	NO unless Ω trivial
$\underline{A}$**-Set**	when $\lvert X\rvert < \aleph_0$: $a_X = \coprod_{x\in X} 1$	$b_X = (X, A \times X)$ uniform rel. on X
EN	when $0 < \lvert X\rvert < \aleph_0$: $a_X = \coprod_{x\in X} 1$	when $\lvert X\rvert = 1$: $b_X = 1$

Reformulations and Examples

Any universal property (for $\mathcal{C}$) introduced so far can be recast in terms of universal arrows to/from a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ by a suitable choice of $\mathcal{D}$ and F.

More Examples

We have given several examples of functors, do the universal arrows to/from these functor exists? For instance, consider the following inclusion functors

- In: **Set** $\longrightarrow$ **Rel**, for each $Y \in$ **Rel** a universal arrow from In to Y is $\langle \mathcal{P}(Y), R_Y \rangle$ where $R_Y \subseteq \mathcal{P}(Y) \times Y$ s.t. $R_Y(Y', y) \Longleftrightarrow y \in Y'$
- In: **Set** $\longrightarrow$ **pSet**, for each $Y \in$ **pSet** a universal arrow from In to Y is $\langle Y + 1, p_Y \rangle$ where $p_Y : Y + \{\perp\} \longrightarrow Y$ s.t. $p_Y(y) = y$ and $p_Y(\perp)$ undefined
- In: **pSet** $\longrightarrow$ **Rel**, for each $Y \in$ **Rel** a universal arrow from In to Y is $\langle \mathcal{P}(Y) - \{\emptyset\}, R'_Y \rangle$ where $R'_Y \subseteq (\mathcal{P}(Y) - \{\emptyset\}) \times Y$ s.t. $R_Y(Y', y) \Longleftrightarrow y \in Y'$

Adjunctions

An *adjunction* $\langle F, G, \phi \rangle$ from $\mathcal{C}$ to $\mathcal{D}$ consists of

- two functors $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{C}$, called $\boxed{\text{left and right adjoint}}$
- a natural isomorphism $\mathcal{C}^{op} \times \mathcal{D} \underset{\mathcal{C}[-, G-]}{\overset{\mathcal{D}[F-, -]}{\underset{\Downarrow \phi}{\longrightarrow}{\longrightarrow}}}$ **Set** (this requires $\mathcal{C}$ and $\mathcal{D}$ to be locally small)

Notation $\boxed{\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left, "G"] \end{tikzcd} \text{ or } F \dashv G \text{ or } G \vdash F}$ for F left adjoint to G (G right adjoint to F)

Prop. An adjunction $\langle F, G, \phi \rangle : \mathcal{C} \longrightarrow \mathcal{D}$ induces two natural transformations

(unit) $\eta : \mathrm{id}_{\mathcal{C}} \longrightarrow GF$ $\boxed{\eta_c \stackrel{\Delta}{=} \phi_{c,Fc}(\mathrm{id}_{Fc})}$ s.t. $\boxed{\langle \eta_c, Fc \rangle \text{ universal from } c \text{ to } G}$

(counit) $\epsilon : FG \longrightarrow \mathrm{id}_{\mathcal{D}}$ $\boxed{\epsilon_d \stackrel{\Delta}{=} \phi^{-1}_{Gd,d}(\mathrm{id}_{Gd})}$ s.t. $\boxed{\langle Gd, \epsilon_d \rangle \text{ universal from } F \text{ to } d}$

moreover $(G\epsilon) \circ (\eta G) = \mathrm{id}_G$ and $(\epsilon F) \circ (F\eta) = \mathrm{id}_F$

Adjunctions

Prop. Given $G{:}\,\mathcal{D} \longrightarrow \mathcal{C}$ and $\boxed{\text{for each } c \in \mathcal{C} \text{ a } \langle u_c, d_c\rangle \text{ universal from } c \text{ to } G}$, exists a unique adjunction $\langle F, G, \phi\rangle$ s.t. $d_c = Fc$ and $u_c = \eta_c$. F and ϕ are given by

