Regularized deconvolution of multiple images of the same object

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In recent publications the problem of deconvolving multiple images of the same object was considered, and the fact that this problem can be well posed in the sense of distributions was stressed. Here we point out that the problem is still ill posed in spaces of square-integrable functions, and we discuss in what cases the use of multiple images rather than a single image can be advantageous.

Key words: single and multiple deconvolution, linear and nonlinear regularization. © 1996 Optical Society of America

1. INTRODUCTION

Images produced by a space-invariant imaging system can be described by the following mathematical model:

\[ g = K \ast f^{(0)} + w, \]  

where \( g \) is the blurred and noisy image that has been detected, \( K \) is the point-spread function (PSF) of the imaging system, \( w \) is a term that is due to the instrumental noise, and \( f^{(0)} \) is the perfect image that should be obtained in the absence of blurring and noise. Then the problem of image restoration consists in obtaining an estimate \( f \) of \( f^{(0)} \).

If the signal-to-noise ratio is sufficiently large, the noise term \( w \) can be neglected, and Eq. (1) becomes a convolution equation:

\[ g = K \ast f, \]  

whose solution is an ill-posed problem. This problem has been widely investigated, and many methods have been proposed for its stable and approximate solution.\(^1\)\(^-\)\(^3\)

In recent papers\(^4\)\(^,\)\(^5\) the case of multiple images of the same object was considered, and the practical relevance of some mathematical results, which apply to this case, was stressed. In such a case the mathematical model [Eq. (1)] is replaced by the following one:

\[ g_i = K_i \ast f^{(0)} + w_i, \quad i = 1, \ldots, m, \]  

and, if the noise terms \( w_i \) are neglected, Eq. (2) is replaced by the following system of convolution equations:

\[ g_i = K_i \ast f, \quad i = 1, \ldots, m. \]  

The problem is to understand how one can take advantage of the redundance of the data. We note that, from a practical point of view, Eq. (4) simply means that the images \( g_1, \ldots, g_m \) of the object \( f \) are obtained by the use of different imaging devices that are characterized by different PSF's \( K_1, \ldots, K_m \) and so, in particular, by different resolving powers.

In the papers mentioned above, only the case of compactly supported PSF was considered. Now, inasmuch as a compactly supported and integrable function defines a convolution operator that is continuous in \( D'(\mathbb{R}^N) \), which is the space of distributions (or generalized functions) on \( \mathbb{R}^N \), the multiple convolution operators \([D'(\mathbb{R}^N)]^m = D'(\mathbb{R}^N) \otimes \ldots \otimes D'(\mathbb{R}^N) \) (\( m \) times). Then the main result, already proved and discussed in other papers,\(^4\)\(^,\)\(^5\) is that, if the PSF \( K_i \) satisfy some suitable conditions, there exists a set of compactly supported distributions \( H_1, \ldots, H_m \) such that

\[ \sum_{i=1}^m K_i \ast H_i = \delta. \]  

This beautiful mathematical result implies that the operator defined by the multiple convolution operators [Eq. (4)] has a continuous inverse from \([D'(\mathbb{R}^N)]^m \) and \( D'(\mathbb{R}^N) \) and, therefore, that the problem in Eq. (4) is well posed in the sense of distributions. This continuity, however, is too weak for practical applications for which a stronger continuity is required, for instance, that of \( L^2(\mathbb{R}^N) \). In fact, the deconvolvers \( H_1, H_2, \ldots, H_m \) do not define a continuous operator in \( L^2(\mathbb{R}^N) \), and therefore one must also use regularization methods in the case of deconvolution of multiple images. A first step in this direction was already performed by the application of the Wiener-filter method\(^6\) to an example indicated in a previous paper.\(^4\)

In the present paper we continue this analysis by considering two regularization methods: the first is the well-known Tikhonov regularization method; the second is a constrained iterative algorithm. We apply these techniques to the example considered in the papers mentioned above.\(^4\)\(^,\)\(^6\) In particular, we compare the restorations obtained by the use of two images with the restorations obtained by the use of only one. Our result is that, although in the case of Tikhonov regularization two images provide better results than a single one, in the case of the iterative method with the additional constraints of positivity or of compactness of the solution's support, one or two images provide essentially the same results. More precisely, single deconvolution performed by the use of this constrained iterative algorithm gives reconstructions that are significantly more
accurate than the ones obtained by double deconvolution with the Tikhonov method.

