Using a Theorem Prover for Reasoning on Constraint Problems

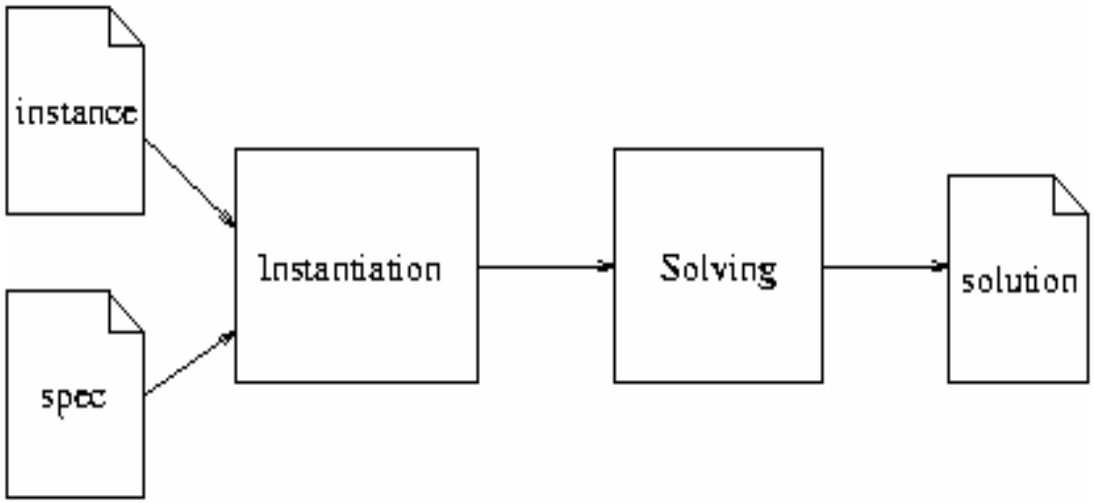
Marco Cadoli and Toni Mancini

Dipartimento di Informatica e Sistemistica
Università degli Studi di Roma “La Sapienza”
{cadoli|tmancini}@dis.uniroma1.it

Outline

- State-of-the-art systems for constraint modelling and programming
- Our goals
- Use of ATP technology for:
  - Detecting & breaking value symmetries
  - Detecting & handling functional dependencies
- Experiments
- Conclusions & future work

State-of-the-art systems architecture



- Clear separate problem spec and instances
- Advantages:
  - Decoupling of the specification from the solver
  - Focus on declarative aspects of specifications

Goals of our research

- Make a more complex reasoning on the spec, in order to:
  - Detect properties of the spec, interesting for the modeller (also for **verification** purposes)
  - Reformulate the spec to improve solver **efficiency**
- Reasoning: done on the problem model (spec), **independently on the instance**

- Undecidable in the general case

Note: spec level optimizations can be composed, after instantiation, with existing techniques

Why reasoning at the spec level

- **Verification:**
  - Problem specs can be viewed as **sw artefacts**. Reasoning helps the modeller during design
- **Optimization:**
  - Many properties amenable to optimizations derive from problem structure (hold for all instances).
  - Additional properties may arise from the instance
  - Current approaches try to infer *all* properties after instantiation (e.g., symmetries, dependencies)
  - But “structural” properties can be much less recognizable there, since problem structure has been destroyed!

Example: Reasoning at the spec level

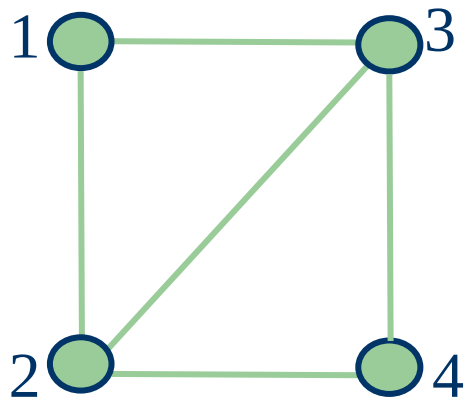
- **Symmetry detection**

  Graph 3-coloring (a problem):

  Is symmetric: color names can be exchanged

  A graph (an instance):

  May be symmetric: further symmetries



- **Functional dependencies**

  Blue nodes can be computed from red and green ones (as complement)

In this work

- Specifications viewed as logical formulae
- Main question:
  Is it possible to infer properties of specs by using Automated Theorem Proving technology?

