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Geometric Modeling of Spatial Entities at Multiple Levels of Resolution

by

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Abstract

The representation of spatial data at different resolution is a topic of relevant interest in spatial information theory. Combinatorial aspects of the multiresolution representation of spatial entities are considered. The target application is geographic map description. The combinatorial structure of geographic maps is mathematically modeled through *abstract cell complexes*, and maps at different levels of detail are related through functions over such complexes. A set of functions is detected that correspond to consistent transformations reducing detail. Properties of such functions have been identified on the basis of requirements of classical generalization operations proposed in the literature. A categorical characterization of geographic maps with such set of functions is provided. The use of category theory allowed to define a minimal and sufficient set of atomic operators that are capable of generating all and only consistent transformations. Thus, intrinsically consistent multiresolution models can be built on the basis of such operators. They also provide means for performing local and gradual changes in detail, and for evaluating the reduction of information between maps at different levels of resolution.

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Table of Contents

List of Figures

Introduction

The analysis and representation of spatial entities is the subject of spatial information theory. The work developed in this thesis relates to the geometric modeling and to the manipulation of spatial objects whose overall structure has the following characteristics:

1. objects are composed of parts (for example: city maps with roads, bridges, neighbourhoods, etc., in Geographic Information Systems; mechanical assembled objects in CAD systems);

2. every component may have a non-elementary topological structure, i.e., it may contain holes, cuts, dangling entities, etc. (for example: in a region of a geographic map, holes may correspond to lakes, cuts to roads, etc.);

3. objects have either an inherent hierarchical structure, or are suitable to be described at multiple levels of detail; hence, it would be useful to have models that provide multiple representations of such objects and/or of each single part; for example, a city in a map may be represented through an aggregate of buildings and roads, or through a region, or through a single point, depending on the level of detail.

Classical representations proposed in the geometric modeling literature are not suitable to describe objects having the first two characteristics (that we call *structured objects*) since they do not provide direct means to maintain their original structure: non-elementary parts composing objects are usually decomposed into simple cells, thus destroying the logical organization of structured objects. Furthermore, most of the previously proposed representations support the description of a restricted class of objects which do not include some important features of spatial entities.

Although this problem is related to more general applications, we have focussed our attention on a specific application context, i.e., Geographic Information Systems (GISs). In such a context, much work has been devoted to providing suitable formalizations and representations of the combinatorial structure of plane maps.

In particular, in this thesis, we are interested in the description of plane maps at different levels of resolution. The concept of resolution in a map involves both the amount of objects explicitly represented in the map and the spatial entities used to describe them, as well as the way in which the shape and location of such objects are approximated. The possibility of modeling spatial data at variable resolution finds applications in spatial representation and reasoning; for example, it can be exploited for efficiently querying large GISs, as a support to map generalization [83], and for solving problems related to wayfinding and planning using different levels of abstraction [164].

Most of the work proposed recently in this context is related either to the development of data models [30, 31, 32, 83, 141], or to preserving the consistency among different representations [58, 70, 71, 75]. However, no unifying mathematical model has been proposed that formally specifies the requirements and properties of multiresolution representations. The main purpose of our work consists of establishing a formal framework as a theoretical basis for tackling the problem.

Three different issues are discussed into details in the following: plane map representation, plane map modeling at different levels of resolution, and the use of multiresolution representations as a support for map generalization. Each such issue plays an important role in the context of the thesis. In particular, although we are not concerned with algorithmic techniques for generalization, we analyse problems related to map generalization because we think our formalization can be exploited as a starting point to solve some of such problems.

Plane Maps

By a *plane map* (also called a *geographic map*, in the following) we mean a model including numerical data, relations between them, semantic information, etc., stored in a spatial database (i.e., not a graphical representation). A map is a structured object, defined as some kind of covering of a portion of the plane by means of three kinds of entities: points, lines and regions. Regions can be multiply connected and they may contain dangling and isolated entities in their interior.

Representations of plane maps within a spatial database cannot be based on classical tabular representations, which are not suitable for describing entities that are located in space and interrelated through spatial relationships [87]. Indeed, large amounts of spatial data are usually dealt with, which users want to analyse, and from which they want to extract information: for instance, GIS users may want to retrieve all private land parcels which are road-accessible. In order to answer this kind of queries efficiently, spatial relations must be embedded in a map representation within a spatial database. Topological relations seem to be the most fundamental [73].

In the last few years, some research has been dedicated to adopting formal approaches for the description and manipulation of sets of spatial entities and their relationships [54, 66, 67, 74, 93, 101, 175, 176]. However, if generic collections of entities are used, the number of topological relations often becomes too large to be stored explicitly. Thus, sets of entities satisfying special properties are obtained by imposing constraints on their intrinsic structure.

A number of models have been proposed in the literature for giving a comprehensive representation of the geometric and topological structure of plane maps [54, 65, 82, 86, 88, 123, 135, 157, 175]. Different models are characterized mostly by their expressive power, defined by the degree of generality of objects and configurations that they are able to represent. However, many of them do not permit a suitable representation of structured two-dimensional objects, since, for example, they cannot describe objects with holes or dangling/isolated parts, without subdividing them further.

In this thesis, we mathematically define a plane map as a *weakly disjoint covering* of a given area, i.e., a collection of pairwise disjoint entities covering such area, and including multiply connected regions, as well as dangling and isolated features.

Multiresolution Representation

With the exception of some application of image processing tools [152, 153] to raster modeling, available GISs do not provide much support for manipulating and relating spatial data at different resolutions. In this thesis, we are interested in multiple representations of vector maps, and in their manipulation in the context of a unified framework.

The resolution of a map is usually intended as the minimum size of a cell represented/visible in the map [32]. However, two map representations of the same area at different resolutions can differ in two basic aspects:

a) *detail* is related both to the amount of objects (information) explicitly represented in the maps and to the set of entities used to describe each object; indeed, some objects can be represented only in the map at higher resolution; objects that appear in both maps can be described by means of entities of lower complexity and/or dimension in the context of the map at lower resolution; any object represented by an entity in the map at lower resolution must be also represented by either an entity or a group of entities in the map at higher resolution (*monotonicity assumption*);

b) *accuracy* corresponds to the maximum error in approximating the spatial extent and location (geometry) of objects in a map by the extent and location of entities representing such objects: the higher the resolution, the lower the approximation error.

While changes in detail mainly involve topological/combinatorial information contained in the representation of a map, changes in accuracy are related to metric aspects. Although in actual implementations combinatorial and metric aspects can be strictly interrelated (together with semantic aspects), the separation of such aspects is important from a modeling point of view. Indeed, they can be studied and analysed through different mathematical tools.

In this thesis, we are interested in modeling changes in detail in geographic maps. Thus, only the combinatorial structure of maps will be studied and formalized, disregarding metric aspects.

When the extraction of less detailed representations is required, the problem of computing such representations on-line arises. In order to avoid the repetition of such a time-consuming process every time it is needed, multiple representations, each corresponding to a given level of detail, are maintained in the database.

Related to the problem of maintaining multiple representations is also the problem of linking different representations of the same objects at different resolutions. Indeed, within a multiple representation database, given either a portion of a map or a single entity, it is often useful to have a fast way to access other representations of the same portion/entity. By enriching multiple representations of a geographic map through information on relations between different representations of the same entity at different levels, we obtain a *multiresolution model.* Different kinds of models can be defined depending both on how the model is built and on which relations are explicited.

Map Generalization

The problem of extracting a less detailed representation of a given area from an available representation of the same area is known as *map generalization* [126]. In cartography, it involves different tasks, such as removing objects, simplifying their geometry, aggregating several objects into one, etc. In GISs, generalization has the broader meaning of *reduction of information density* in a spatial database, while preserving its overall structure and semantics. More specifically, a distinction has been made between model-oriented and cartographic generalization [126]. Model-oriented generalization manipulates data independently of their graphical representation. Indeed, spatial objects may require multiple internal representations independently of their visualization (e.g., in the context of a spatial database). Model-oriented generalization may be driven by analytical queries, while cartographic generalization is mainly driven by graphical requirements (legibility, graphical clarity). In this thesis, we are interested in the model-oriented aspects of generalization.

It must be noted that both geometric aspects (e.g., shape, topology, metrics) and non-geometric aspects (e.g., the role/relevance in a particular context) of spatial entities in a geographic map are involved in a generalization process. In this thesis, we are mainly interested in detecting changes in the topology of entities implied by map generalization. In particular, the approach to multiresolution modeling we develop can be exploited as a support to generalization processes involving topological changes.

In the literature, a set of basic operations applied in a generalization process has been detected [147]. Although attempts have been made to provide formal frameworks for such operations (see, e.g., [8, 122, 126, 151, 168]), a few major problems are still far from being solved. The process usually requires interaction between *ad hoc* programs and expert cartographers. As a consequence of the lack of a formalization, an important problem related to map generalization, as well as to any other method involving the extraction of multiple representations, is consistency [58, 70, 75, 165]. Two maps of the same area at different resolution must be consistent, i.e., objects that appear in both maps must maintain compatible spatial relationships. For example, entities that share some boundary cannot be mapped onto disjoint entities. By maintaining consistent object characteristics through different representations, query evaluations at a coarser level return results compatible with the evaluation at the most detailed level [6]. If multiple representations are not intrinsically consistent, *a posteriori* checks must be performed on the map at reduced resolution in order to guarantee the fulfillment of consistency constraints. This is again a time-consuming process.

The above considerations reveal the importance of providing methods for generating intrinsically consistent multiple representations. Such methods should be based on sound theoretical results, i.e., on a rigorous formalization, that is still lacking in the literature. Our work addresses such a theoretical issue.

Finally, another important topic relates to the possibility of evaluating to what degree a geographic map is a generalization of another one. In other words, methods should be detected to measure the *diversity* in resolution between two different representations of the same area. Such a diversity is not easily quantifiable, except by counting the number of entities explicitly represented at both resolutions: more meaningful evaluations are needed. In our theoretical framework, we also propose alternative methods for measuring the degree of diversity in detail between maps.

Contribution of the Thesis

The major contribution of this thesis consists of providing a sound formal framework for the multiresolution representation of the combinatorial structure of geographic maps, and for modeling changes of detail in a map. In the following we list and analyse basic steps of our work.

- By following the formalization proposed by Puppo and Dettori [141], the combinatorial structure of a geographic map is described through an *Abstract Cell Complex (ACC)*, i.e., a set of cells with an ordering relation (*bounding* relation), independent of any Euclidean embedding. In this way, the drawback of classical representations based on Euclidean cell complexes (i.e., the assumption that cells are homeomorphic to disks: regions cannot be multiply connected and they cannot contain dangling/isolated entities in their interior) has been overcome.

- *Combinatorial map transformations*, i.e., operations reducing the detail of geographic maps, are modeled as functions between cell complexes. A set of such functions is characterized, which correspond to consistent transformations, on the basis of requirements of classical generalization operations that act on the combinatorial structure of a map.

- A categorical characterization of ACCs with such set of functions (also called *morphisms*) is provided. Mathematical principles from category theory provide an appropriate formal basis for obtaining innovative results.