- $Fc \triangleq d_c$ and $F(f{:}\,a \longrightarrow b) \triangleq$ the unique $f'{:}\,d_a \longrightarrow d_b$ s.t.
$$\begin{tikzcd} a \arrow[r, "u_a"] \arrow[d, "f"'] \arrow[dr, phantom, "\text{comm.}"] & Gd_a \arrow[d, dashed, "Gf'"] \\ b \arrow[r, "u_b"] & Gd_b \end{tikzcd}$$
- $\phi_{c,d}{:}\,\mathcal{D}[d_c, d] \longrightarrow \mathcal{C}[c, Gd]$ is the mapping $g \longmapsto (Gg \circ u_c)$

(functoriality of F, naturality and bijectivity of ϕ follow from the properties of universal arrows)

Dual. Given $F{:}\,\mathcal{C} \longrightarrow \mathcal{D}$ and $\boxed{\text{for each } d \in \mathcal{D} \text{ a } \langle c_d, u_d\rangle \text{ universal from } F \text{ to } d}$, exists a unique adjunction $\langle F, G, \phi\rangle$ s.t. $c_d = Gd$ and $u_d = \epsilon_d$.

Corr. If both F_1 and F_2 are left (right) adjoint to G, then they are naturally isomorphic.

Follows from the fact that universal arrows from c to G (from G to c) are determined up to unique iso.

Prop. Given $F{:}\,\mathcal{C} \longrightarrow \mathcal{D}$ equivalence, exists $\langle F, G, \phi\rangle$ adjunction s.t. η and ϵ are isos.

One has to make use of choice, in order to pick c_d and i_d s.t. $i_d{:}\,F(c_d) \longrightarrow d$ iso.

Logic and Adjunctions

Dogma (Lawvere): every logical constant corresponds to an adjunction.

- when $\mathcal{C}$ and $\mathcal{D}$ are preorders, adjunctions $\langle F, G, \phi\rangle$ amounts to *Galois connections*, i.e. pairs of monotonic maps $\langle F, G\rangle$ s.t. $\boxed{\forall c \in \mathcal{C},\ d \in \mathcal{D}.\ Fc \leq d \iff c \leq Gd}$ Moreover, $Gd = \vee\{c|Fc \leq d\}$ and $Fc = \wedge\{d|c \leq Gd\}$
- **Ent**$[\tau]$ preorder of *syntactic formulas* $A(x)$ ordered by entailment $x\colon t \vdash A \implies B$
- $P[X]$ partial order $(\mathcal{P}(X), \subseteq)$ of *semantic predicates* over X ordered by inclusion

All logical structure on $P[X]$ (the same holds for **Ent**$[\tau]$) is induced by adjuntions:

- $\bot \dashv ! \dashv \top$ where $!\colon P[X] \longrightarrow 1$ is the unique functor into the one object category 1
- $\vee \dashv \Delta \dashv \wedge$ where $\Delta\colon P[X] \longrightarrow P[X] \times P[X]$ is the diagonal functor
- $- \wedge a \dashv a \supset -$ (in general there is no L_a s.t. $L_a \dashv - \wedge a$)
- $\exists_f \dashv Pf \dashv \forall_f$ for $X \xleftarrow{f} Y$, with $Pf\colon P[X] \longrightarrow P[Y]$ s.t. $Pf(X') \triangleq \{y|f(y) \in X'\}$ $\exists_f(Y') \triangleq \{x|\exists y.f(y) = x \wedge y \in Y'\}$ and $\forall_f(Y') \triangleq \{x|\forall y.f(y) = x \supset y \in Y'\}$

From $\exists_f$ and $\forall_f$ one can define the usual quantifiers $\exists_Y, \forall_Y\colon P[X \times Y] \longrightarrow P[X]$ and $=_X \in P[X \times X]$

Logic and Adjunctions

Dogma (Lawvere): every logical constant corresponds to an adjunction.

Bi-rules for entailment inspired by adjunctions

- $\Gamma, \bot \vdash_X B$ $\quad$ $\Gamma, A \vdash_X \top$ $\qquad \begin{array}{c} \Gamma, A_1 \vdash_X B \\ \Gamma, A_2 \vdash_X B \\ \hline\hline \Gamma, A_1 \vee A_2 \vdash_X B \end{array} \qquad \begin{array}{c} \Gamma, A \vdash_X B_1 \\ \Gamma, A \vdash_X B_2 \\ \hline\hline \Gamma, A \vdash_X B_1 \wedge B_2 \end{array}$