An explanation of this fact is attempted, and an example in which the use of two images provides substantial improvement with respect to the case of only one, independently of the method adopted, is provided. This example does not correspond to that of compactly supported convolution kernels, but this is not a difficulty because we believe that the use of multiple images is a topic of great practical relevance beyond the cases for which precise mathematical results can be obtained.

In Section 2 we formulate the problem, and we recall the main results discussed by other authors. In Section 3 we outline the regularization methods that we consider in this paper. In Section 4 we discuss our numerical results in the case of the example considered in previous papers. Finally, in Section 5, we discuss a new example in which the use of two images provides a substantial improvement of the restorations.

2. FORMULATION OF THE PROBLEM

In this section we formulate problem (4) as the solution of the operator equation of the first kind and introduce the functional spaces that we shall consider. To each convolution kernel $K_i$ we can associate a linear operator $A_i$ defined by

$$A_if = K_i * f.$$  

If $K_i$ is compactly supported and integrable, then $A_i$ is a continuous operator in $L^2(\mathbb{R}^N)$ and is also a continuous operator in $L^2(\mathbb{R}^N)$ (for simplicity we do not use different notation for the two operators). However, we do not restrict our analysis to the case of compactly supported kernels. If this condition is not satisfied, then $A_i$, in general, is not a continuous operator in $D'(\mathbb{R}^N)$, whereas it is a continuous operator in $L^2(\mathbb{R}^N)$ if $K_i$ is integrable. A necessary and sufficient condition for the continuity of $A_i$ in $L^2(\mathbb{R}^N)$ is the boundedness almost everywhere (a.e.) of the Fourier transform $\hat{K}$ of the kernel.

Given $m$ kernels $K_1, \ldots, K_m$, let us assume that they satisfy conditions such that the corresponding operators $A_i$ are all continuous in the same space $X$ [$D'(\mathbb{R}^N)$ or $L^2(\mathbb{R}^N)$]. Then the set of $m$ operators defines a continuous operator $A$ from $X$ into $Y = X^m = X \oplus \ldots \oplus X$ (m times) as follows:

$$Af = \{A_1f, \ldots, A_mf\}.$$  

Inasmuch as the $m$ images of Eq. (4) define a multiple image $g = \{g_1, \ldots, g_m\} \in X^m$, Eq. (4) can be written in the synthetic form

$$g = Af.$$  

Note that the ranges of the operators $A_i$ are never closed subspaces of $L^2(\mathbb{R}^N)$ when $K_i$ is integrable and that therefore the range of $A$ is not a closed subspace of $[L^2(\mathbb{R}^N)]^m$. It follows that the inverse operator $A^{-1}$, when it exists, is not continuous. If $A$ does not exist, then it is possible to define a generalized inverse $A^\dagger$ of $A$, but $A^\dagger$ is also not continuous. In other words, problem (8) is ill posed in $L^2$ spaces when the kernels $K_i$ are integrable.

The situation is different when we consider Eq. (8) in a space of distributions, i.e., $X = D'(\mathbb{R}^N)$, $Y = [D'(\mathbb{R}^N)]^m$, and the kernels $K_i$ are compactly supported. Then, as follows from the results already proved and discussed, there may exist a set of compactly supported distributions $H_1, \ldots, H_m$ such that Eq. (5) holds true. By Fourier transforming both sides of this equation one obtains the so-called Bezout equation:

$$\sum_{i=1}^m \hat{K}_i(\omega) \hat{H}_i(\omega) = 1,$$  

which, thanks to an analytical continuation, must hold true for any $\omega \in \mathbb{C}^N$. Therefore the existence of the set of deconvolvers $H_1, \ldots, H_m$ is related to the existence of analytical solutions of the Bezout equation. These solutions may not exist for arbitrary kernels $K_i$, because, for instance, the functions $\hat{K}_i(\omega)$ have a common zero. A rigorous discussion of this problem has been the object of several papers, all based, in some way, on a theorem of Hörmander about the ring structure of the Paley–Wiener space. Here we merely remind the reader of the necessary and sufficient condition that must be satisfied by the kernels $K_i$ to ensure the existence of compactly supported deconvolvers $H_i$ satisfying Eq. (9): there exist positive constants $A$ and $B$ and a positive integer $N$ such that

$$\left[ \sum_{i=1}^m |\hat{K}_i(\omega)|^2 \right]^{1/2} \geq A \exp(-B |\operatorname{Im} \omega|/(1 + |\omega|)^N),$$  

$$\omega \in \mathbb{C}^N.$$  

Among other things, this condition excludes the case in which all the kernels $K_i$ are in $C_{\mathbb{C}}^\infty$. A set of convolvers satisfying condition (10) is referred to as strongly coprime.