- Since not all ACCs correspond to a geographic map, a subcategory of the category of ACCs is defined that captures specific characteristics of geographic maps (called the category of *Geographic ACCs, GACCs*). Morphisms in the category of GACCs are all consistent map transformations. Within such a category we define minimal morphisms. A minimal morphism is a morphism that cannot be obtained as composition of any two non-identical morphisms in the category. Conversely, every non-minimal morphism in the category can be obtained as finite composition of minimal morphisms. The set of minimal morphisms can be partitioned into classes, by restricting their domain to the set of entities that are modified, while disregarding entities that are not affected by the transformation. Each class is called a *minimal operator*.

- The major result of this thesis is the explicit detection of minimal operators in the category of GACCs. Such operators are extremely important for several reasons:

  - they are necessary and sufficient to generate all consistent combinatorial transformations; thus, implicitly consistent models are built on the basis of such operators;
  - they perform local changes within a map;
  - they guarantee the graduality of changes in detail during a map transformation;
  - they provide a way to formalize the concept of diversity between two compatible maps: each consistent transformation is obtained as a sequence of minimal operators; thus, the number of operators applied to the more detailed map in order to obtain the less detailed one provides an evaluation of the computational

effort involved in the process (each minimal operator is considered to have unit cost), and of the resulting reduction of information.

Summary

The thesis consists of five chapters, and it is organized as follows.

In Chapter 1, we provide an overview of previous work proposed in the literature, and related to the multiresolution representation of spatial entities. Although the application we have focussed on is two-dimensional, we have analysed the problem of describing spatial objects composed of parts from a more general point of view. Classical models for entities in any dimension are described, with a special attention to models developed for GIS applications. We remark here that, however, the part related to models describing entities in any dimension is not essential for understanding the formalism and results described in the thesis. Then a survey on the various methods for defining spatial relations among such entities is presented. Finally, multiresolution models previously developed are described and discussed. Such models can be subdivided into two classes: instrumental models and structural models. Although instrumental models (mainly developed for describing surfaces/terrains) are not suitable for describing objects composed of parts at different levels of resolution, the overview on such models (Section 1.3.1) has been included in this Chapter because they represent the most part of multiresolution models proposed in the literature, and, historically, the first ones. Also, on the basis of properties/requirements of such models, there have been attempts to define structural models, as well as to develop data structures and algorithms to manipulate them. However, the reader can easily skip such Section without compromising the reading of the thesis. The critical analysis of the literature we provide also includes the discussion of limits of classical graph-based formalisms (i.e., plane graphs with Euler operators), as opposed to the approach followed in this thesis.

Chapter 2 introduces formal notions related to spatial entities and their relations within a geographic map. The definition of a map as a *weakly disjoint covering* of a portion of the plane is provided. A way to obtain such a covering from a generic set of entities by means of a regularization/overlay process is also described.

In Chapter 3, the combinatorial structure of plane maps is formalized through *abstract cell complexes (ACCs)*. Consistent map transformations are characterized in this Chapter as functions between ACCs. On the basis of the resulting set of functions, the category of ACCs is defined. After analysing properties of such category and defining related notions (e.g., minimal morphisms, minimal operators, etc.), we refine it into the subcategory of *Geographic ACCs (GACCs)*, i.e., ACCs representing geographic maps.

Chapter 4 is dedicated to the definition of *atomic operators* as functions between ACCs. Such operators are shown to correspond to minimal operators of the category of GACCs. A proof of the minimality, as well as of the sufficiency of atomic operators is provided. The importance of having a set of operators that guarantee both the locality and the graduality of changes in detail is highlighted. Furthermore, a method for evaluating the diversity between two different representations of the same area is provided by counting the number of atomic operators to be applied to a given map in order to obtain a less detailed one.

In Chapter 5, a multiresolution model for geographic maps is considered, which is built on the basis of a sequence of consistent map transformations. Data structures for encoding both individual plane map representations and the multiresolution model are proposed. A first attempt to model changes in accuracy in a map and to integrate them with changes in detail formalized in the previous chapters is also briefly discussed.

Finally, some concluding remarks, as well as future developments of this work are outlined. The realization of a prototype system implementing the proposed theoretical work is one of our major objectives.

A few appendixes have been added for introducing basic notions and results from point-set topology, graph theory, and category theory.

Chapter 1

Previous and Related Work

In this Chapter, we will review classical representations proposed in the literature. This survey includes not only representations for entities embedded in the plane but more general representations for entities in any dimension. Although we will focus on geographic map representation as application of this thesis, it is interesting to have a broader view of what has been proposed in the literature. Furthermore, many representations used in two dimensions can be seen as specializations of more general representations. However, as already remarked in the Introduction, the reading of the overview on arbitrary-dimensional representations is not essential for understanding the problems developed in this thesis.

Most representations proposed in the literature are based on Euclidean cell complexes, which decompose objects into elementary parts without maintaining their original structure: non-elementary parts (i.e., containing holes and/or cuts in their interior) composing the objects are further decomposed into cells that are homeomorphic to disks. In two dimensions, more general objects (including holes and cuts) have been modeled by using plane graphs.

Since our aim is providing representations for spatial databases, we must take into account the peculiarities in dealing with spatial data. Classical tabular representations are not suitable for describing spatial properties. Spatial concepts must be supported by the query language: to this aim, relations (topological, metric, etc.) among spatial entities are usually embedded in the database itself. In particular, topological relations provide fundamental information exploited in querying a spatial database. Thus, in this Chapter, we also provide an overview of commonly used formalisms for defining such relations.

Notwithstanding the importance of providing representations of spatial entities at different levels of detail in several application fields, no unifying mathematical model has been proposed that formally specifies requirements, and properties of such representations. Many multiresolution models have been proposed in the literature, each targeted to a specific application. They are based on different decompositions of the represented object. We describe previously proposed models and analyse their applicability to the purpose of this thesis. More specifically, we try to detect important properties of multiresolution models and we propose a classification of such models into *instrumental* and *structural* models, depending on the kind of decompositions they are based on.

In structural multiresolution models, both the level of resolution and the used decomposition are strictly related to the inherent structure of the represented object. On the contrary, in instrumental multiresolution models, each representation arbitrarily decomposes the object into simple parts. In this case, a finer decomposition yields a better approximation of the shape of the object. Examples of instrumental models are multiresolution models for the description of surfaces. A variety of instrumental models have been proposed in the literature, while structural models are very few.

In this thesis, we are interested in the definition of structural multiresolution models that support the description of geographic maps at different levels of detail, thus the majority of available models are not suitable to our purpose.

The Chapter is organized as follows: Section 1 reviews representations for spatial entities, with particular emphasis for representations based on decompositions, being a geographic map a decomposition of a portion of the plane into points, lines and regions. Section 2 presents formalisms for describing spatial relations among entities, while in Section 3, a survey of multiresolution models proposed in the literature is provided. We describe both instrumental and structural models (Sections 1.3.1 and 1.3.2, respectively). However, as already pointed out in the Introduction, Section 1.3.1 can be skipped without compromising the understanding of the rest of the thesis. Section 4 is dedicated to a critical analysis of the state-of-the-art that explains the reasons why previously proposed models are not suitable for our purposes.

1.1 Representations for Spatial Entities

The problem of representing spatial entities arises in various contexts and it can be tackled from different points of view. From a theoretical point of view, researchers in the field of geometric modeling aim at defining abstract models that describe generic spatial entities (e.g., with complicated shapes and/or in arbitrary dimensions, etc.) in order to provide independence from any specific application. Formal models must not only provide a mathematical definition of the objects described, but also permit to check the consistency of data structures designed to encode such objects within a computer system.

The research in the field of spatial entities representation is currently aimed at two main objectives:

- define models (and representations) that are powerful enough to describe all objects of interest within a given application domain;
- define data structures that support an efficient and consistent manipulation of such objects.

Such requirements may be in contrast with each other: in order to cover wide and general representation domains, hardly checkable, and quite inefficient representations are often defined. Indeed, very wide representation domains have very few characteristics from a mathematical point of view and, thus, less theoretical results exist. Conversely, efficient representations based on theoretical results can be obtained for specific domains which can, however, be insufficient for describing objects involved in some applications.

Thus, the main efforts of geometric modeling as a research field are devoted to attempts in finding trade-offs between the possibility of dealing with specific representation domains, on whose basis it is possible to develop sound and efficient representation schemes, and the extension of such representations to more general domains.

A general classification distinguishes between *decomposition* and *constructive* representations. More precisely:

- *decomposition* representations describe an object in terms of a collection of primitive objects combined through a glueing operation; they are subdivided into
  - *object-based* representations: an object is subdivided into pairwise quasi-disjoint elementary parts (see next Section);
  - *space-based* representations: the space occupied by the object is subdivided into regular elements, and only approximate representations can be obtained; examples are *spatial enumerations* and adaptive subdivisions like the *quadtree*, the *octree* or the *bintree* [100, 152, 153, 163][1];
- *constructive* representations describe an object as the Boolean combination of predefined primitive point sets [144, 145, 146]; they are typical of a design environment since they simulate the construction process; thus their main application is in CAD systems.

Both object-based and space-based decompositions have been used in GISs for representing spatial entities. They are usually referred to as *vector* and *raster* approaches, respectively. In this thesis, we are interested in providing *vector* representations; thus, we will treat into detail only object-based decomposition representations. Indeed, we intend to model

[1]Hybrid models have also been defined which combine object- and space-based representations [16, 34].

geographic maps, defined as decompositions of a portion of the plane (the domain of the map) into points, lines and regions. Thus, a decomposition, where each piece corresponds to a part of the described object is quite a natural representation tool.

We do not consider constructive representations since they are usually based on very simple primitives, while the parts composing the objects we want to represent can have more complicated structures.

In the following Section, we will briefly review the most relevant representations based on decompositions proposed in the literature.

1.1.1 Representations Based on Decompositions

Object-based decomposition representations describe an object as the combination of pair-wise quasi-disjoint elementary entities whose union covers the object.

In most representations proposed in the literature, an object is thought of as a unique subset of the space, not further structured (i.e., composed of a single "piece"). The (arbitrary) decomposition of the object in a representation scheme is usually aimed at obtaining pieces having a simple topological structure, in order to make manipulation easier. Hence, in this case, the decomposition is only intended for representation purposes: we will call it an *instrumental* decomposition.

Decomposition representations have been employed in solid modeling and in finite element methods. In both fields, volumetric[2] approaches have been applied: the point set covered by the object (domain) is represented through a collection of basic elements.

In geometric modeling, also boundary representations, describing objects through their boundaries, have been used. In the two-dimensional case (i.e., for objects in the plane), this approach is trivial since it consists of representing polygons. In three dimensions, it corresponds to providing a two-dimensional decomposition of the surface bounding the object [7, 89, 120, 139, 169, 173].

Although classical decomposition representations have been developed first as tools for representing single-piece objects, sometimes they have also been applied as *structural* decompositions, i.e., for representing objects implicitly composed of meaningful parts. Being based on components having a simple topological structure, such representations permit to have each part represented as a collection of elements. For example, in boundary representations used in solid modeling, multiply connected faces of the object boundary are

[2]The term *volumetric representation* is used to mean an explicit representation of the point set corresponding to the object as opposed to other representations describing the object only implicitly (e.g., through its boundary). This is not referred only to three-dimensional objects but, more generally, to objects in any dimension.