What does Γ, A means? $\boxed{\Gamma \wedge A \text{ in intuitionistic logic, } \Gamma \otimes A^{a} \text{ in linear logic}}$

- $\top' \begin{array}{c} \Gamma \vdash_X B \\ \hline\hline \Gamma, \top \vdash_X B \end{array} \qquad \wedge' \begin{array}{c} \Gamma, A_1, A_2 \vdash_X B \\ \hline\hline \Gamma, A_1 \wedge A_2 \vdash_X B \end{array} \qquad \supset' \begin{array}{c} \Gamma, C, A \vdash_X B \\ \hline\hline \Gamma, C \vdash_X A \supset B \end{array}$
- $\begin{array}{c} \Gamma, A \vdash_{X,x} B \\ \hline\hline \Gamma, \exists x.A \vdash_X B \end{array} x \notin \Gamma, B \qquad \begin{array}{c} \Gamma, A \vdash_{X,x} B \\ \hline\hline \Gamma, A \vdash_X \forall x.B \end{array} x \notin \Gamma, A \qquad =' \begin{array}{c} [x/y]\Gamma \vdash_{X,x} [x/y]B \\ \hline\hline \Gamma, x = y \vdash_{X,x,y} B \end{array}$

Note. P hyperdocrine and the adjunctions are *indexed*, i.e. *commute* with re-indexing.
Correspondingly logical constants *commute* with substitution, e.g. $[M/y](\exists x.A) \equiv \exists x.[M/y]A$ ($x \notin M$).

---

[a] $\otimes$ bifunctor and I object with natural isos $I \otimes a \cong a$, $a \otimes (b \otimes c) \cong (a \otimes b) \otimes c$ and $a \otimes b \cong b \otimes a$.

Addendum: Monads and Adjunctions

Every adjunction $\langle F, G, \phi\rangle : \mathcal{C} \longrightarrow \mathcal{D}$ induces a *monad*[a] on $\mathcal{C}$, i.e. a triple (T, η, μ) where

- $T : \mathcal{C} \longrightarrow \mathcal{C}$ $\boxed{Tc \triangleq G \circ F}$
- $\eta : \mathrm{id}_{\mathcal{C}} \longrightarrow T$ and $\mu : T^2 \longrightarrow T$ $\boxed{\eta \text{ unit and } \mu \triangleq G\epsilon F \text{ with } \epsilon \text{ counit}}$ s.t.

$$\begin{tikzcd}
T^3 \arrow[r, "\mu T"] \arrow[d, "T\mu"] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[r, "\mu"] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, "\eta T"] \arrow[dr, "\mathrm{id}_T"'] & T^2 \arrow[d, "\mu"] & T \arrow[l, "T\eta"'] \arrow[dl, "\mathrm{id}_T"] \\
& T &
\end{tikzcd}$$

commute

Every monad T on $\mathcal{C}$ is induced by an adjunction, there are two *canonical* choices

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend right, "In"'] \arrow[r, phantom, "\top"] & \mathcal{C}_T \arrow[l, bend right, dashed]
\end{tikzcd}$$

Kleisli

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend right, dashed] \arrow[r, phantom, "\top"] & \mathcal{C}^T \arrow[l, bend right, "U"']
\end{tikzcd}$$

Eilenberg-Moore

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[a] There is a dual notion of *comonad* on $\mathcal{D}$. Different adjunctions may induce the same monad/comonad.

Addendum: Monads and Adjunctions

The Kleisli construction $$\begin{tikzcd} \mathcal{C} \arrow[r, bend right, "In"'] \arrow[r, phantom, "\top"] & \mathcal{C}_T \arrow[l, dashed, bend right, "G"'] \end{tikzcd}$$

unless stated otherwise $a \xrightarrow{f} b$ means $f \in \mathcal{C}[a,b]$ and id_a and $g \circ f$ are identity and composition in $\mathcal{C}$

- $\mathcal{C}_T$ has the same objects of $\mathcal{C}$, $\mathcal{C}_T[a,b] \triangleq \mathcal{C}[a,Tb]$, the identity on a is η_a and the composite of $f \in \mathcal{C}_T[a,b]$ and $g \in \mathcal{C}_T[b,c]$ is $g^* \circ f$, where $g^* \triangleq \mu_c \circ Tg$
- $In(a) = a$ and $In(f\colon a \longrightarrow b) = \eta_b \circ f$
- G (right adjoint to In) is $G(a) = Ta$ and $G(f\colon a \longrightarrow Tb) = f^*$.