In Section 4 we consider a well-known example of a strongly coprime set of convolvers. Because the explicit expressions of the corresponding deconvolvers contain derivatives of the delta function, they do not define continuous operators in $L^2$, as should be obvious from our analysis above.

3. REGULARIZED SOLUTIONS

As we remarked in Section 2, Eq. (8) is always ill posed when data and solution belong to $L^2$ spaces. Because this equation has the general form of a first-kind operator equation, the application of regularization methods to this problem is straightforward. For completeness, however, we give the main formula.

We do not restrict our analysis to the case of compactly supported kernels, but we require that the operator $A$ be continuous. If we introduce the function

$$\hat{K}(\omega) = \left[ \sum_{i=1}^m |\hat{K}_i(\omega)|^2 \right]^{1/2},$$  

then $A$ is continuous from $L^2(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)^m$ if and only if $\hat{K}(\omega)$ is bounded a.e., in which case

$$\|A\| = \sup_{\omega \in \mathbb{R}^N} |\hat{K}(\omega)|.$$  

It is easy to verify that the adjoint operator $A^*$ is given by

$$A^*g = \sum_{i=1}^m K_i^* * g_i,$$  

where $K^\dagger(x) = \overline{K(-x)}$ (the bar denotes complex conjugation). Finally, the generalized inverse $A^\dagger$ of $A$ is given by

$$
(A^\dagger g)(x) = \frac{1}{(2\pi)^N} \sum_{i=1}^{m} \int_{\mathbb{R}^N} H_i(\omega) \hat{g}_i(\omega) \exp(i\omega x) \, d\omega,
$$

where $B$ is the support of $\hat{K}(\omega)$ in $\mathbb{R}^N$ (band of the imaging system). This equation also defines the domain of the operator $A^\dagger$, i.e., the set of all the elements $g \in L^2(\mathbb{R}^N)$ such that $A^\dagger g \in L^2(S)$.

The necessary and sufficient condition is that all the functions

$$
\hat{g}_i(\omega) = \frac{\overline{K_i(\omega)}}{|\hat{K}(\omega)|^2} \hat{g}_i(\omega)
$$

be square integrable. If $B = \mathbb{R}^N$, then the inverse operator $A^{-1}$ exists, and it is again given by Eq. (14), with the integration now being extended to $\mathbb{R}^N$.

We note that the condition $B = \mathbb{R}^N$ is satisfied if the strongly coprime condition (10) is satisfied. In such a case, the inverse operator $A^{-1}$ can also be written in terms of the compactly supported deconvolvers $H_i$:

$$
(A^{-1}g)(x) = \frac{1}{(2\pi)^N} \sum_{i=1}^{m} \int_{\mathbb{R}^N} H_i(\omega) \hat{g}_i(\omega) \exp(i\omega x) \, d\omega.
$$

Obviously, Eqs. (14) and (16) must be equivalent because the inverse operator is uniquely defined. The reason for this ambiguity is that the solution of the Bezout equation is not unique. In fact, this equation has an infinity of analytical solutions (when the strongly coprime condition is satisfied) and also of nonanalytical solutions, such as $\hat{H}_i(\omega) = \overline{K_i(\omega)}|\hat{K}(\omega)|^2$, whose corresponding deconvolvers $H_i$ are not compactly supported (by the Paley–Wiener–Schwartz theorem). These various solutions provide equivalent representations of the inverse operator $A^{-1}$.

We consider two methods for obtaining regularized solutions of Eq. (8): Tikhonov regularization and constrained successive approximations.