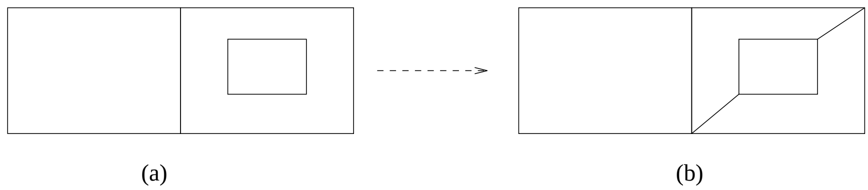


Figure 1.1: (a) The face on the right is multiply connected; (b) such face is partitioned into two simply connected faces.

segmented into simply connected faces (see Figure 1.1), being based on Euclidean cell complexes (see next Section).

Another method consists of representing component parts separately (possibly through a decomposition representation), and then glueing them through some assembling process [113, 155]. Examples are the so-called *Modular Boundary Models (MBMs)* which describe solids as a collection of face-abutting object parts, each represented by its enclosing surfaces [19, 50, 51]. In particular, the *Face-to-Face Composition (FFC)* model [19, 33] is a specific model belonging to this class which contains explicit information about connection and interference among object components. The elementary components in an MBM consist of planar-faced r-sets, i.e., subsets of $\mathbb{E}^3$ not containing dangling/isolated entities. This represents the main limitation of this kind of models for application to our purposes.

In the following, we describe the most widely used representations based on decompositions, i.e., representations based on Euclidean cell complexes.

1.1.1.1 Representations Based on Cell Complexes

Cell complexes in $\mathbb{E}^n$ (also Euclidean cell complexes) are collections of k-cells ($0 \leq k \leq n$). A k-cell is any subset of $\mathbb{E}^n$ ($k \leq n$) homeomorphic to a k-dimensional disk. The boundary of each k-cell ($k > 0$) is the union of cells which have a dimension lower than k. In a cell complex, pairs of cells must be either completely disjoint, or meet along their boundary. A cell complex is said to be n-dimensional if n is the maximum dimension of its cells. A *regular* n-dimensional complex is a complex in which each k-cell (for each $k < n$) belongs to the boundary of a $(k+1)$-cell (this excludes the presence of dangling/isolated entities). When the cells are simplices, i.e., each k-cell is a linear convex combination of $k+1$ points, the cell complex is said to be a *simplicial complex*. In three dimensions, a simplex is a tetrahedron. Then, a simplicial complex in the space is usually called *tetrahedralization*.

Simplicial complexes are widely used in practical applications, since they have nice combinatorial properties. Indeed, a regular n-simplicial complex is fully characterized by the collection of its n-simplices, while all other s-simplices, for $s < n$, can be obtained as a linear convex combination of vertices of n-simplices. Efficient algorithms have been proposed in the literature for computing simplicial complexes.

Originally, cell complexes have been defined in algebraic topology with the precise aim of simplifying the computation of topological invariants of continuous spaces. Indeed, the study of such invariants through point-set topology has been proven to be largely intractable due to the arbitrary nature of point sets and to the difficulty in enumerating all possible point sets. The solution to such intractability has been found by "approximating" continuous topological spaces through finitely describable complexes of topologically *simple* cells, i.e., cells homeomorphic to disks (see, for example, [2, 29, 166]).

The advantage of using cell complexes lies in the fact that theoretical results of algebraic topology provide means to check the consistency of representations defined on the basis of such mathematical concepts. Furthermore, in practical applications such as solid modeling, the main advantage consists of the possibility of separating topological aspects (concerning topological relations involving entities of the complex) from geometric information. Topology in a cell complex is based on combinatorial rules among the cells of the complex, while geometry is a metric description attached to any single cell, which embeds the complex in a Euclidean space. Actually, the geometric embedding may be the simplest part of the structure, since it may consist only of point coordinates (for example, for simplicial complexes embedded in $\mathbb{E}^3$, such information is sufficient). In geographic information systems, for instance, separating topology from geometry allows the system to maintain topological consistency among points and lines to be represented, in spite of numerical approximation concerning geometric computation [141].

Other application fields, in which representations based on cell complexes have been successfully applied, are computer graphics, finite element analysis, computer vision, robotics, image understanding, etc. (see [20, 25, 106], for examples of applications of such representations).

Several representations have been defined in recent years describing three-dimensional [60], and even n-dimensional cell complexes [26, 28, 64, 115, 116, 117, 148]. We must remark that many such representations describe cell complexes in $\mathbb{E}^n$ decomposing domains which are either regular sets or have $(n-1)$-manifold boundary[3]. In other words, such classical representations do not support the description of situations like cells with holes, cells with isolated/dangling entities in their interior, etc. A few proposals have also been provided for describing objects with non-manifold boundary [91, 128, 135, 136, 148, 170, 171].

Representations based on cell complexes can be basically subdivided into two main groups [26, 116] with respect to the way cells are "represented": either implicitly (*implicit-cell*

[3]An regular set or r-set in $\mathbb{E}^n$ is a set such that $A \equiv \overline{i(A)}$ (where $i(A)$ and $\overline{A}$ denote the interior and the closure of a set A, respectively), i.e., a subset having homogeneous n-dimensionality (and no dangling parts). An $(n-1)$-manifold is a subset of $\mathbb{E}^n$ where each point has a neighborhood homeomorphic to an $(n-1)$-dimensional disk.

representations) or explicitly (*explicit-cell representations*).

Implicit-cell representations are composed of a set of basic elements (e.g., *darts* [114, 115], *cell-tuples* [26], etc.) plus a set of functions acting on such elements (e.g., *involutions* [116], *switch operators* [26], etc.). Note that the implicit definition of cells may be explicit in a data structure derived from such representation scheme, and, however, it is always possible to retrieve all the cells composing the complex from such representations. Examples of implicit-cell representations are the *cell-tuple structure* [26] and *n-dimensional generalized maps*.

The *cell-tuple structure* [26, 27, 28] has been defined for modeling the topology of subdivisions of orientable/non-orientable manifolds in arbitrary dimensions (also called *subdivided manifolds*). The cells composing the complexes described by such representation are obtained as union of simplicial cells. Indeed, a correspondence between cell-tuple structures and such complexes has been established through the so-called *barycentric triangulation* (see [28] for further details).

Another representation for the description of n-dimensional cell complexes are the so-called n-G-maps [116]. More precisely, they allow the representation of *simplicial complexes structured into cells* (i.e., cell complexes which allow a barycentric triangulation). Here, the elementary cells are more general since they can have also incomplete boundary. N-G-maps are composed of a set of basic elements, called *darts*, on which element functions (*involutions*) act. A small set of basic operations for constructing such representations has been defined and, based on them, a modeler has been implemented [23]. Such representation has also been completely specified through algebraic specifications thus obtaining a safe and reliable software package [22, 23, 24, 62, 63]. Extensions of n-G-maps for dealing with more general domains as well as specializations for more restricted complexes are currently under study [79, 110, 111, 117].

Within explicit-cell representations, cells are maintained, together with some adjacency and incidence relations among them [64, 80, 148]. The topology of the subdivision can be schematically represented through a graph (or hypergraph), whose nodes correspond to cells of the subdivision, and whose arcs correspond to incidence/adjacency relations among such cells. The *incidence graph* proposed by Edelsbrunner [64] is the most classical among explicit-cell representations. Such structure is as well used for representing n-dimensional complexes with manifold domain. This model explicitly represents all i-cells of the subdivision (for $0 \leq i \leq n$) as nodes of the graph and, for each k-cell γ ($1 \leq k \leq n-1$), the node corresponding to it is joined to all nodes corresponding to $(k-1)$-cells as well as all nodes corresponding to $(k+1)$-cells incident at γ thus generating arcs of the graph.

An extension of the incidence graph has been proposed in [148] for describing non-manifold objects. In such representation, a semantic attribute is associated with each cell (i.e., with each node of the graph): a cell can be either *active* or *non-active*. Point sets associated with active cells are included in the point set defined by the represented complex. Such a point set is thus the union of all point sets associated with active cells (this can be, of course, a non-manifold set).

Within CAD/CAM applications, representations of non-manifold objects are very useful since they permit to represent objects of mixed dimensionality (objects which are not r-sets) possibly having internal structures and incomplete boundaries. Thus, they support modeling of mechanical parts, faces and curves (representing either construction aids or regions of contact between solids and interior faces): they decompose solids either into finite elements or into subsets exhibiting different physical properties.

An attempt to investigate pathological situations which are not captured by classical models and representations has been proposed by Pigot [136]. His intention is to detect a class of complexes with more general cells and domains. In particular, he analyses properties and limits of CW-complexes, in which only the interior of cells is homeomorphic to a (open) disk, thus allowing the presence of dangling lower-dimensional entities inside cells.

1.1.1.2 Decomposition Representations for GIS Applications

Decomposition representations have also been used in GIS applications, where some of the above mentioned structures developed within the context of solid modeling (for describing decompositions of the boundaries of solid objects) have been applied for representing plane maps.

In [87], decomposition representations for spatial entities in two dimensions have been classified into *cell-complex-based* representations and *graph-based* representations.

In particular, the need for a sound mathematical theory underlying representations for spatial entities involved in GIS applications was pointed out first by Corbett in [47] and White in [172], where the authors discuss how mathematical concepts from algebraic topology (e.g., cell complexes) help in solving specific problems arising in automated cartography.

As already remarked, in the majority of practical applications, the most commonly used complexes are simplicial complexes (triangulations), for their combinatorial and geometric simplicity. In [82], a structure based on simplicial complexes, the *cell graph*, has been exploited to solve the problem of preserving topological relations during affine transformations.

Although structures for generic complexes can be used also for triangulations, some special structures have been also developed in the literature, which can exploit nice characteristics of such decompositions. A widely used data structure for triangulations that exploits their combinatorial properties consists of storing, for each triangle, its three vertices and its three edge-adjacent triangles, in a consistent order [64]. Also, if augmented with one triangle incidence per vertex, it can be navigated efficiently: all topological relations between vertices and triangles are retrieved in optimal time.

A similar data structure has been proposed recently in the context of GISs to encode a trapezoidal map, which is called the *quad-view* data structure [81]. Such a data structure has been used to implement efficient traversal and overlay algorithms.

A few data structures have been proposed to encode general subdivisions with features such as holes and dangling lines. They are basically extensions of incidence graphs to non-regular situations. TIGRIS [96] is a commercial system whose underlying data structure is similar to the DCEL (*Doubly-Connected-Edge-List* [139]). A similar structure was also adopted in [54] by extending the DCEL with lists of isolated points and dangling edges attached to faces.

More recently, the problem of representing more general and complex geographic entities has been faced and, thus, the need to formally describe them arises. Some models have been proposed attempting to provide formal approaches to the description and manipulation of spatial entities.

In particular, in [176], a canonical model for areal objects is proposed. A generic area is represented by means of a tree structure. Set-theoretic and topological operations on areal objects are also presented. The model relies on an object-oriented approach (see also [72, 177]). Such a model is effective and allows to represent objects with holes, but it does not allow the presence of dangling and/or isolated entities inside them.