Prop [Manes76]. There is a bijection between monads and *Kleisli triples* $(T, \eta, -^*)$, i.e.

- an operation $T\colon \mathcal{C}_0 \longrightarrow \mathcal{C}_0$, a family $\eta_a \in \mathcal{C}[a,Ta]$ of arrows and a family $-^*\colon \mathcal{C}[a,Tb] \longrightarrow \mathcal{C}[Ta,Tb]$ of operations s.t.
- $\boxed{f^* \circ \eta_a = f \ ,\ \eta_a^* = \mathrm{id}_{Ta} \ ,\ (g^* \circ f)^* = g^* \circ f^*}$ where $f\colon a \longrightarrow Tb$ and $g\colon b \longrightarrow Tc$.

A Kleisli triple induces a monad, in particular $Tf \triangleq (\eta_b \circ f)^*$ and $\mu_a \triangleq \mathrm{id}^*_{Ta}$

Conversely a monad induces a Kleisli triple, in particular $f^* \triangleq \mu_b \circ Tf$.

Addendum: Monads and Adjunctions

The Eilenberg-Moore $$\begin{tikzcd} \mathcal{C} \arrow[r, dashed, bend right, "F"'] \arrow[r, phantom, "\top"] & \mathcal{C}^T \arrow[l, bend right, "U"'] \end{tikzcd}$$ construction

- $\mathcal{C}^T$ is the category whose objects are T-*algebras* (a, α), i.e. $\alpha\colon Ta \longrightarrow a$ s.t.

  $$\begin{tikzcd} T^2a \arrow[r, "\mu_a"] \arrow[d, "T\alpha"] & Ta \arrow[d, "\alpha"] \\ Ta \arrow[r, "\alpha"] & a \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "\eta_a"] \arrow[dr, "\mathrm{id}_a"'] & Ta \arrow[d, "\alpha"] \\ & a \end{tikzcd}$$

  and $(a, \alpha) \xrightarrow{f} (b, \beta) \overset{\Delta}{\iff}$

  $$\begin{tikzcd} Ta \arrow[r, "Tf"] \arrow[d, "\alpha"] & Tb \arrow[d, "\beta"] \\ a \arrow[r, "f"] & b \end{tikzcd}$$

  identities and composition are *inherited* from $\mathcal{C}$, e.g. $\mathrm{id}_{(a,\alpha)} = \mathrm{id}_a$
- $U(a, \alpha) = a$ and $U(f) = f$
- F (left adjoint to U) is $Fa = (Ta, \mu_a)$ and $Ff = Tf$

Part 5 - [AspertiLongo91, Ch 6]

Cones and Limits

Given a diagram D in $\mathcal{C}$ (i.e. a morphism from a graph $\mathcal{G}$ to the underlying graph of $\mathcal{C}$)

- a *cone* to D consists of an object $c \in \mathcal{C}$ and a family $f_i \in \mathcal{C}[c, D(i)]$ of arrows indexed by nodes $i \in \mathcal{G}$ s.t. $\boxed{D(e) \circ f_i = f_j \text{ for any arc } i \xrightarrow{e} j \text{ in } \mathcal{G}}$
- **Cones**(D) is the category whose objects are cones $(a, \langle f_i | i \rangle)$ and whose arrows $h \colon (a, \langle f_i | i \rangle) \longrightarrow (b, \langle g_i | i \rangle)$ are $h \in \mathcal{C}[a, b]$ s.t. $\boxed{g_i = f_i \circ h \text{ for any node } i \in \mathcal{G}}$ identities and composition are *inherited* from $\mathcal{C}$
- a *limit* for D is a terminal object in **Cones**(D).
- Dual notions: cocone $(f_i \in \mathcal{C}[D(i), c])$, **Cocones**$(D)$, colimit (initial in **Cocones**(D)).

I-indexed products, equalizers and pullbacks are instances of limits:

(products) are limits for diagrams whose shape is a *discrete graph* (i.e. without arcs)

(equalizers) are limits for diagrams of shape $\cdot \rightrightarrows \cdot$

(pullbacks) are limits for diagrams of shape $\cdot \rightarrow \cdot \leftarrow \cdot$

Dogma 4: a diagram D in $\mathcal{C}$ can be seen as a system on constraints, and then a limit for D represents all possible solutions of the system.