### A. Tikhonov Regularization

As is well known, the method of Tikhonov regularization consists in minimizing the functional

$$
\Phi_\lambda[f] = \|Af - g\|_2^2 + \lambda \|f\|_2^2,
$$

where $\lambda$ is a positive number called the regularization parameter. If we denote by $\hat{f}_\lambda$ the function that minimizes this functional, then by means of straightforward computations one finds that its Fourier transform is given by

$$
\hat{f}_\lambda(\omega) = \sum_{i=1}^{m} \frac{\overline{K_i(\omega)}}{|\hat{K}(\omega)|^2 + \lambda} \hat{g}_i(\omega).
$$

If we neglect the noise term in Eq. (3) so that $g_i$ is given by Eq. (4) with $f = f^{(0)}$, we find the following relationship between $\hat{f}_\lambda(\omega)$ and $\hat{f}(\omega)$:

$$
\hat{f}_\lambda(\omega) = \frac{|\hat{K}(\omega)|^2}{|\hat{K}(\omega)|^2 + \lambda} \hat{f}(\omega).
$$

It follows that, in the absence of noise, the combined effect of imaging and restoration can be described by a global PSF whose Fourier transform is given by

$$
\hat{T}_\lambda(\omega) = \frac{|\hat{K}(\omega)|^2}{|\hat{K}(\omega)|^2 + \lambda}.
$$

We call this the global transfer function of the imaging system with restoration. We also note that we can obtain regularized restorations of $f^{(0)}$ by using the images $g_i$ separately; these restorations, denoted $\hat{f}_{\lambda,i}$, are given by

$$
\hat{f}_{\lambda,i}(\omega) = \frac{\overline{K_i(\omega)}}{|\hat{K}_i(\omega)|^2 + \lambda} \hat{g}_i(\omega),
$$

and the corresponding global transfer functions are given by

$$
\hat{T}_{\lambda,i}(\omega) = \frac{|\hat{K}_i(\omega)|^2}{|\hat{K}_i(\omega)|^2 + \lambda}.
$$

The knowledge of the global transfer function allows us to estimate the resolution limit provided by the imaging device. To show this we consider the case of a single imaging system whose action is represented by Eq. (1) (the noise term is again taken into account). By Fourier transforming both sides of this equation, we obtain

$$
\hat{g}(\omega) = \hat{K}(\omega)\hat{f}(\omega) + \hat{\omega}(\omega).
$$

The effective band of the image $\hat{g}(\omega)$ is defined as the set of frequencies such that

$$
|\hat{K}(\omega)| \cdot |\hat{f}(\omega)| = |\hat{\omega}(\omega)|.
$$

The effective band contains all the values of $\omega$ for which the image can provide information about $f^{(0)}(\omega)$. Although this band cannot be exactly determined, it can be approximately evaluated if one assumes that the exact image and the noise are two uncorrelated white stationary processes with power spectra of $E^2$ and $\epsilon^2$, respectively. Therefore relation (24) can be replaced by

$$
|\hat{K}(\omega)| \geq \epsilon/E,
$$

where $E/\epsilon$ is the signal-to-noise ratio. Now, it can be proved that a good choice of the regularization parameter is given by $\lambda = (\epsilon/E)^2$. In such a case, because the global transfer function is a monotonic function of $|\hat{K}(\omega)|$, as follows from Eq. (20) or (22), condition (25) is completely equivalent to the following one:

$$
|\hat{T}_\lambda(\omega)| \geq 1/2.
$$

This means that, if $|\hat{T}_\lambda(\omega)|$ is a decreasing function of $\omega$, an estimate of the superior extreme $\omega_{\text{eff}}$ of the effective band is given by the frequency such that the global transfer function is equal to 1/2. Thanks to the Shannon sampling theorem, the corresponding resolution limit achievable by the instrument is approximately equal to $\delta = \pi/\omega_{\text{eff}}$. Note that the solution of inequality (26) represents merely an easy way to determine the effective band of the image, but, as is clearly shown by relation (24), this band depends on the PSF, the signal, and the noise and is not modified by the introduction of regularization. Nevertheless, the regularization method provides an improvement in the quality of
the image, and this generally also permits improvement
of the restoration of details to an amount of the order of
the resolution limit.