A number of other representations have been proposed for giving a comprehensive representation of the geometric structure of plane geographic maps (see, e.g., [54, 65, 82, 123, 135, 88, 175]).

In particular, when using a graph-based formalism (see Appendix C) [54, 88, 125, 157, 160], a map is defined as a decomposition of a portion of the plane into points, lines and regions, induced by embedding of a (possibly non-connected) planar graph in the plane: points correspond to vertices, lines correspond to edges, while regions correspond to faces of the graph. The use of plane graphs permits to model regions with holes as well as isolated entities inside regions with no need to apply any further decomposition on such regions.

A meaningful example of representation based on plane graphs is the so-called *realm* [86, 88, 157]. Realms are sets of points and line segments such that the coordinates of points are finite values.

By using the planar graph formalization it is possible to exploit related theoretical results. In particular, as seen in Appendix C, Euler formula relates the number of vertices, edges, faces and connected components of the graph:

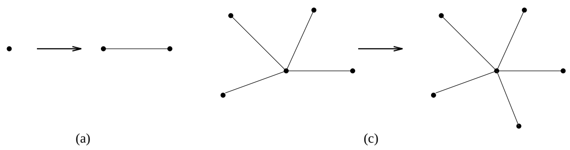


Figure 1.2: The MEV operator.

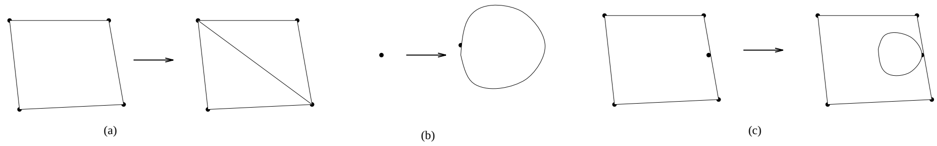


Figure 1.3: The MEF operator.

$$v - e + f = c.$$

On the basis of this formula, a set of operators have been defined, called *Euler operators*, which permit to build/destroy any plane graph, and ensure the topological consistency of the structure which is incrementally built, through the respect of the Euler formula at any step [120]. In other words, if an Euler operator is applied to a plane graph (in which Euler equation is satisfied), the result of the operation is another graph in which such equation is still satisfied. Euler operators only ensure the topological consistency (such as connectivity) of the graph, while they do not ensure the geometrical consistency (such as avoiding edge intersection). Three constructive Euler operators are defined:

- MVC (*Make-Vertex-and-Component*): it creates a new connected component, composed only of one vertex.
- MEV (*Make-Edge-and-Vertex*): given an existing vertex v, and a new vertex v', it adds the new vertex and an edge having v and v' as endpoints. No new faces are created (see Figure 1.2).
- MEF (*Make-Edge-and-Face*): given a graph, and two of its vertices v_1, v_2, belonging to the same face f, it adds an edge e joining v_1 and v_2. Face f is split in two faces (see Figure 1.3).

Such operators are necessary and sufficient to generate any plane graph. Converse destructive operators are called KVC, KEV, KEF (*Kill-Vertex-and-Component*, *Kill-Edge-and-Vertex*, *Kill-Edge-and-Face*, respectively).

By extending Euler formula, sets of operators for building boundary representations of solids have also been defined (see [120]).

Note that the advantage of using graphs lies in the availability of several data structures, as well as algorithms for efficiently encoding and manipulating them, developed in the computational geometry and geometric modeling literatures.

1.2 Formalisms for Topological Relations

There are major differences in dealing with spatial and non-spatial data. Classical databases organize and retrieve data through their values, while spatial databases organize data which are embedded in concrete geometric spaces and retrieve them through their spatial relationships (see [87] for a survey on spatial databases). Thus, tabular representations are not suitable for describing properties of entities geometrically defined and located in space. Spatial query languages must support spatial concepts. Among other properties of spatial data, relations based on topology, metric, order and directions seem to be fundamental concepts to be exploited when querying a spatial information system [73].

Topological relations are preserved under continuous transformations (e.g., translation, scaling, rotation). Metric relations exploit the existence of measurements such as distances, spatial proximity, etc. Order relations rely upon the definition of an ordering, while directional relations are characterized in terms of position of entities with respect to some oriented direction (e.g., north, south, right/left-of, etc.).

The topology of a set of spatial entities is a qualitative concept, independent of quantitative measures such as distances, sizes, etc. The basic topological relations are *neighborhood* and *overlapping* from which connectivity can be derived. Topological relations can of course be derived from positions of entities (with respect to some fixed coordinate system); however, positioning of objects relative to one another can be known without information on the exact (absolute) object positions and, above all, the amount of processing required to extract topological relations from absolute positions can be very large, while the amount of information needed to describe topology can be relatively small.

In geographic information systems, topological relations are usually explicitly stored for reasons of efficiency and consistency [93]. For example, in spatial information processing systems, an important criterion for determining whether a database state is correct or not is topology: neighbouring countries must have a common border, seas and lakes do not overlap, etc.

A great amount of work has been done related to spatial entities and their topological relations [54, 66, 67, 70, 74, 93, 101, 133, 175, 176].

In the following Sections, we describe the most widely used formalisms for representing topological relations and models for efficiently performing spatial queries based on such formalisms.

1.2.1 The 4- and 9-Intersection Formalisms

Topological relations between pairs of simply connected regions are usually classified according to the so-called 4-*intersection relations* [67], which are obtained combinatorially from a simple scheme involving the mutual intersections between interiors and borders.

In the following, we review a formal definition of such relations as proposed in [67] (as an extension of a previous work [140] on relations between one-dimensional intervals). Essentially, the model adopted by Egenhofer and Franzosa describes spatial relations between two subsets A and B of a topological space X on the basis of the four intersections of the borders and interiors of the two sets (denoted $b(A)$, $b(B)$, $i(A)$, and $i(B)$, respectively), i.e., $b(A) \cap b(B)$, $b(A) \cap i(B)$, $i(A) \cap b(B)$, and $i(A) \cap i(B)$.

The first intersection is called the *border-border* intersection, the second intersection the *border-interior* intersection, the third intersection the *interior-border* intersection, the fourth intersection the *interior-interior* intersection.

Definition 1.2.1 *Let A, B be a pair of subsets of a topological space X. A topological relation between A and B is described by a four-tuple of topological invariants associated to each of the four sets $b(A) \cap b(B)$, $b(A) \cap i(B)$, $i(A) \cap b(B)$, and $i(A) \cap i(B)$, respectively.*

Note that a topological relation between two sets is preserved under homeomorphism of the underlying space X. Specifically, if $f : X \to Y$ is a homeomorphism and $A, B \subset X$, then $b(A) \cap b(B)$, $b(A) \cap i(B)$, $i(A) \cap b(B)$, and $i(A) \cap i(B)$ are mapped homeomorphically onto $b(f(A)) \cap b(f(B))$, $b(f(A)) \cap i(f(B))$, $i(f(A)) \cap b(f(B))$, and $i(f(A)) \cap i(f(B))$, respectively. Since a topological relation is defined in terms of topological invariants of such intersections, it follows that the relation between A and B in X is identical to the relation between $f(A)$ and $f(B)$ in Y.

A commonly used representation for a topological relation is by means of a 2×2 matrix:

$$\begin{pmatrix} b(A) \cap b(B) & b(A) \cap i(B) \\ i(A) \cap b(B) & i(A) \cap i(B) \end{pmatrix}$$

The entries of the matrix correspond to properties of the four set-intersections that are invariant under homeomorphism. For example, the property empty/non-empty is set-theoretic, and therefore, topologically invariant. A lot of work and results obtained on the 4-intersection relations has been developed considering such simple, though general invariant [65, 66, 67, 70, 74].

By assigning the appropriate value empty/non-empty to the entries of the four-tuple, we have $2^4 = 16$ possible relations (see table 1.4).

	$b(A) \cap b(B)$	$b(A) \cap i(B)$	$i(A) \cap b(B)$	$i(A) \cap i(B)$
r_0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
r_1	$\emptyset$	$\emptyset$	$\emptyset$	$\neg\emptyset$
r_2	$\emptyset$	$\emptyset$	$\neg\emptyset$	$\emptyset$
r_3	$\emptyset$	$\emptyset$	$\neg\emptyset$	$\neg\emptyset$
r_4	$\emptyset$	$\neg\emptyset$	$\emptyset$	$\emptyset$
r_5	$\emptyset$	$\neg\emptyset$	$\emptyset$	$\neg\emptyset$
r_6	$\emptyset$	$\neg\emptyset$	$\neg\emptyset$	$\emptyset$
r_7	$\emptyset$	$\neg\emptyset$	$\neg\emptyset$	$\neg\emptyset$
r_8	$\neg\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
r_9	$\neg\emptyset$	$\emptyset$	$\emptyset$	$\neg\emptyset$
r_{10}	$\neg\emptyset$	$\emptyset$	$\neg\emptyset$	$\emptyset$
r_{11}	$\neg\emptyset$	$\emptyset$	$\neg\emptyset$	$\neg\emptyset$
r_{12}	$\neg\emptyset$	$\neg\emptyset$	$\emptyset$	$\emptyset$
r_{13}	$\neg\emptyset$	$\neg\emptyset$	$\emptyset$	$\neg\emptyset$
r_{14}	$\neg\emptyset$	$\neg\emptyset$	$\neg\emptyset$	$\emptyset$
r_{15}	$\neg\emptyset$	$\neg\emptyset$	$\neg\emptyset$	$\neg\emptyset$

Figure 1.4: The 16 specifications of binary topological relations based on the criteria of empty/non-empty intersections of borders and interiors.

Note that since a set is either empty or non-empty, such 16 relations provide complete coverage (i.e., for each pair of sets A and B in X, there is always one of the above topological relations between A and B). Also, since a set cannot be simultaneously empty and non-empty such relations are mutually exclusive (i.e., for each pair of sets A and B in X, there is exactly one of the above topological relations between A and B).

In general, each of the 16 relations can occur between two sets. Depending on various restrictions on the sets and on the underlying topological space X, the actual set of existing relations may be reduced to a subset of them. For general pointsets in $\mathbb{R}^2$, all 16 relations can be realized.

In [67], such relations have been classified into 6 groups (depending on the dimension of the involved sets): area/area, line/area, point/area, line/line, point/line, point/point.

Among such groups, the most widely studied and analysed has been the area/area group, especially when the involved regions are simply connected and the codimension (i.e., the difference between the dimension of the space and the dimension of the embedded spatial objects) is zero [66, 67, 70, 74, 76]. Only 8 of the 16 relations have been proved to be possible in such case, namely relations r_0, r_3, r_5, r_8, r_9, r_{11}, r_{13}, and r_{15} [67]. Such relations have been called: *disjoint* (r_0), *contains* (r_3), *inside* (r_5), *meets* (r_8), *equals* (r_9), *covers* (r_{11}), *coveredBy* (r_{13}), and *overlaps* (r_{15}), see Figure 1.5.