- In this work we focus on:
  - Value symmetries
  - Functional dependencies

ESO specifications

In general, an ESO spec is a formula ψ: $\exists \mathbf{S}\ \phi(\mathbf{S},\mathbf{R})$ where:

- $\mathbf{S}$ is the set of ***guessed* predicates**
- $\mathbf{R}$ is the **input schema** (set of relations)
- $\phi(\mathbf{S},\mathbf{R})$ is a first-order formula on the relational vocabulary $\mathbf{S} \cup \mathbf{R} \cup \{=\}$

The **input instance** is given **as a *relational DB*** on the schema $\mathbf{R}$

Example: Graph 3-Col in ESO

- EDB schema: binary relation *edge*
- Herbrand domain: graph nodes

Choose 3 subsets of the graph nodes...

$$\exists RGB \quad \forall X \;\; R(X) \vee G(X) \vee B(X) \;\wedge$$

s.t. they are a covering...

...disjoint...

$$\forall X \;\; R(X) \rightarrow \neg G(X) \;\wedge$$
$$\forall X \;\; R(X) \rightarrow \neg B(X) \;\wedge$$
$$\forall X \;\; B(X) \rightarrow \neg G(X) \;\wedge$$

...and every edge links nodes with diff. colors

$$\forall XY \; X \neq Y \wedge R(X) \wedge R(Y) \rightarrow \neg edge(X,Y) \wedge$$
$$\forall XY \; X \neq Y \wedge G(X) \wedge G(Y) \rightarrow \neg edge(X,Y) \wedge$$
$$\forall XY \; X \neq Y \wedge B(X) \wedge B(Y) \rightarrow \neg edge(X,Y),$$

Detecting and breaking symmetries

Why it is useful?

- **Efficency:** It is well known that adding additional constraints that break symmetries can significantly reduce solving time

- **Verification:**
  - Does the model have symmetries? (Does it really have an expected symmetry?)
  - Does a given formula correctly break a symmetry? (correctness & completeness)

Why at the specification level?

- To help user during modelling
- Easier to detect “structural” symmetries

Detecting value symmetries [Cadoli & M., 2003]

Given ψ: $\exists \boldsymbol{S}\ \phi(\boldsymbol{S},\boldsymbol{R})$ *(Hyp: S_i all monadic)*:

Definition: Unif. value transformation (UVT)

A mapping σ: $\boldsymbol{S} \rightarrow \boldsymbol{S}$ which is total and onto (i.e. a permutation)

Definition: Unif. value symmetry (UVS)

A u.v. transformation σ s.t.:

$$\forall \boldsymbol{S}\ \forall \mathcal{I}\ \ \phi(\boldsymbol{S},\mathcal{I}) \ \leftrightarrow\ \phi^{\sigma}(\boldsymbol{S},\mathcal{I})$$

with ϕ^{σ} defined as

$$\phi[S_1/\sigma(S_1),\ldots,S_n/\sigma(S_n)]$$

Example: Graph 3-Col (cont.)

- Function σ s.t. $\sigma(R)=R$, $\sigma(G)=B$, $\sigma(B)=G$ is a *u.v. transformation* for the graph 3-Col spec

- Function σ is also a *u.v. symmetry* for it, since it is true that

$$\forall \boldsymbol{S}\ \forall \mathcal{I}\quad \phi(\boldsymbol{S},\mathcal{I})\ \leftrightarrow\ \phi^{\sigma}(\boldsymbol{S},\mathcal{I})$$

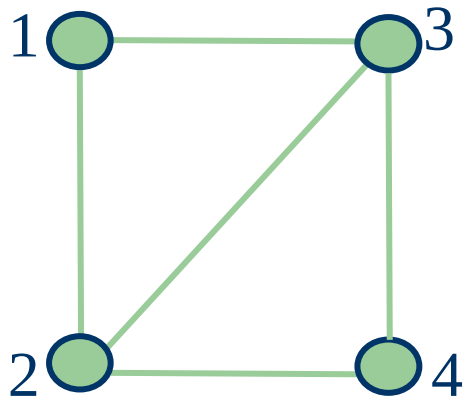
  (i.e., *for every* input instance, solutions are mapped into solutions and vice versa)

Why monadic guessed predicates?