B. Constrained Successive Approximations

The method of successive approximation is also known as
the Landweber-Bialy method. 10 It is an iterative tech-
nique for solving least-squares problems associated with
first-kind operator equations. In the case of a linear op-
erator equation such as Eq. (8) the iteration scheme is as
follows:

\[ f_{k+1} = f_k + \tau (A^* g - A^* A f_k), \]

(27)

where \( \tau \) is a relaxation parameter whose value must sat-
ify the following conditions:

\[ 0 < \tau < \frac{2}{\|A\|^2}. \]

(28)

If we take \( \tau = 0 \), then it is easy to show that, in the
case of the operator equation (7), whose adjoint is given
by Eq. (13), the result of the 4th iteration is equivalent to
a filtering of the generalized solution [Eq. (14)], because
we have

\[ \hat{f}_4(\omega) = \left(1 - [1 - \tau |\hat{K}(\omega)|^2]\right)^4 \sum_{i=1}^{\infty} \frac{\hat{K}(\omega) \hat{g}_i(\omega)}{|\hat{K}(\omega)|^2}, \]

(29)

with \( \hat{K}(\omega) \) being defined by Eq. (11). This result is
analogous to that which holds true in the case of a single
convolution operator. 3 Therefore, if we neglect the noise
term, for a fixed number of iterations \( k \) the combined ef-
fect of imaging and iteration can be described by a global
transfer function given by

\[ T_k(\omega) = 1 - [1 - \tau |\hat{K}(\omega)|^2]^k. \]

(30)

Equation (30) has a behavior similar to that of \( T_4(\omega) \),
Eq. (20), because it is a filter that does not transmit the
frequencies corresponding to small values of \( \hat{K}(\omega) \).

As in the case of Tikhonov regularization, if we decon-
volv the various images separately, at the 4th iteration
we obtain the following approximations:

\[ \hat{f}_{k,i}(\omega) = \left(1 - [1 - \tau |\hat{K}(\omega)|^2]\right) \frac{\hat{K}(\omega) \hat{g}_i(\omega)}{|\hat{K}(\omega)|^2}, \]

(31)

whose corresponding global transfer functions are

\[ T_{k,i}(\omega) = 1 - [1 - \tau |\hat{K}(\omega)|^2]^k. \]

(32)

It is known that, in the absence of noise, this iterative
method converges (strongly) to the generalized solu-
tion of the problem 13; moreover, in the case of noisy
data, it behaves as a regularization algorithm, with the
number of iterations playing the role of a regularization parame-
ter. 10

However, the most interesting feature of this method
is that it can easily be modified to take into account addi-
tional constraints on the solution. A typical constraint
is that of positivity. In general one can assume that the
unknown image \( f^{(0)} \) belongs to a closed and convex set \( C \).
In such a case it is quite natural to look for least-squares
solutions that belong to \( C \), i.e., for solutions of the problem

\[ \|A f - g\| = \text{minimum}, \quad P_C f = f, \]

(33)

where \( P_C \) is the (generally nonlinear) projection operator
onto the set \( C \).

Let us consider the following modification of algorithm
(27):

\[ f_{k+1} = P_C [f_k + \tau (A^* g - A^* A f_k)], \]

(34)
i.e., at each iteration we project the result onto \( C \). Then
it is possible to prove, at least in the case in which the
operator \( A \) has a bounded inverse, 14 that algorithm (34)
converges to the unique solution of problem (33). How-
ever, problem (33) is, in general, ill posed. In such a case
algorithm (34) (with \( f_0 = 0 \)) has, presumably, a regulari-
ad effect. Evidence that this may be true is given by
numerical experiments. 3 Therefore by stopping the
iterations it is possible to obtain a stable and constrained
approximate solution.

In this paper we consider two kinds of constraint:

(i) The solution of the problem is positive (nonneg-
ate). This is a very natural requirement in image
restoration or in spectroscopy. Now, the set of nonneg-
ative (a.e.) functions is a closed and convex set in \( L^2(\mathbb{R}^n) \),
and therefore algorithm (34) can be used. The action of
the projection operator \( P_C \) consists in replacing by zero
the negative values of the function \( f \).

(ii) The solution of the problem has a bounded sup-
port. This is a constraint that is widely used in the
problem of out-of-band extrapolation and superresolution. 15
The set of all the functions whose support is interior to
a given and fixed set \( D \subset \mathbb{R}^n \) is closed linear subspace
of \( L^2(\mathbb{R}^n) \). In this case the action of the projection opera-
tor \( P_C \) (which is linear) consists in replacing by zero the
values of the function \( f \) in points exterior to the set \( D \).