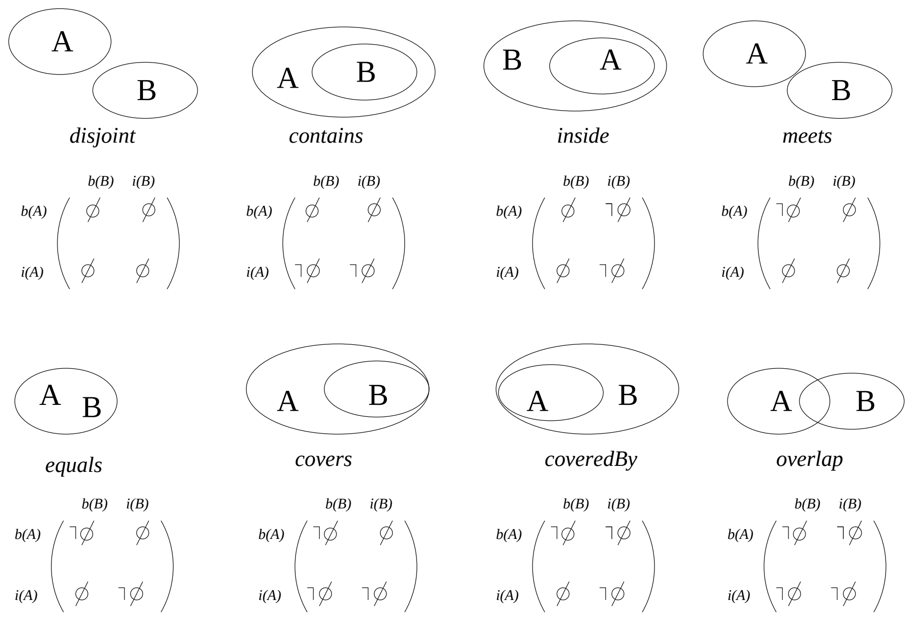


Figure 1.5: Binary topological relations for simply connected regions.

The same names are also suitable for other groups of topological relations; however, for example, also relations *bounds* and *boundedBy* are possible between entities of different dimensions, and such that one is contained in the border of the other [54] (see, for example, Section 2.2 for relations between entities in a map).

In [69, 70], properties of the area/area relations have been analysed: in particular, it has been shown that each of the eight possible relations between regions has a *converse* relation (which can coincide with the relation itself). More precisely:

- $A\ disjoint\ B$ iff $B\ disjoint\ A$
- $A\ meets\ B$ iff $B\ meets\ A$
- $A\ equals\ B$ iff $B\ equals\ A$
- $A\ overlaps\ B$ iff $B\ overlaps\ A$
- $A\ inside\ B$ iff $B\ contains\ A$
- $A\ covers\ B$ iff $B\ coveredBy\ A$

Since there are two pairs of converse relations (*inside* and *contains*, *covers* and *coveredBy*), the number of different types of relations is 6.

In [70], given a set of n objects, the $n \times n$ *connectivity matrix* has been defined whose elements correspond to the relations between pairs of objects. As a consequence of the mentioned converseness property, such matrix is symmetric. Furthermore, by considering other properties of the matrix, it is possible to find topological inconsistencies in the scene composed of such objects.

Over the last few years, the 4-intersection formalism has obtained considerable interest within GIS and database communities. Several researchers have developed models and techniques based on it: for example, Smith and Park [159] showed that the point-set relations form a *relation algebra*. The possible topological relations between regions are compared with order relations in [101, 102]. In [162] the 4-intersection model was incorporated into a spatial-analysis query language, while Abdelmoty et al. [1] have exploited it as a representation framework to be implemented in a deductive database. Herring [97] discussed extensions of the model to cover all possible topological relations between lines in $\mathbb{R}^2$ (i.e., with codimension one). Egenhofer and Herring [67], Hadzilacos and Tryfona [93], and Clementini et al. [45] investigated how the 4-intersections behave when applied to model all relations between regions, lines and points in $\mathbb{R}^2$. In particular, in [68] and [45], by applying simple geometric considerations (similar to those in [67] for the area/area case), it has been shown that line/area cases are only 11 because there are 5 impossible relations; line/line distinct cases are 12 because there are 4 pairs of converse relations. The possible relations are 3 for the point/area and the point/line cases and only 2 for the point/point case. Furthermore, Pigot [134] and Hazelton et al. [94] applied the model to 3- and 4-dimensional objects.

The sound model also provided a good basis for advanced theoretical studies such as the formal derivation of the *composition table* for this set of relations [67], the design of comprehensive reasoning systems to detect inconsistencies in topological descriptions [70]. In particular, in [75], the importance of providing consistency checking tools within multiple representations has been highlighted. Recently, Papadimitriou et al. [133] have shown that the closure (under appropriate logical operators) of the topological relations between pairs of simply connected regions permits to define languages which are complete for topological properties.

The 4-intersection model with the content invariant of empty/non-empty intersections is very simple, and thus it has obtained great success; however, it is sometimes too generic since it does not allow to distinguish among situations that humans usually consider different [45, 97, 98].

As an example, let us consider the *overlap* relation. Figure 1.6 shows situations that have the same 4-intersection matrix with empty/non-empty values but correspond to topologically distinct configurations. Other topological invariants may enable the distinction of more details about such relations (i.e., they can be seen as more restrictive classifiers). For example, the dimension and the number of connected components of non-empty inter-

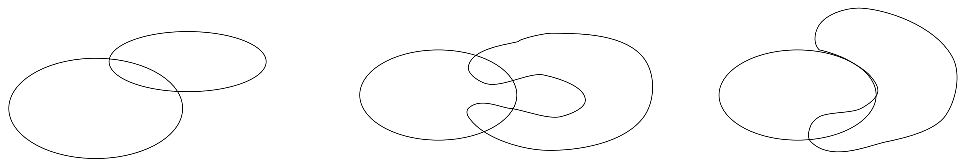


Figure 1.6: Topologically distinct configurations having same 4-intersection matrix.

sections, as well as the type of border-border component intersections (i.e., whether the connected arcs of border components run through different/equal parts of the other object) may be used as additional invariants [66, 75, 76]. In particular, in [46] the dimension of non-empty intersections has been considered for both the 4- and the 9-intersection model (described below). In [76], a set of invariants for border-border intersections has been detected that provides a classifying invariant for simply connected regions in the plane. This is relevant to spatial query processing since any query about topological relations between simply connected regions can be derived from it.

Egenhofer and Herring [67, 69] extended the principle of border and interior intersections with exterior intersections which provides a generic model (called *9-intersection model*) for describing topological relations among n-dimensional objects embedded in higher dimensional spaces.

The following 3×3 matrix with nine set intersections is thus considered for classifying a topological relation between two objects A and B:

$$\begin{pmatrix} b(A) \cap b(B) & b(A) \cap i(B) & b(A) \cap B^- \\ i(A) \cap b(B) & i(A) \cap i(B) & i(A) \cap B^- \\ A^- \cap b(B) & A^- \cap i(B) & A^- \cap B^- \end{pmatrix}$$

where A^- and B^- denote the exterior of A and B, respectively. By considering the empty or non-empty content of such nine entries, the total number of combinations is $2^9 = 512$. Excluding the impossible cases, only 68 relations correspond to real cases: 8 cases (6 if we do not consider converse relations) for the area/area group, 19 cases for the line/area group, 3 cases for the point/area group, 33 cases for the line/line group (reduced to 23 due to converse relations), 3 cases for the point/line group, and 2 cases for the point/point group. A comparison of the various 4-intersection and of the 9-intersection models (both with the content and with the dimension invariant) has been given in [46], where all possible cases for the various groups of relations have been illustrated, and the situations in which either model is more suitable have been described. An equivalent calculus-based method has also been proposed in the same paper.

Parts of the extensions to the 4-intersection model proposed in the literature have also been tested with human subjects to identify how closely the formalism represents human cognition [77]. The model has also been used for reasoning and investigating about gradual temporal changes of topological relations [71, 77]: a partial order (i.e., a reflexive, antisymmetric and transitive relation) is established on the set of relations and, by means of a graph-based formalism, an inference system (traversing the graph) predicts the topological relation existing between two objects after a slight change. In particular, in [71] the area/area case is analysed, while in [77] the so-called *conceptual neighborhood* for line/region relations is taken into account.

Most of the work developed around the 4- and 9-intersections was related to simply connected regions. However, in GISs, objects must be handled with more complex structures (i.e., containing holes, etc.). Some investigations about the relations between more complex geographic objects exist: for example, topological relations between lines that have more than two endpoints have been studied in [68], while topological relations between areal objects generalized to their convex hull have been defined in [48]. Also, a first attempt to defining topological relations between multiply connected regions has been performed in [67] by using the 4-intersection model (9 possible relations were identified between such regions: all relations occurring between simply connected regions, plus relation r_{14} that is called *overlap with disjoint borders*); in [68], the 9-intersection method is adopted for relations including also non-simple lines: 18 relations have been proved to be possible for the area/area case (among them, 10 can be realized only for regions with holes), 20 cases for the line/area group (one of which possible only if non-simple lines are allowed), and 33 cases for the line/line group if the considered lines are simple, plus another 24 cases occurring only for non-simple lines.

A more comprehensive study of topological relations between regions with holes can be found in [74]. For such regions, the classification is obtained by combining the 4-intersections among the simply connected regions interior to the outer borders (i.e., those obtained by eliminating the holes), and the simply connected regions corresponding to the holes [74]. More precisely, let A be a region with n holes $H_1^A, \ldots, H_n^A$. The generalized region A^* is defined as the union of A and all holes contained in A, i.e.,

$$A^* = A \cup (\cup_{i=1}^{n} H_i^A).$$

Thus, if A has no hole, $A^* = A$. By considering a hole as the closure of an inner interior, we can associate a region A with n holes to the group of $n+1$ simply connected regions $A^*, H_1^A, \ldots, H_n^A$ and consider relations between such regions. Thus, being holes treated as separate objects, topological relations between regions with holes are expressed in terms of topological relations between simple regions. In particular, in [74] rules for detecting only information on the relationship between such simple regions strictly necessary to completely retrieve the actual relation between the original multiply connected regions have been established.

1.2.2 Relation-Based Models

As we have seen, several models for spatial data (in particular geographic maps) are based on Euclidean cell complexes, where all entities are represented as cells. Such models combine both metric and topological information (represented by the coordinates of points and the geometry of lines in the plane, and the relations between pairs of entities, respectively) implicit in a geographic map. Representations of spatial configurations, which do not require a graphical presentation are complementary to cell-complex models. Such models are sufficient if no visual information are available: for instance, when spatial information are collected from verbal descriptions. The need to define this kind of models arises from recent studies developed in the context of a new research area called *Naive Geography* [78]. This field of study relates to defining formal models of the common-sense geographic world. The aim is the design of GISs that can be used without major training by wider and wider classes of users. The effort should be addressed towards the development of models representing geographic space in a way which is close to the conception/perception of such a space by common people.

Relation-based models are attempts towards such purpose. They focus more on the representation of spatial relations among entities than on the geometry of the individual entities. In other words, they employ qualitative considerations more than quantitative properties of objects [95, 179]. They are represented through a graph-based formalisms. Such models also support the representation of incomplete information and efficient spatial querying.

In [131] both cell complex models and relation-based models have been analysed and compared. A relation-based model for geographic maps has also been proposed that stores regions as nodes and relations among them as arcs. Geographic maps considered in [131] are partitions of the plane into (possibly multi-connected) regions. Thus, only *meet* relations can occur between regions. Nodes in the model correspond to simply connected regions. However, regions with holes can be represented in this models by defining a hierarchical representation where hierarchical links represent containment relations (see Figure 1.7 for an example).