- *Monadic predicates* ⇔ *domain values* in CSPs
- *Binary predicates* handled by unfolding

Example: CSP u.v. transformation

Graph 3-Coloring on:



- $\boldsymbol{V} = \{X_1, \ldots, X_4\}$
- $\boldsymbol{D} = \{D_1, \ldots, D_4\}$, $D_i = \{R,G,B\}$ for each i
- $\boldsymbol{C}$ = ...

The set of functions $\{\sigma_0, \sigma_1, \ldots, \sigma_4\}$ s.t.:

- $\sigma_0 = Id$,
- $\sigma_1(R) = \ldots = \sigma_4(R) = R$
- $\sigma_1(G) = \ldots = \sigma_4(G) = B$
- $\sigma_1(B) = \ldots = \sigma_4(B) = G$

is a u.v. transformation for this CSP (cf., e.g., [Meseguer&Torras, 1999])

Some results [Cadoli & M., 2003]

Given:

- ψ: $\exists \boldsymbol{S}\ \phi(\boldsymbol{S},\boldsymbol{R})$ (S_i *all monadic*) and
- u.v. transformation σ for ψ

Theorem: σ is a u.v. symmetry iff $\phi \equiv \phi^{\sigma}$

→ Can use a first-order theorem prover ☺

Theorem: Checking whether σ is a u.v. symmetry for ψ is ***undecidable*** ☹

Nonetheless, experimental analyses show that first-order theorem provers usually perform quite well in these cases

Experiments with ATP (OTTER)

Given:

- ψ: $\exists \boldsymbol{S}\ \phi(\boldsymbol{S},\boldsymbol{R})$ *(S_i all monadic)* and
- u.v. transformation σ for ψ

We checked with a first-order thm prover **(OTTER v. 3.3)**

whether: $\phi \equiv \phi^{\sigma}$

Specification	u.v.transf.	Is symm?	Time (s)
3-Coloring	R ↔ G	**Yes**	**0.27**
Not-All-Eq SAT	T ↔ F	**Yes**	**0.22**
Not-All-Eq 3-SAT	T ↔ F	**Yes**	**4.71**
Social golfer (unfolded)	Swap 2 groups in the same week	**Yes**	**0.96**

What about non-UVSs?

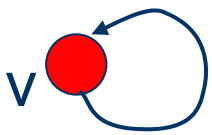
What if the given σ is not a UVS for ψ?

- Finding a **counter-example** suffices
- Can use a **finite model finder**

Example: 3-Coloring with red self-loops

$$\psi\colon \exists \boldsymbol{S}\ \phi_{3\text{-Col}}(\boldsymbol{S},\boldsymbol{R}) \wedge \forall \boldsymbol{X}\ \boldsymbol{edge(X,X) \rightarrow R(X)}$$

MACE v2.2 was able to show that σ: R ↔ G is not a UVS in **< 1 sec**, by finding the following counter-example:



edge	
From	To
v	v

R	G	B
v	–	–

Breaking symmetries [Cadoli & M., 2003]

Goal: Given symmetry σ, find $\beta(\boldsymbol{S})$ such that

$$\psi\text{: } \exists \boldsymbol{S}\ \phi(\boldsymbol{S},\boldsymbol{R})\ (S_i \textit{ all monadic})$$

can be safely transformed into:

$$\exists \boldsymbol{S}\ (\phi(\boldsymbol{S},\boldsymbol{R}) \wedge \beta(\boldsymbol{S}))$$

$\beta(\boldsymbol{S})$ is called **symmetry-breaking formula**

Breaking symmetries: SB formulae

Given ψ: $\exists \boldsymbol{S}\ \phi(\boldsymbol{S},\boldsymbol{R})$ (S_i *monadic*) and u.v. symmetry σ for ψ:

Definition: Symmetry-breaking formula f

Formula $\beta(\boldsymbol{S})$ s.t.:

1. σ is no longer a symmetry fo $\exists \boldsymbol{S}\ (\phi \wedge \beta(\boldsymbol{S}))$:

$$(\phi(\boldsymbol{S},\boldsymbol{R}) \wedge \beta(\boldsymbol{S})) \not\equiv (\phi(\boldsymbol{S},\boldsymbol{R}) \wedge \beta(\boldsymbol{S}))^{\sigma}$$

2. No solutions, except symmetric ones, are lost:

$$\phi(\boldsymbol{S},\boldsymbol{R}) \models \bigvee_{\boldsymbol{\sigma} \in \sigma^*} (\phi(\boldsymbol{S},\boldsymbol{R}) \wedge \beta(\boldsymbol{S}))^{\boldsymbol{\sigma}}$$

Can use a finite model finder!