4. Example of compactly supported kernels: numerical results

In this section we discuss an example that has already
been considered in the literature. 16 This is the particular
case in which the kernels are given by the character-
istic functions of two intervals, that is,

\[ K_1(x) = \chi_{[-r_1,r_1]}(x), \quad K_2(x) = \chi_{[-r_2,r_2]}(x), \]

(35)

the corresponding transfer functions of which are sinc
functions:

\[ \hat{K}_1(\omega) = 2 \frac{\sin(r_1 \omega)}{\omega}, \quad \hat{K}_2(\omega) = 2 \frac{\sin(r_2 \omega)}{\omega}. \]

(36)

Both transfer functions have countable set of zeros,
which are at points \( \omega_{1,n} = n \pi / r_1 \) and \( \omega_{2,n} = n \pi / r_2 \)
\((n = \pm 1, \pm 2, \ldots)\), respectively. These zeros never co-
incide if \( r_1 / r_2 = \sqrt{p} \) (\( p \) is not a perfect square). This is
precisely the condition that ensures that these convolvers
are strongly coprime. From now on we consider the case
in which \( r_1 = 1 \) and \( r_2 = 2 \), because this is the case
considered in the above-mentioned papers.

In Fig. 1 we plot the functions \( \hat{T}_1(\omega) \) \{Eq. (20)\}
and \( \hat{T}_3,1(\omega) \) \{Eq. (22)\} (in both cases with \( \lambda = 10^{-3} \)),
that are associated with convolvers (35). The global transfer
function corresponding to single deconvolution [Fig. 1(a)]
has precisely the zeros of \( \hat{K}_1(\omega) \), whereas \( \hat{T}_3,1(\omega) \) [Fig. 1(b)]
is never zero, even if, in correspondence with particular
frequencies, it assumes very small values. This happens when the zeros of $T_{s,1}(\omega)$ and $T_{s,2}(\omega)$ occur close to each other. As already noted, this is why, from a general point of view, it is convenient to choose the strongly co-prime PSF in such a way that the zeros of their Fourier transforms are close to each other at the highest possible frequencies. The functions $T_{d}(\omega)$ [Eq. (30)] and $T_{s,1}(\omega)$ [Eq. (32)] have a very similar behavior. In fact, $T_{s,1}(\omega) = 0$ when $K_{d}(\omega) = 0$.

As already explained in Subsection 3.A, the knowledge of the global transfer functions allows us to determine the effective bands of the single and the double images. In the case of $T_{s,1}(\omega)$ this function is not monotonic in $\omega$. Nevertheless, it becomes definitely smaller than $1/2$ for $\omega \geq 64.4$. This suggests that we assume this to be the superior extreme $\omega_{d,1}$ of the effective band. The corresponding resolution limit is $\delta_{1} = 0.048$. We must, however, observe that this effective band is not completely full because some small intervals around the zeros of $K_{1}(\omega)$ are also excluded by the condition $|T_{s,1}(\omega)| > 1/2$. In a similar way we find that the effective band of the double images and the corresponding resolution limit is $\delta = 0.037$. Also in this case the effective band is not completely full, owing to the relationship already observed between zeros of $K_{1}(\omega)$ and zeros of $K_{2}(\omega)$. Anyway, some improvement in resolution by the use of double images has been obtained, even if this example does not seem to be particularly meaningful for showing the improvement of resolution that double deconvolution can provide in particular cases.

The presence of zeros in the global transfer functions corresponding to single deconvolution implies that linear filtering methods applied to the case of one image are not able to recover the Fourier transform of the object in the neighborhoods of the zeros of the transfer function.

As we will show, this is not true in the case of the (generally nonlinear) iterative method of Eq. (34), which can be used when, for instance, we know that the object must be nonnegative (this is a rather natural condition in spectroscopy and imaging) or when the object can be considered zero outside a bounded interval (see the remarks at the end of Section 3). In the case of positivity, because this algorithm is nonlinear it cannot be described by means of a global transfer function as can the linear methods. Moreover, in the case of compact support the algorithm is linear, but its global transfer function must be expressed in terms of the singular functions of the integral operator. Unfortunately, the computation of these functions is not easy. Therefore we justify the efficiency of these methods in interpolating the Fourier transform by means of a numerical simulation.