Spatial relations between regions at the same level in the hierarchy are characterized by the following *component invariants* [76]:

- *boundary-boundary sequence*: it captures the order of the components of the boundary-boundary intersection;
- *boundary dimension*: it represents the dimension of each boundary-boundary component;
- *complement relationship*: it describes how a boundary-boundary component is related to the complement of the union of two regions.

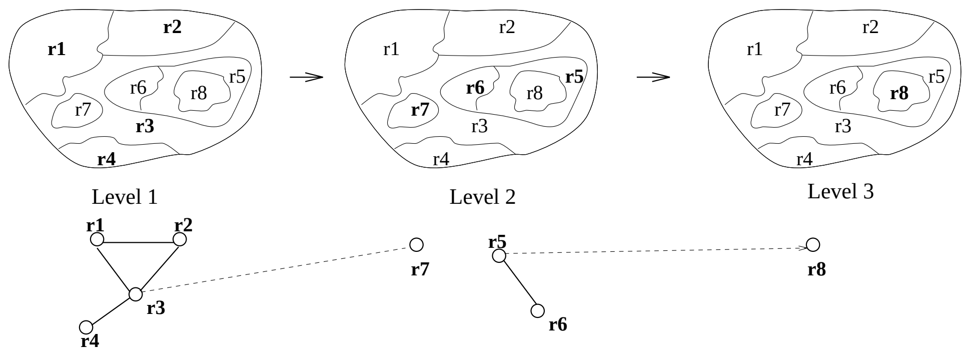


Figure 1.7: Hierarchical representation of a geographic map containing regions with holes: hierachical links (dashed arrows) correspond to containment relation between the surrounding region and the region filling the hole.

Since metric information are not stored in relation-based models, each such model corresponds to a class of cell complexes obtained by defining an equivalence relation that identifies complexes having the same topology. During this identification, some information (such as the shape of lines) on the represented map are lost.

In order to make a reasonable comparison of relation-based models and cell complex models, as well as to define mappings to generate one kind of models from the other, metric information should be added to relation-based models. However, such models are not efficiently extendible by enriching them through metric information.

An analysis has been performed on both kinds of models in terms of storage and computational complexity. The major results of such an analysis are the following: relation-based models are useful when they can be used as such (i.e., without adding extra information) and allow efficient spatial querying. They are more suitable as cognitive models to be used within Naive Geography since they allow the representation of incomplete information which is not, on the contrary, allowed when cell complexes are used. Obviously, since no positional information is stored, relation-based models have a lower space complexity. However, also cell-complex based representations can be defined independently of the specific embedding (i.e., metric information), as representation of the topological structure of a map (see *abstract cell complexes* defined in Chapter 3). Such models can be easily enriched through metric information by attaching them to each entity.

Although the entire work has been based on representation of two-dimensional entities in a map (regions), results can be extended to situations where also dangling/isolated entities are included in the map; such entities are stored as nodes in the graph.

The model described in this Section focuses on relations more than on entities. However, in this thesis we are interested in defining models that allow representations of single entities at different levels of detail, so it is essential to explicitly represent all entities in the geographic map, including their boundaries.

A relation-based model might be added to a spatial database as an auxiliary model for speeding up some queries but it is not sufficient in an environment where the representation of all entities of the map (with their boundaries) is required.

1.3 Multiresolution Models

Another topic of relevant interest in modeling spatial entities is related to the representation of data at multiple levels of resolution. In some applications, the possibility of analysing objects at different levels is useful, for example, for providing more detailed and/or accurate descriptions of parts of major interest. The importance of using multiple representations in GIS contexts was underlined in [35, 36]. Indeed, multilevel modeling offers interesting capabilities for spatial representation and reasoning: from support to automated cartography [83], to efficient browsing over large datasets, to structured solutions in wayfinding, planning and design [164], to rendering virtual reality environments [43, 55, 149].

More precisely, with the term *multiple representations* of a given object we mean sequences of representations of the object itself based on different decompositions. Each decomposition in the sequence corresponds to a representation at a given level of resolution. Such sequences can be built according to two different criteria, which depend on the way decompositions are intended and used. The refinement technique adopted to produce decompositions can be driven by either a *structural* or an *instrumental* point of view. In the former case, the decomposition is inherent the structure of the object represented. Different decompositions have different semantics and the level of resolution can be strictly related to the intrinsic topological structure of the described object. As an example, we mention multiple representations of geographic maps [141]. In such case, some objects that appear in two maps of the same area at different resolution can be described through entities of different dimension. However, the accuracy of the representation is also taken into account; thus both aspects, detail (which is concerned with changes in either the dimension or the topology of a given entity), and accuracy (which is concerned with metric changes of an entity) can contribute in changing resolution.

Instrumental multiple decompositions are used only as representation tools. Thus different decompositions do not have different semantics; they are distinguished only on the basis of their accuracy in the representation. As an example, in multiple decompositions used as a basis for approximations of a given object, the most refined decomposition corresponds to the lowest error in approximating the object shape (i.e., it corresponds to the most accurate representation). This is, for example, the case of decompositions used as domain discretizations for the definition of digital terrain models.

The great majority of multiresolution models proposed in the literature are related to *instrumental* multiple representations. This is certainly due to the fact that decompositions (essentially, cell complexes) have almost exclusively been defined and used with *instrumental* purposes.

Related to the problem of having multiple representations is the problem of linking different representations of the same object at different levels of resolution. By enriching multiple representations through this kind of information, we obtain a *multiresolution model*. By explicitly storing such information, efficient techniques can be developed to navigate the model "vertically" (i.e., through different resolutions).

Although deeply different, the instrumental and structural multiresolution models share some properties:

- each level is represented as a collection of elements related through connection/topological links;
- there exist interference links between different levels; interferences can correspond to
  - relations between different representations of a given portion of the domain (in the case of instrumental multiresolution)
  - conceptual links between different representations of a given part of the object (in the case of structural multiresolution)

Generally speaking, multiresolution models can be qualitatively distinguished according to relevant issues that have impact from modeling and computational points of view. Such issues must be taken into account in the analysis/comparison of multiresolution models. The main modeling issues for instrumental multiresolution models (applied to terrain representation) have been highlighted in [56]. They include:

- *shape of elements in the decomposition*: decompositions in a multiresolution model can be either *regular* (e.g., the elementary parts composing them are cubes) or *irregular*; regular decompositions are very advantageous from a computational point of view; however, such decompositions cannot be generally used when the structure of the object must be preserved;
- *adaptivity* and *data distribution*: multiresolution models are often used as compression mechanisms of the available data (on the basis of which they are built); indeed, the amount of data available in practical applications is actually quite high and thus, it is useful to have flexible methods for reducing the number of information required for describing the considered entities; some models more naturally adapt to the characteristics of the object to be described thus achieving better data compression;

- *level of resolution* and *size*[4]: usually it is not possible to achieve a given resolution with a given amount of data (i.e., a given size); a tradeoff must usually be found between the two aspects.

Processing issues are related to the detection of which set of operations must be (efficiently) performed on the model. Such issues have great impact on the design and implementation of the models through prototypal systems. Some basic operations can be detected that each multiresolution model should support. Of course, not all models efficiently support all such operations. The fundamental operations include:

- construction operations;
- extraction of a representation at an arbitrary level of resolution;
- navigation of the model both through different levels of resolution, and across a single level (this usually implies the evaluation of topological links between entities of the decomposition at such level);
- geometric queries (such as location of points or parts of the domain, etc.).

Different multiresolution models usually support different kinds of operations. An analysis of which operations are typical of instrumental multiresolution models has been done only for the specific application to surface multiresolution representation [56]. A systematic study of multiresolution models in more general cases which are applicable to other contexts has not yet been undertaken.

However, it seems quite reasonable to require the following two properties for a multiresolution model to be effective:

- the storage cost of a multiresolution model achieving a given maximum degree of resolution must not introduce a serious overhead with respect to a corresponding (single resolution) representation at the same highest resolution;
- algorithms for accessing and manipulating such model must be efficient.

In the following, an overview of the state-of-the-art of multiple representations is presented. Most of the work related to multiple representations of spatial entities has been developed within GIS applications.

[4]This issue can be applicable also to structural multiresolution.

1.3.1 Instrumental Multiresolution

In instrumental multiresolution models, the level of resolution is based on the accuracy in the representation. Among them, some work has been done in the development of multiresolution representations for the description, manipulation, and visualization of surfaces described by multivariate scalar fields. Two main applications are the representation of topographic surfaces (terrains) within GISs, and the visualization of volumetric datasets in volume rendering.

Multiple representations for surface approximation combine several descriptions of a given surface based on decompositions of the domain. They support description at different resolutions, and implicitly provide compression mechanisms for the available data.

Indeed, surfaces are usually represented by means of digital models: a digital surface model is characterized by a domain partition (a cell complex which provides a discretization tool for a continuous domain), and by a space of functions defined on such partition (each approximating the scalar field on a cell of the complex). Such models are intended as approximations of surfaces measured through some sampling process.

Thus, a "complicated" surface is piecewise represented using approximations through elementary patches (which can be either linear or polynomial). Here, decompositions are used only as means of "simplification" in the representation. The refinement of a given decomposition is just a tool for reaching the required accuracy in the representation: very small patches (hence, a great amount of patches) implies a more accurate approximation within the corresponding portion of the domain. The same approach is used in finite element analysis (see [158]) where a given domain is subdivided into simpler parts in order to compute some given properties with a certain accuracy. Thus, what is important is not which decomposition is used, but how refined it is: a better accuracy in the approximation is generally paid in terms of a finer decomposition of the domain, yielding higher storage cost and computation time.

The topology of the decomposition underlying a digital surface model can be either regular or irregular, hence classifying such models into *regular* (i.e., into cubic cells) and *irregular* models. A major drawback of regular models (although quite efficient from a computational point of view) is the requirement on data distribution that restricts their applicability to gridded data. Irregular models, on the contrary, do not impose any requirement on both data distribution and domain shape. Their advantage is in being more adaptive to irregularity of the described surface. Among them, those based on simplicial complexes are the most commonly used because they support linear interpolation within each simplex, thus guaranteeing the continuity of the approximation, and for their geometric and combinatorial properties.

From a computational point of view, the most important operations that instrumental multiresolution models should support are: the extraction of a representation at a given accuracy, the location of points or portions of the domain, etc. Of course, depending on the application, different models support different sets of operations to be efficiently performed.

Multiresolution surface models can be classified into two main classes, namely *Hierarchical Models* and *Pyramidal Models*. Hierarchical models are based on a recursive subdivision of the domain using either regular or irregular cells. Each cell is considered as an individual entity and refined independently of the others. A different approach to multiresolution surface description is provided by pyramidal models, in which the structure is not hierarchical but stratified. Here the refinement is globally performed on the whole domain instead of only on portions of it, thus allowing to preserve global properties.