Cones and Limits

- $\mathcal{C}$ is *$\mathcal{G}$-complete* $\overset{\Delta}{\iff}$ every diagram D of shape $\mathcal{G}$ in $\mathcal{C}$ has a limit
- $\mathcal{C}$ is *complete* $\overset{\Delta}{\iff}$ every *small*[a] diagram D in $\mathcal{C}$ has a limit
- $\mathcal{C}$ is *finitely complete* $\overset{\Delta}{\iff}$ every *finite* diagram D in $\mathcal{C}$ has a limit

Given a graph $\mathcal{G}$ and a category $\mathcal{C}$, the category **Diagram**$(\mathcal{G},\mathcal{C})$ consists of

(objects) diagrams D of shape $\mathcal{G}$ in $\mathcal{C}$

(arrows) $\langle f_i | i \in \mathcal{G}\rangle : D_1 \longrightarrow D_2 \overset{\Delta}{\iff}$

$$\begin{tikzcd}
D_1(i) \arrow[r, "f_i"] \arrow[d, "D_1(e)"'] \arrow[dr, phantom, "\text{comm.}"] & D_2(i) \arrow[d, "D_2(e)"] \\
D_1(j) \arrow[r, "f_j"] & D_2(j)
\end{tikzcd}$$

for any arc $e: i \to j$ in $\mathcal{G}$

Let $\Delta: \mathcal{C} \longrightarrow$ **Diagram**$(\mathcal{G},\mathcal{C})$ be s.t. $\Delta(c)(i) = c$, $\Delta(c)(e) = \mathrm{id}_c$ and $\Delta(f: a \to b)_i = f$.

Prop. A limit for a diagram D of shape $\mathcal{G}$ amounts to a universal arrow from Δ to D.

---

[a] The shape of D is a small graph, i.e. the collections of nodes and arcs are sets.

Existence of Limits

Thm. Given a graph $\mathcal{G}$, if $\mathcal{C}$ has all $\mathcal{G}_0$-indexed and $\mathcal{G}_1$-indexed products and equalizers for any pair of parallel arrows, then any diagram D of shape $\mathcal{G}$ in $\mathcal{C}$ has a limit.

- Let $c_0 \triangleq \prod_i D(i)$, $c_1 \triangleq \prod_{e:i\to j} D(j)$ and $g_1, g_2 \colon c_0 \longrightarrow c_1$ be the unique arrows s.t.

$$\begin{tikzcd}
D(i) \arrow[d, "D(e)"] & c_0 \arrow[l, "\pi_i"'] \arrow[d, "g_1"] & c_0 \arrow[d, "g_2"] \arrow[dr, "\pi_j"] & \\
D(j) & c_1 \arrow[l, "\pi_e"] & c_1 \arrow[r, "\pi_e"'] & D(j)
\end{tikzcd}$$

commute for any $e \colon i \to j$ in $\mathcal{G}$

- let $l \colon c \rightarrowtail c_0$ be an equalizer of $g_1, g_2 \colon c_0 \longrightarrow c_1$ and $l_i \triangleq \pi_i \circ l \colon c \longrightarrow D(i)$
- then $(c, \langle l_i | i \rangle)$ is a limit for D.

In fact, given $(a, \langle f_i | i \rangle$ cone to D the arrow $f \triangleq \langle f_i | i \rangle \colon a \longrightarrow c_0$ is s.t. $g_1 \circ f = g_2 \circ f$, thus $\exists! f' \colon a \longrightarrow c$ s.t. $f = l \circ f'$ (or equivalently $f_i = l_i \circ f'$ for any $i \in \mathcal{G}$).