The example that we consider is that of a Gaussian function with zero mean and standard deviation $\sigma = 0.1$, whose analytical form is

$$f(x) = \frac{2}{\sqrt{2\pi} \sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right).$$

Note that the standard deviation of this Gaussian is approximately twice the resolution distance associated with $K_{1}(\omega)$. This function and its Fourier transform are plotted in Fig. 2. We obtain the simulated images by convolving $f$ with the two convolution kernels [Eqs. (35)] (a fast-Fourier-transform routine is used to this end) and then by affecting the results with Gaussian noise (the relative error is equal to 5%). These images and the corresponding Fourier transforms are represented in Fig. 2. The behavior of the Tikhonov inversion method is shown in Fig. 3, where the Fourier transforms of the regularized solutions are plotted in the case of the inversion of only one image $g_{1}$ [Fig. 3(a)] and in the case of the simultaneous inversion of $g_{1}$ and $g_{2}$ [Fig. 3(b)]. It is evident that, owing to the linearity of the method, the reconstruction provided by Eq. (21) is characterized by regular holes corresponding to the zeros of $K_{1}(\omega)$. We can partly compen-

Fig. 1. (a) Global transfer functions as defined in Eq. (22) in the case of the convolver $K_{1}(x)$ of Eqs. (35) with $r_{1} = 1$. The value of the regularization parameter is $\lambda = 10^{-3}$. (b) Global transfer function of Eq. (20), corresponding to double deconvolution of $K_{1}(x)$ and $K_{2}(x)$, with $r_{1} = 1$ and $r_{2} = \sqrt{2}$, respectively [Eqs. (35)], for the same value of $\lambda$.

Fig. 2. (a) Input Gaussian function, with $\sigma = 0.1$. (b) Fourier transform of the input function. (c) Image $g_{1}$ obtained by convolving the Gaussian function in (a) with the convolver $K_{1}$ [Eqs. (35)], with $r_{1} = 1$, and by affecting the result with 5% Gaussian noise. (d) Fourier transform of the image $g_{1}$. (e) Image $g_{2}$ obtained by the use of the convolver $K_{2}$ [Eqs. (35)], with $r_{2} = \sqrt{2}$. (f) Fourier transform of $g_{2}$.
for single deconvolution and
e.g., double deconvolution provided by Eq. (18)
of g with the constraint of positivity
cated for this lack of information by adding the information
contained in g through Eq. (19). Note that we can obtain
the optimum value of the regularization parameter λ by minimizing the function
\[ \epsilon(\lambda) = \frac{\|f - f_\lambda\|}{\|f\|}. \]  
(38)
The improvement of the restoration accuracy that is
due to double deconvolution is measured by the value
of the minimum of function [Eq. (38)], which is \( \epsilon = 43.1\% \)
for single deconvolution and \( \epsilon = 16.5\% \) for double
deconvolution.

The greater efficiency of double deconvolution is closely
related to the linearity of the algorithm that is adopted
for the inversion. Things are completely different if the
nonlinear regularization technique is applied. In fact,
the constrained Landweber–Bialy algorithm, Eq. (34), al-
 lows us to interpolate the values of the Fourier trans-
form of the regularized solution in the neighborhood of
the zero of \( K_1(\omega) \) also, so that the performance of the
method is not improved by the use of more convolvers. In
Fig. 3(c) we give the reconstruction of the Fourier trans-
form of function (37) that is obtained by application of the
Landweber–Bialy method with the constraint of the posi-
tivity of the solution to the single inversion of \( g_1 \). As
is evident, the insertion of the a priori information into
the inversion algorithm makes the regularized solution
extremely stable and permits great restoration accuracy.
As in the case of the Tikhonov method, we obtain the opti-
 mum value of the number of iterations (which here plays
the role of the regularization parameter) by minimizing
\[ \epsilon(n) = \frac{\|f - f_n\|}{\|f\|}, \]  
(39)
and the reconstruction error is given by the minimum of
this function, which is \( \epsilon = 5.4\% \). As is clearly shown
in Fig. 3(d), similar results are obtained if we modify
the Landweber–Bialy method by imposing the condition
that the regularized solution be zero outside a bounded
interval (in this case \( \epsilon = 6.5\% \)). Here we have cho-
 sen the interval \([-0.5, 0.5]\) because outside this interval
function (37) is essentially zero. In fact, the compactness
prescription on the support is tantamount to imposing the
analyticity of the Fourier transform of the regularized
solution. We do not report the results obtained by double
deconvolution because they do not differ significantly from
those obtained by a single convolution.