Most literature on multiresolution models deals with the two-dimensional case. Complete and detailed surveys of multiresolution terrain models can be found in [57, 142]. Examples of hierarchical models are: the *quadtree* [152] (and its variants [5, 154, 167]) and the *quaternary triangulation* [4, 85], both based on regular domain subdivisions; the *ternary triangulation* [49, 137], the *adaptive hierarchical triangulation* [156], and the *hierarchical Delaunay triangulation* [53], which are based on non-regular domain subdivisions.

The first pyramidal model proposed in the literature is the *Delaunay pyramid* [52], which is based on a sequence of Delaunay triangulations satisfying progressively finer levels of accuracy. Other works following this approach have been proposed only recently. In [15] efficient data structures for the Delaunay pyramid were proposed. In [43] a multiresolution model similar to the Delaunay pyramid was presented, which also supports extraction of continuous surfaces at variable resolution. An alternative pyramidal model based on Delaunay triangulations has been proposed in [10], that is built according to a simplification technique (i.e., by following a bottom-up approach).

Recently, dimension-independence is becoming a more and more relevant issue in geometric modeling. Handling generic n-dimensional entities is, however, a complex task. Very few extensions of the multiresolution terrain models, in the direction of the representation of scalar fields in arbitrary dimension, have been proposed in the literature.

Three-dimensional generalizations of surface quadtrees can be obtained on the basis of *octrees* [42, 153], which require regularly distributed data and provide a domain subdivision into regular three-dimensional cells (cuboids) of variable dimensions.

A few other proposals include generalizations of two-dimensional models based on triangulations to the three- and general n-dimensional case. Indeed, multiresolution representation of three-dimensional scalar fields is a research issue currently under investigation, while the area is almost completely unexplored for higher dimensions. A hierarchical model for the representation of three-dimensional scalar fields based on tetrahedralizations has been proposed in [13] for applications to volume visualization. This model is an extension of the

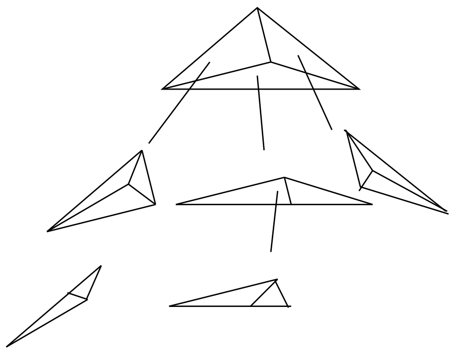


Figure 1.8: A hierarchical 2-simplicial complex.

hierarchical model proposed in [53]. The hierarchical model is described by means of a tree structure. Each level in the tree corresponds to a given approximation error. A 2-dimensional example is shown in Figure 1.8.

First attempts to multiresolution description in the generic n-dimensional case have been presented in [11, 12, 17]. Finally, a model, called *multilevel model* [18], has been proposed for giving a unifying framework for multiresolution hypersurface description. Such model allows to formally describe both hierarchical and pyramidal models based on simplicial complexes and is independent of the dimension of the described entities as well as of the specific application considered.

1.3.2 Structural Multiresolution

The problem of providing multiple representations in which the decomposition is coherent with the intrinsic structure of the represented object has not been widely studied. Here, the attention is focussed on the choice of the decomposition rather than on its level of accuracy.

The basic functionalities required in the model include: the extraction of a representation at a given level of detail; the navigation of each decomposition at a fixed level (through topological links) as well as the possibility of efficiently accessing more detailed description of a part of the object from a less detailed one (through inter-level links).

Structural multiresolution has been investigated especially within GIS applications. Some research has been undertaken in the last few years concerning the development of data models for the multiresolution representation of geographic maps [14, 30, 31, 32, 83, 141] (which can be seen as structured objects whose parts correspond to regions of the map). Some proposals have been presented that address multiple representations of spatial data in the context of GIS, concerning either the development of data models [14, 30, 31, 32, 83], or the assessment of consistency among different representations [70, 71, 75].

In particular, in [14] a structural multiresolution model for representing geographic maps has been defined. A further development of the model was presented by Puppo and Dettori in [141].

Such a model is represented by means of a tree of maps at different resolutions, where each map is the refined description of a simple region of its parent node in the tree. The basic representation for each component is the *Plane Euclidean Graph* (PEG), defined in [54], for describing maps containing generic polygonal regions with holes, isolated points and lineal features (some restrictions, however, have been imposed on possible relationships between such entities). Each node in the tree is a PEG corresponding to the refined map of a polygonal region in its parent node, also containing the refinements of its lineal features. The refinement criterion is based on the notion of homotopy, and imposes a consistency of the overall shape of a region, of its holes, its adjacencies in the map, and its features, throughout the whole hierarchy (see also Section 5.3, where we explicitely describe the refinement criterion and try to combine it with changes of detail formalized in this thesis).

The major contribution of such work consists of approaching the problem from a theoretical point of view by using formal tools. Also, the possibility to separate metric aspects from topological aspects in a map as well as to model them by means of different mathematical concepts has been underlined. However, some restrictions have been imposed on the model. In particular, only changes in accuracy were considered. Also, lines and regions can be refined but they cannot be represented through entities of different dimensions in different decompositions of the model. Furthermore, hierarchical links are maintained only between regions but not between lines.

A limitation of this work is the lack of a characterization of which functions correspond to consistent map transformations: no set of operators could be defined that generate all consistent transformation. Consequently, intrinsically consistent models could not be defined. Also, no data structure for the proposed model was provided.

Another example of multiple representations where the decomposition technique is dictated by the intrinsic natural structure of the described object has been proposed in [149]. Such a representation is aimed at very complex and fairly irregular CAD representations of mechanical parts. Thus, the considered objects are composed of parts, and their overall structure must be maintained throughout the sequence. A technique for generating multiresolution approximations of arbitrary polyhedra through triangulations is presented. Such a method is based on a simplification criterion that produces a sequence of approximation decompositions of the boundary of the polyhedron having a decreasing number of faces and vertices. The simplification process clusters vertices into groups; for each such group, a representative vertex is chosen according to some given criterion. Thus, original faces are approximated with fewer faces defined in terms of selected vertices. In other words, at a given level of resolution, i.e., in a given decomposition of the sequence, a group of faces will be represented through a single face (i.e., at a low accuracy), and then their representation will be refined (i.e., represented at a higher accuracy) at a further level. Thus, the decomposition is not only a means of approximating/representing the object at a given accuracy, but it also has a semantic meaning related to the structure of the object. By means of such multiple representations, it is possible to display some objects of interest at a higher resolution with respect to other less relevant objects within a composite scene.

Finally, another model based on simplicial complexes, that makes use of a more general refinement transformation, allowing the use of arbitrary simplicial complexes (e.g., in any dimension, non-orientable, non-manifold, non-regular), was recently presented in [138].

1.4 Analysis of the State-of-the-Art

In this Section, we try to critically analyse what has been proposed in the literature and to relate it with the main objectives of this thesis.

As remarked in Section 1.1.1, classical representations developed in geometric modeling were defined by assuming that the represented objects are composed of a single block (not further structured); thus, the corresponding mathematical models are point sets which capture the intuitive notion of a real object such as finiteness, rigidity, boundary determinism, and finite describability.

The most commonly used representations are based on *Euclidean cell complexes.* The main reasons for using cell complexes in describing geometric objects are the following:

- they provide a formal basis for developing sound descriptions of spatial objects; they give the possibility of exploiting various theoretical results and properties, thus providing means for checking the consistency of the developed representations;
- they permit to separate topological information from geometric information, thus allowing to study the two aspects independently of each other.

Since in classical representations an object is considered as composed of a single piece, and no intrinsic structure must be preserved, cell complexes are used only as instrumental decompositions (thus they can be arbitrary: it is not the specific decomposition, but the object it represents to be important).

However, in this thesis, we are mainly interested in applications where objects must be handled, which may have the following characteristics:

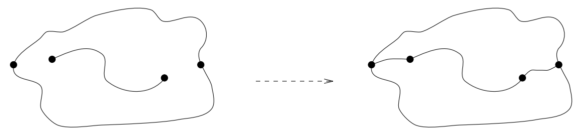


Figure 1.9: (a) The region contains a cut; (b) such a region is partitioned into two simple regions.

- they may be intrinsically structured, i.e., they correspond to (possibly non regular) point sets partitioned into parts;
- such parts may be topologically non-elementary and dimensionally non-homogeneous; for example, geographic maps may have multiply connected regions (e.g., regions with lakes) as well as cuts inside regions (e.g., rivers or roads crossing a region); similarly, mechanical pieces may have holes (e.g., bolts and nuts) and isolated/dangling parts (e.g., aerials and doors, which are three-dimensional pieces that can be treated as lower-dimensional ones).

Actually, the use of a decomposition (in which each element corresponds to a part of the object) seems to be quite a natural way for describing such objects. In this case, however, the decomposition has a semantic meaning and it is said to be used for a structural purpose; it cannot be arbitrarily chosen but it must preserve the structure of the object.

Note that the mathematical model should be changed from a "simple" point set into a decomposed point set. Such a model should permit to develop a both unambiguous and unique representation. If we had a bijective correspondence between pieces in the model and pieces in the object, then the representation would obviously be unique. Such a kind of model is still lacking in the literature. Indeed, cell complexes are not sufficiently powerful to model all relevant structures.

Representations based on cell complexes are unambiguous but not unique, since being only instrumental decompositions, different complexes can be used to describe the same object.

The major drawbacks and limitations in applying Euclidean cell complexes as models for structured objects arise from being composed of cells homeomorphic to disks. Therefore, if a single meaningful piece in the context of a structured object is not homeomorphic to a disk (e.g., it has holes and cuts), it will not be possible to represent such a piece with a single cell (see again Figure 1.1, and Figure 1.9, for another example). Besides imposing an arbitrary decomposition to the object, decompositions into simple cells can be expensive in terms of both computation and storage.

In Figure 1.10, a taxonomy of classes of cell complexes is depicted. Arrows go in the direction of classes of complexes having a greater expressive power. We recall that Euclidean

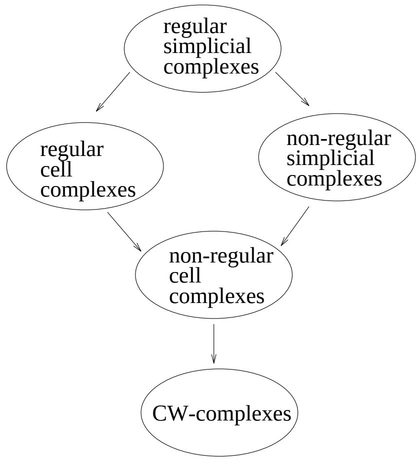


Figure 1.10: Classes of cell complexes.