Preservation and Creation of Limits

Given a functor $F\colon\mathcal{A} \longrightarrow \mathcal{B}$

- if $D\colon\mathcal{G} \longrightarrow \mathcal{A}$ is a diagram in $\mathcal{A}$, then $F \circ D$ is a diagram in $\mathcal{B}$ (of the same shape)
- if $(a, \langle f_i|i\rangle)$ is a cone for D, then $(Fa, \langle Ff_i|i\rangle)$ is a cone for $F \circ D$
- F *preserves* limits for $D \overset{\Delta}{\Longleftrightarrow} (Fa, \langle Ff_i|i\rangle)$ limit for $F \circ D$ when $(a, \langle f_i|i\rangle)$ limit for D
- F *creates* limits for $D \overset{\Delta}{\Longleftrightarrow} (b, \langle g_i|i\rangle)$ limit for $F \circ D$ implies
  - $\exists!(a, \langle f_i|i\rangle)$ cone for D s.t. $b = Fa$ and $\forall i.g_i = Ff_i$
  - moreover this unique cone is a limit for D

Thm. If F has a left adjoint, then F preserves limits for any diagram D in $\mathcal{A}$.

Corr. Given an object a in a CCC, $-^a$ preserves limits and $- \times a$ preserves colimits.

Thm.[a] If the category $\mathcal{A}$ is locally small and complete, then F has a left adjoint $\Longleftrightarrow$

- F preserves limits and satisfies the *solution set condition*
- $\forall b \in \mathcal{B}.\exists\langle g_i\colon b \to Fa_i|i \in I_b\rangle$ small family of arrows s.t.
  $\forall b \xrightarrow{g} Ga.\exists i \in I_b.g = (Ff) \circ g_i$ for some $a_i \xrightarrow{f} a$.

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[a] It is called Adjoint Functor Theorem and is due to Peter Freyd.

Preservation and Creation of Limits

Given a functor $F\colon \mathcal{A} \longrightarrow \mathcal{B}$

- if $D\colon \mathcal{G} \longrightarrow \mathcal{A}$ is a diagram in $\mathcal{A}$, then $F \circ D$ is a diagram in $\mathcal{B}$ (of the same shape)
- if $(a, \langle f_i | i \rangle)$ is a cone for D, then $(Fa, \langle F f_i | i \rangle)$ is a cone for $F \circ D$
- F *preserves* limits for $D \overset{\Delta}{\Longleftrightarrow} (Fa, \langle F f_i | i \rangle)$ limit for $F \circ D$ when $(a, \langle f_i | i \rangle)$ limit for D
- F *creates* limits for $D \overset{\Delta}{\Longleftrightarrow} (b, \langle g_i | i \rangle)$ limit for $F \circ D$ implies
  - $\exists!(a, \langle f_i | i \rangle)$ cone for D s.t. $b = Fa$ and $\forall i. g_i = F f_i$
  - moreover this unique cone is a limit for D

Thm. If T is a monad on $\mathcal{C}$, then $U\colon \mathcal{C}^T \longrightarrow \mathcal{C}$ creates limits for any diagram D in $\mathcal{C}^T$.

The category **Top** is complete and cocomplete. The forgetful functor $U\colon$ **Top** $\longrightarrow$ **Set** preserves limits and colimits. However, U **does not** create limits (nor colimits).

Consider a diagram of shape $\cdot$ in **Top**, i.e. a topological space $\underline{X} = (X, \tau)$.

A limit cone for $U(\underline{X})$ in **Set** is $X \xrightarrow{\mathrm{id}_X} X$, but there are several cones $\underline{Y} \xrightarrow{f} \underline{X}$ in **Top** s.t. $Uf = \mathrm{id}_X$,

i.e. take $\underline{Y} = (X, \tau')$ with $\tau' \supseteq \tau$ (and $f = \mathrm{id}_X$). Only when taking $\underline{Y} = \underline{X}$ one has a limit cone in **Top**.

Part 6 - Informal concepts (defined by examples) vs Mathematical notions
Possibility Modalities vs Monads
Collection Types vs Monads

Modalities and Monads

We consider modalities only in propositional logic[a]:

- W set of possible worlds
- $P \triangleq (\mathcal{P}(W), \subseteq)$ complete boolean algebra of propositions

An *accessibility relation* $R \subseteq W \times W$, induces two operators $\Diamond_R, \Box_R \colon P \longrightarrow P$

- $\Diamond_R(p) \triangleq \{u \in W | \exists v.uRv \wedge v \in p\}$ possibility modality
- $\Box_R(p) \triangleq \{u \in W | \forall v.uRv \supset v \in p\}$ necessity modality

Properties of $\Diamond_R$ ($\Box_R$ satisfies *dual* properties)

- monotonicity (also called functoriality): $\dfrac{A \implies B}{\Diamond_R A \implies \Diamond_R B}$
- sup-preservation: $\bigvee_{i \in I} \Diamond(p_i) \iff \Diamond_R(\bigvee_{i \in I} p_i)$ ($\implies$ follows from functoriality)

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[a] For more general settings see [ReyesZolfaghari91], [GhilardiMeloni88].