5. EXAMPLE OF NONCOMPACTLY
SUPPORTED KERNELS

The substantial equivalence between single and double
deconvolution when nonlinear filtering is adopted refers
mainly to the example introduced in previous papers.4,6
In this case the Fourier transforms of the convolution
kernels, Eqs. (36), are not characterized by superimposed
zeros but provide information about approximately the
same region of the frequency spectrum, as is shown in
Fig. 1. In other words, the complementarity of the in-
formation provided by the two images is localized only
in the neighborhood of the zeros of the kernels, so the
application of a nonlinear method that is able to inter-
polate the Fourier transform of the regularized solution
in these intervals makes double deconvolution practically
useless. However, the availability of two (or more) simulta-
neous images of the same object becomes advantageous
when the constrained iterative method is applied, if the
convolvers provide information about different domains of
the spectrum. This is, for instance, the case for the two
noncompactly supported PSF’s:

\[ K_1(x) = \frac{2}{\sqrt{2\pi \sigma_1}} \exp\left( -\frac{x^2}{2\sigma_1^2} \right), \]  
(40)
\[ K_2(x) = \frac{4}{\sqrt{2\pi \sigma_2}} \cos(\omega_0 x) \exp\left( -\frac{x^2}{2\sigma_2^2} \right). \]  
(41)
PSF's is particularly advantageous, as it allows the different kinds of information contained in the two images to be exploited. The usefulness of double deconvolution in this case is particularly clear if one considers the global transfer functions $T_{1,1}(\omega)$ and $T_{1,2}(\omega)$ [Eq. (22)] and $T_{1}(\omega)$ [Eq. (20)] represented in Fig. 5 (again for $\lambda = 10^{-3}$) for the choice of the parameters $\sigma_1 = 0.3$, $\omega_0 = 14$, and $\sigma_2 = 0.5$. In fact, $T_1(\omega)$ is equal to 1, in correspondence to a frequency range that is approximately the union of the intervals where $T_{1,1}(\omega)$ and $T_{1,2}(\omega)$ are equal to 1. In the case of two images the resolution distance is $\delta = 0.16$.

To show the different performances of single and double deconvolution in this case, we consider the source function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-x_0)^2}{2\sigma^2}\right] + \exp\left[-\frac{(x-x_0)^2}{2\sigma^2}\right],$$

which is plotted in Fig. 6(a) for $\sigma = 0.03$ and $x_0 = 0.15$. This distance is of the order of the resolution distance estimated above. Figure 6(b) is a plot of the result obtained by application of the Landweber–Bialy algorithm, with the positivity constraint, to invert the image $g_1$. We obtained this image by convolving $f(x)$ with $K_1(x)$ [Eq. (40)] and by affecting the result again with 5% Gaussian noise. As one can see, an imaging device with $K_1(x)$ as the PSF is unable to resolve the two peaks of the input function (the reconstruction error is huge: $\epsilon = 76.8\%$). The use of $K_2(x)$ permits achievement of this resolution even if the lack of information about the Fourier transform of the solution in the central part of the spectrum implies the presence of artifacts [which are evident in Fig. 6(c)] and a large restoration error ($\epsilon = 74.2\%$). Finally, as is clearly shown in Fig. 6(d), we find that a great improvement in the restoration accuracy ($\epsilon = 14.7\%$) together with the achievement of high resolution power can be obtained only by the use of double deconvolution, as it allows us to exploit the information about the source function contained in both images.

6. CONCLUDING REMARKS

We have considered the deconvolution of multiple images of the same object. In recent papers it has been shown that, if the convolvers are compactly supported functions, this problem can be well posed in the sense of distributions. Nevertheless, we show here that, if a stronger continuity such as the one in $L^2$ is required, the problem is still ill posed. This justifies the use of the regularization techniques to improve the restoration accuracy. In our numerical applications we point out that an example of double deconvolution widely discussed in the literature is not useful for demonstrating the advantage of double deconvolution over single convolution. Moreover, we show that in this case single deconvolution performed by a constrained iterative method, which allows for a priori information about the solution, provides better results than does double deconvolution performed by a well-known linear filter. At the same time, however, we note that multiple deconvolution is always advantageous, independent of the kind of algorithm adopted for the regularization, when the different images contain information about different regions in the Fourier spectrum. We provide an example supporting this assertion. A potential applica-
tion of this example would be in confocal microscopy, if it were possible to combine images provided by a conventional confocal microscope with images provided by a 4Pi confocal microscope.16

REFERENCES