Experiments: SB with ATP

Given:

- $\psi : \exists \boldsymbol{S}\ \phi(\boldsymbol{S},\boldsymbol{R})$ *(Si all monadic)*, UVS σ for ψ
- formula $\beta(\boldsymbol{S})$

Question: Is $\beta(\boldsymbol{S})$ a SB formula for ψ wrt σ?

Spec	Symm	Formula $\beta(\boldsymbol{S})$	SB?	CPU time (sec)	
				Cond 1 (MACE)	Cond 2 (OTTER)
3-Col	R ↔ G	$R(v_1) \vee B(v_1)$	**Yes**	**1.47**	**1.89**
3-Col	R ↔ G	$\forall w\ G(w) \rightarrow \exists v\ R(v) \wedge v \leq w$	**Yes**	**2.67**	**?**
SocGolf	swap	*as above*	**Yes**	**0.04**	**?**

2nd order SB formulae

Example: 3-Coloring, σ: R $\leftrightarrow$ G,

$$\beta(R,G,B) : |R| \leq |G|$$

(ESO formula, guesses a total inj. function T from R to G)

- Conditions 1 & 2 become 2nd order (non-)equivalences. For $\beta(\boldsymbol{S})$ of the form $\exists \boldsymbol{T}\ \gamma(\boldsymbol{R},\boldsymbol{G},\boldsymbol{B})$

$$1.\ \exists \boldsymbol{T}\ (\phi(\boldsymbol{S},\boldsymbol{R}) \wedge \gamma(\boldsymbol{S},\boldsymbol{T})) \not\equiv \exists \boldsymbol{T}\ (\phi(\boldsymbol{S},\boldsymbol{R}) \wedge \gamma(\boldsymbol{S},\boldsymbol{T}))^{\sigma};$$
$$2.\ \phi(\boldsymbol{S},\boldsymbol{R}) \models \bigvee_{\boldsymbol{\sigma} \in \sigma^*} (\exists \boldsymbol{T}\ \gamma(\boldsymbol{S},\boldsymbol{T}))^{\boldsymbol{\sigma}}.$$

- How to use ATP technology?
  - Use a 2nd order thm prover (under investigation)
  - Find 1st order conditions that imply 2nd order equiv

ESO SB formulae: Checking cond 1

Example: 3-Coloring, σ: R $\leftrightarrow$ G, $\beta(R,G,B) : |R| \leq |G|$

In ESO: guess total inj. funct. T from R to G:

$$\beta(R,G,B) : \exists T\ \gamma(R,G,B,T)$$

Finite model finder

Theorem prover

Cond. 1: $\exists T\ (\phi(S,R) \wedge \gamma(S,T)) \not\equiv \exists T\ (\phi(S,R) \wedge \gamma(S,T))^{\sigma}$

1. Find

$$\langle edge, \overline{R}, \overline{G}, \overline{B}, \overline{T} \rangle \models \phi(edge, \overline{R}, \overline{G}, \overline{B}) \wedge \gamma(\overline{R}, \overline{G}, \overline{B}, \overline{T}) \wedge \neg\gamma(\overline{R}, \overline{G}, \overline{B}, \overline{T})^{\sigma}$$

A graph plus a coloring s.t. $|\overline{R}| < |\overline{G}|$, e.g.: v

2. If $\gamma(\overline{R}, \overline{G}, \overline{B}, T)^{\sigma} \equiv \bot$

i.e., $|\overline{G}| \leq |\overline{R}|$
Contraddiction!

...then, this interpretation shows the non-equivalence. Overall computation: **< 0.3 sec**

Detecting functional dependencies among guessed predicates

Why it is useful?