Modalities and Monads

We consider modalities only in propositional logic[a]:

- W set of possible worlds
- $P \triangleq (\mathcal{P}(W), \subseteq)$ complete boolean algebra of propositions

Given a monotonic map (functor) $F \colon P \longrightarrow P$

- $\overline{F}(p) \triangleq \bigwedge\{q | p \leq q \wedge F(q) \leq q\}$ is the smallest *closure* (monad) generated by F, i.e. $p \leq \overline{F}(p) = \overline{F}^2(p)$ and $F(p) \leq \overline{F}(p)$
- if F preserves countable sups, then $\overline{F}(p) = \bigvee\{F^n(p) | n \in N\}$
- if $F = \Diamond_R$, then $\overline{F} = \Diamond_{R^*}$ (R^* is the reflexive and transitive closure of R)

Concluding remarks

- some possibility modalities $\Diamond_R$ are not closures (monads).
- possibility modalities $\Diamond_R$ satisfied properties not valid for arbitrary closures.

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[a] For more general settings see [ReyesZolfaghari91], [GhilardiMeloni88].

Collection Types and Monads [Manes98]

Collection types in the setting of database languages [Buneman&al][a]

- $M\tau$ type of *collections* c s.t. the *elements* of c have type τ

Comprehension notation - M has (at least) the structure of a strong monad

$$\frac{\Gamma, x_1\colon\tau_1, \ldots, x_{j-1}\colon\tau_{j-1} \vdash e_j\colon M\tau_j \quad 1 \le j \le n \qquad \Gamma, x_1\colon\tau_1, \ldots, x_n\colon\tau_n \vdash e\colon\tau}{\Gamma \vdash \{e|x_1 \leftarrow e_1, \ldots, x_n \leftarrow e_n\}\colon M\tau}$$

where $\boxed{x \leftarrow e \text{ generalizes } x \in e}$

- unit $\eta\colon\tau \longrightarrow M\tau$ is $\boxed{x\colon\tau \vdash \{x\}\colon M\tau}$
- if $f\colon\tau \longrightarrow M\tau'$, then $f^*\colon M\tau \longrightarrow M\tau'$ is $\boxed{c\colon M\tau \vdash \{x'|x \leftarrow c, x' \leftarrow f(x)\}\colon M\tau'}$
- *strength* $t\colon\tau \times M\tau' \longrightarrow M(\tau \times \tau')$ is $\boxed{x\colon\tau, c\colon M\tau' \vdash \{(x, x')|x' \leftarrow c\}\colon M(\tau \times \tau')}$

A collection $c \in M\tau$ should have a finitely many elements. There should be

- an empty collection $0\colon M\tau$ and
- the union $c_1 + c_2\colon M\tau$ of two collections $c_1, c_2\colon M\tau$

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[a] Share some features with computational types [Moggi, Wadler].

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Monads $T_{(\Omega,E)}$ on **Set** induced by *algebraic theories* are called *finitary monads*[b]

- (Ω, E) *algebraic theory* $\overset{\Delta}{\Longleftrightarrow}$ Ω algebraic signature and E set of Ω-equations
- **Alg**$_{(\Omega,E)}$ full subcategory of **Alg**$_{\Omega}$ of Ω-algebras satisfying the equations in E
- $U\colon$ **Alg**$_{\Omega} \longrightarrow$ **Set** has a left adjoint, T_{Ω} monad on **Set** induced by the adjunction
- also $U\colon$ **Alg**$_{(\Omega,E)} \longrightarrow$ **Set** has a left adjoint, the monad $T_{(\Omega,E)}$ monad on **Set** is s.t. $T_{(\Omega,E)}(X) = T_{\Omega}(X)/=_E$ with $=_E$ equivalence on $T_{\Omega}(X)$ induced by E.

[Manes98] characterizes *collection monads* in **Set** in terms of algebraic theories

- M collection monad $\Longleftrightarrow$ induced by a *balanced* algebraic theory (Σ, E), i.e. $\mathrm{FV}(M_1) = \mathrm{FV}(M_2)$ for any equation $M_1 = M_2 \in E$

Concluding remark: collection types correspond to a special class of strong monads.

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[a] Share some features with computational types [Moggi, Wadler].

[b] There is also a purely category-theoretic definition.

Alcuni Esercizi

Esercizio 1

Sia $(A,\cdot)$ una struttura applicativa parziale con due elementi I e B t.c.

- $I\ x = x$
- $B\ x\ y \downarrow$ e $B\ x\ y\ z \simeq x\ (y\ z)$

questo basta per avere una categoria $\underline{A}$**-Set**, e tale categoria ha oggetti iniziali e terminali ed equalizzatori.

Reminder

$\underline{A}$**-Set** is the category of sets with an $\underline{A}$-*realizability* relation

(objects) $\underline{X} = (X, \Vdash)$ with $\Vdash \subseteq A \times X$ *onto* $\boxed{\forall x \in X . \exists a . a \Vdash x}$

(arrows) $\underline{X}_1 \xrightarrow{f} \underline{X}_2 \overset{\Delta}{\iff} X_1 \xrightarrow{f} X_2$ has a *realizer* r $\boxed{a \Vdash_1 x \text{ implies } r\ a \Vdash_2 f(x)}$

Esercizio 2

Far vedere che se $(A,\cdot)$ e' una pCA, allora $\underline{A}$**-Set** ha prodotti e coprodotti binari (quindi ha prodotto e coprodotti finiti) e ha esponenziali. Sappiamo gia' che $\underline{A}$**-Set** ha oggetti iniziali e terminali ed equalizzatori. Si frutti la *completezza combinatoria* di una pCA, data dall'esistenza di un algoritmo di astrazione $[x]M$ t.c.

$$x \notin \mathrm{FV}([x]M) \qquad ([x]M)\downarrow \qquad ([x]M)x \simeq M$$

per far vedere che esistono combinatori per codificare coppie ed inclusioni disgiunte.

Sottocategorie piene di $\underline{A}$-Set

- $\underline{X} = (X, \Vdash)$ *effective* $\overset{\Delta}{\Longleftrightarrow}$ $\forall a, x, x'$. if $a \Vdash x$ and $a \Vdash x'$, then $x = x'$
  cioe' un a realizza al piu' un x (quindi a identifica univocamente x)
  tali oggetti sono chiusi per prodotti, coprodotti, equalizzatori, e sono un *exponential ideal*, cioe' $\underline{Y}^{\underline{X}}$ effective when $\underline{Y}$ effective
- $\underline{X} = (X, \Vdash)$ *uniform* $\overset{\Delta}{\Longleftrightarrow}$ $\exists a$ s.t. $\forall x.\ a \Vdash x$
  cioe' esiste un a che realizza tutti gli x (quindi a non fornisce alcuna informazione)
  tali oggetti sono chiusi per prodotti (ma non per coprodotti), equalizzatori, e sono un *exponential ideal*

Esercizio 3

Usando un argomento diagonale si dimostra che non esiste una *funzione unviersale* $U \colon N \times N \longrightarrow N$ per le funzioni ricorsive totali. Questo fatto si usa per dimostrare che in **EN** non e' cartesianamente chiusa.
Si osservi che

- $e \in$ **EN**$[(\mathrm{id}_N, N), (e, X)]$, infatti e' realizzata da id_N
- i prodotti in **EN** si possono definire usando una codifica effettiva e *bigettiva* di $N \times N$ in N, cioe' codifica $c \colon N \times N \longrightarrow N$ e proiezioni $p_i \colon N \longrightarrow N$ sono ricorsive (totali)
- che un potenziale candidato per N^N in **EN** deve essere della forma (e, R) con R insieme delle funzioni ricorsive totali.

Errata Corrige (parziale) [AspertiLongo91]

- pag 8, Ex 1 falso con la definizione di Top data a lezione, il problema e' dato dagli spazi con la topologia caotica.

  correzione: give an epic which is not surjective in Top_0 (la sottocategoria piena di Top dei T_0-spazi), p.e. l'inclusione dello spazio dei razionali in quello dei reali e' epic poiche' e' densa.

- pag 8, Ex 3 falso.

  correzione: prove that an epic which is also a split monic is an iso.