- **Efficency:** A search strategy can be (automatically) synthesized in order to exploit such a dependence (e.g., by **avoiding branches on dependent predicates**)

- **Verification:**
  - **Dependencies by design:** Do they really hold?
  - **Unexpected dependencies:** Do they exist?
    - Do they evidence a **model bug**?
    - Do they evidence a **bad design choice** (e.g., redundancy)?

Why at the specification level?

- They are intrinsically “structural” properties!

Dependencies: existing approaches

Only at the instance level, e.g.:

- Detecting “defined” vars in SAT instances due to:
  - Clausification process
  - Equivalence clauses

  and using modified versions of the DP procedure

- SAT solvers that avoid branches on these vars, e.g., **EQSATZ**

Dependencies: our approach

Detect dependent guessed predicates at the spec level

Definition: Given spec $\psi : \exists \boldsymbol{SP}\ \phi(\boldsymbol{S},\boldsymbol{P},\boldsymbol{R})$,
set *P functionally depends* on *S* if, for each instance *I* and each pair of interpretations *M* and *N* of (***S***,***P***), s.t:

1. $M \neq N$,
2. $M,I \models \phi$, *and*
3. $N,I \models \phi$

then: $M_{|S} \neq N_{|S}$

i.e., no two distinct solutions to ϕ exist that coincide on extension for predicates in ***P***.

Dependencies (cont)

Theorem: Given spec $\psi : \exists SP\ \phi(S,P,R)$,

set *P* functionally depends on *S* iff the following formula is valid

$$[\phi(S,P,R) \wedge \phi(S',P',R) \wedge \neg(SP \equiv S'P')] \rightarrow \neg(S \equiv S')$$

→ Can use a first-order theorem prover ☺

Theorem: Checking whether set ***P*** functionally depends on ***S*** is **undecidable** ☹

Nonetheless, experimental analyses show that first-order theorem provers usually perform quite well in these cases

Experiments with ATP (OTTER)

Given ψ: $\exists SP\ \phi(S,P,R)$

We checked with a first-order thm prover whether

$$[\phi(S,P,R) \wedge \phi(S',P',R) \wedge \neg(SP \equiv S'P')] \rightarrow \neg(S \equiv S')$$

was a valid formula.

Specification	*S*	*P*	Time (s)
3-Coloring	R,G	B	**0.25**
NAE-3-SAT	T	F	**0.38**
Sailco	regBoat, extraBoat	inventory	**0.21**
Protein Folding	Move	X,Y,Hits	**569.55 (*)**

(*) Simplified version, with Hits omitted.

Automated synthesis of a search proc

Question: Is explicitly avoiding branches on dependent predicates effective in practice?

Experiments with Ilog OPL Studio:

1. **2D HP-Protein folding** on benchmark instances
2. **Blocks world** on random instances
   (results not shown, but similar to those of 2D HP-Protein folding)

2D HP-Protein folding with OPL

Length	Time no search proc	Time with search proc	Saving %
30	–	6	**99.9**
36	3078	13	**85.1**
37	3386	17	**93.6**
45	–	99	**>97.3**
48	–	156	**>95.7**
50	–	215	**>94.0**
60	–	128	**>96.4**
64	–	233	**>93.5**
102	1242	1007	**18.9**
123	914	913	**0.0**
136	761	760	**0.0**

Timeout = 1 hour

Conclusions

- We have proven that reasoning at the spec level is effective for:
  - Detecting and breaking “structural” symmetries
  - Detecting and handling functional dependencies in the problem model

  Positive consequences both for the modelling (verification) and solving stages (efficiency)

- Reasoning automatable by using ATP technology
- The wide availability of problem specs constitutes a **brand new set of benchmarks for ATP systems**

Current work

- Experiments with other thm provers and on diff. specs
- Investigate the applicability of the approach for **other forms of reasoning**, e
  - Symmetries on variables

For symmetry reasons, can be reduced to **"r1 < r2"**, i.e. **"ordered"**

Example: N-Queens

```
int+ N = ...; range Row      range Col
1..N;
var Col Queen[Row];
solve {
  forall (r1, r2 in Row : r1 <> r2) {
    Queen[r1] <> Queen[r2];
    Queen[r1]+r1 <> Queen[r2]+r2;
    Queen[r1]-r1 <> Queen[r2]-r2;
}};
```

Proved by OTTER in **< 1 sec**

Thank you! ☺

Questions?