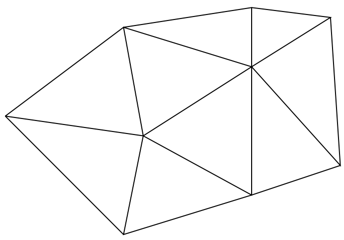


Figure 1.11: A regular two-dimensional simplicial complex.

cell complexes are composed of closed cells homeomorphic to disks. Regular simplicial complexes are the most widely used [28, 80, 82, 116] and do not allow the direct representation of both regions with holes and dangling/isolated entities (see Figure 1.11 for an example). Regular cell complexes are more powerful since they allow simply connected cells of any shape [64, 148] (see Figure 1.12 for an example). Non-regular cell (simplicial) complexes allow the presence of lower dimensional entities dangling outside highest dimensional cells [82, 136, 148] (see Figure 1.13). CW-complexes can be composed of more general cells. Indeed, only the interior of cells must be homeomorphic to an open disk. Thus, dangling entities inside regions are allowed [136] (see Figure 1.14). However, they do not allow the presence of either isolated entities inside regions or regions with holes. A more powerful tool is represented by *abstract cell complexes (ACC)* [141] which, as we will see Chapter 3, provide only a combinatorial description, independently of any possible embedding. However, we are interested in modeling geographic maps, which are objects embedded in the plane, so we need to characterize them by means of some embedding. We will consider ACCs corresponding to plane graphs (possibly having more than one connected component). Indeed, as already remarked, they support the representation of multiply connected regions, possibly with isolated and dangling entities in their interior. Furthermore, the use

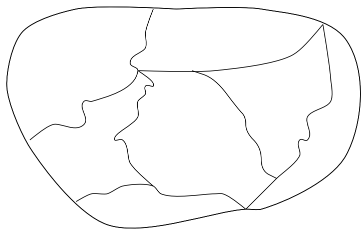


Figure 1.12: A regular two-dimensional cell complex.

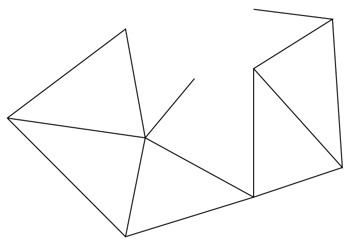


Figure 1.13: A non-regular two-dimensional simplicial complex.

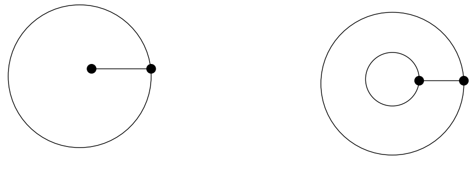


Figure 1.14: Cells allowed in a CW-complex, which are not allowed in a Euclidean cell complex.

of plane graphs permits not only to exploit sound theoretical results, but also to choose among a variety of data structures and algorithms already developed in the literature.

Another important subject in this thesis is concerned with multiresolution modeling. Historically, the first multiresolution models have been used for providing representations at different accuracies. Several proposals have been provided for the description of surfaces. However, for the purpose of this thesis, multiresolution models based only on accuracy are not sufficient. We are indeed interested in multiresolution in which the level of detail is of major importance. Very few models have been proposed related to this topic.

Basic representations in classical multiresolution models are based on cell complexes (used as instrumental decompositions) which, as we have seen, are not suitable for representing structured objects. Also, the multilevel organization is built on the basis of refinement/transformation of highest-dimensional cells (i.e., in two dimensions, regions). However, we also want to allow entities to change dimension when passing from one level to the other (e.g., a city is represented as a region at a more detailed level and as a point at a coarser level). Thus, the model we intend to define must allow transformation/refinement of entities of any dimension.

The first step in the direction of structural multiresolution is to define a suitable formal model for structural decompositions. The extension to the multiresolution case is the second step.

As we have seen, for the two-dimensional case, the most suitable approach seems to be the use of plane graphs. Using plane graphs as representations for maps, we can model changes in detail by means of basic transformations defined by Euler operators. However, Euler operators are not sufficient to formally describe all transformations for modifying the level of detail in a map.

More precisely, Euler operators allow to obtain every plane graph from a given plane graph, i.e., to perform any transformation between maps. However, when we apply Euler operators, we are not provided with a method to keep track of the relations between entities of the original graph and entities of the resulting graph. On the other hand, in a multiresolution model, such relationships must be explicitly maintained. Indeed, while browsing over a map at a given level of detail, more detailed representations of some entities in the map maybe required (i.e., zooming in certain portions of the map). Thus, it is necessary to have links between different representations of the same entities in order to navigate through the levels efficiently. For example, in the case of Figure 1.15, we want to keep a link between the line representing the *bridge* at the more detailed level, and the point representing it at the less detailed level; the same holds for the region and the line representing the *river* at the two different levels.

If the transformation is performed by means of Euler operators, it is possible to maintain a link between entities at different levels, but only in a canonical way (i.e., by establishing

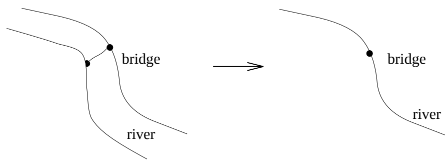


Figure 1.15: Transforming the representations of a bridge and a river.

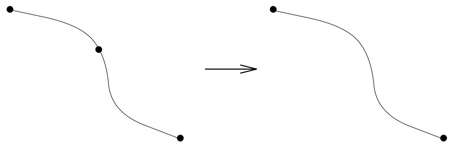


Figure 1.16: An example of a map transformation.

a convention for assigning an image to entities that are eliminated). In other words, Euler operators are not powerful enough to allow us to distinguish between situations that we consider as conceptually different. For example, the transformation shown in Figure 1.16 could be performed only with a *Kill-Edge-and-Vertex* operator (described in Section 1.1.1.2) applied to the corresponding graph.

However, such a transformation could correspond to two different situations (corresponding to different changes in detail): one in which the connecting point is eliminated and the two lines are merged (Figure 1.17(a)), and one in which a line is contracted to a point (Figure 1.17(b)).

In our formalization of changes in detail in a geographic map, we want to maintain these situations as separate, being conceptually different: merging two lines into one while eliminating the connecting point may correspond to the deletion of a *city* (represented by the point) from a map where the *road* (previously segmented into two lines for representation needs and then unified into a single line) crossing such a city is maintained (see Figure 1.18).

On the other side, the contraction of a line to a point may correspond to a situation in which a *bridge* represented by a line becomes a point when the underlying *river* is thinned

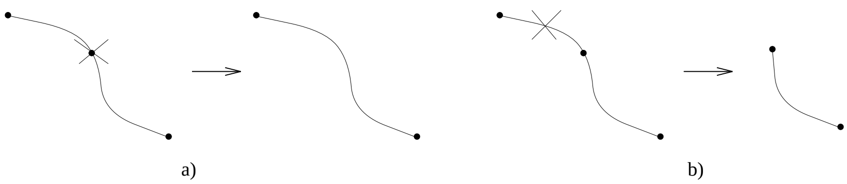


Figure 1.17: (a) Eliminating the point connecting two lines; (b) eliminating one line (contracting it to a point).

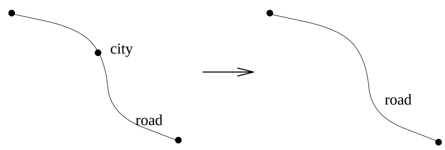


Figure 1.18: Eliminating a city from a map and merging the two branches of the road crossing it.

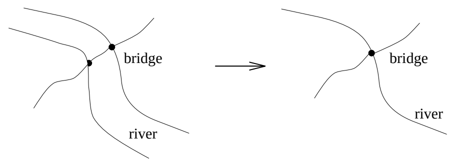


Figure 1.19: Contracting the line representation of a bridge into a point representation.

from an elongated region into a line representation (see Figure 1.19).

Thus, a different formalization must be provided, defining different sets of operators for manipulating graphs.

An important requirement that such operators must fulfill is the preservation of consistency. As mentioned in Chapter 1, map generalization provides a set of operations for abstracting details away from a map. Thus, it can be seen as a tool for generating multiresolution models for geographic maps. We must remark that, when updating operations are applied to maps, it is fundamental that they preserve the consistency of the map. Indeed, a critical point in geographic databases maintaining representations of the same area at different resolution levels is the preservation of topological consistency between the represented entities: it is essential that a query evaluation at a coarser level gives a result consistent with that obtained from the evaluation at the more detailed level. If the operators applied do not guarantee to generate consistent maps, *a posteriori* check must be performed. This is the approach commonly adopted in the map generalization literature [70, 75, 93], while current GISs lack methods to build intrinsically consistent multiple representations.

Chapter 2

Spatial Entities and Maps

In this Chapter, we provide a model for geographic maps based on concepts of point-set topology.

A plane map is defined as some kind of *covering* of a portion D of the plane $\mathbb{R}^2$, called the *domain* of the map, with a collection of spatial entities (points, lines, regions). It is possible to consider collections in which the composing entities are related in different ways (e.g., with different degrees of disjunction).

One of the most widely studied situations is when the entities are pairwise disjoint (*disjoint covering*) and form a plane subdivision, i.e., for each entity of the subdivision, also its border belongs to the subdivision, regions are simply connected and they fit properly together (i.e., they can meet only along common bounding lines). More general collections can be of interest for some applications. For example, in geographic maps, also multiply connected regions, as well as isolated, dangling points and lines inside regions must be considered.

We will define a plane map as a *weakly disjoint covering* which can be obtained by starting from any collection of entities through a regularization operation followed by an overlay process. In other words, we begin with a non-structured generic collection of spatial entities that we try to arrange into a consistent structure. Here we consider sets of entities having the topological structure of a planar graph, i.e., all relevant points are treated as nodes, all important connections joining such nodes as arcs, and the resulting plane graph satisfies Euler's formula.

The purpose of this thesis relates to modeling changes in resolution between maps. As already pointed out, two map representations at different resolution can differ in detail and/or accuracy. Here we are interested in describing changes in detail. The combinatorial concepts related to entities in a map defined in this Chapter will then be used in Chapter

3 where a formalization of the combinatorial structure of geographic maps is provided.

The Chapter is organized as follows: Section 1 defines the classes of spatial entities composing a geographic map, while in Section 2 topological relations between spatial entities are presented. In Section 3, the reasons for using sets of entities having particular structures instead of generic sets are discussed. Plane maps are defined in Section 4 as disjoint coverings of a portion of the plane. A method to obtain a plane map from any generic set of entities is proposed in Section 5, which consists of a regularization process followed by an overlay process. Section 6 is dedicated to describing possible topological relations between entities in a plane map. Finally, in Section 7, some combinatorial concepts are defined on entities of a plane map.

2.1 Spatial Entities

We consider collections of spatial entities of three classes, namely *points*, *lines*, and *regions* embedded in the Euclidean plane (i.e., each entity corresponds to a 0-, 1- or 2-dimensional subset of $\mathbb{R}^2$). We provide definitions of each spatial entity, and of related concepts (interior, border, closure, etc.: see also Appendix A for basic notions).

Definition 2.1.1 *A point is uniquely defined by its coordinates in a coordinate system on* $\mathbb{R}^2$.

The border of a point is empty while its interior coincides with the point itself[1].

Definition 2.1.2 *A (simple) open line* l *is a subset of* $\mathbb{R}^2$ *that is homeomorphic to the standard open unit interval* $(0,1)$ *on the real line.*

The (relative) *interior* of an open line l, denoted $i(l)$, is the line itself. In the following, whenever no ambiguity arises, we will denote both the homeomorphism and the corresponding line by l. By extending the homeomorphism by continuity to the closed interval $[0,1]$, we define $l(0)$ and $l(1)$ to be the *endpoints* of a line l (such points define the *border* of l, denoted $b(l)$). The line keeps on being simple if it is injective on $[0,1]$ or on both $[0,1)$ and $(0,1]$. If $l(0) = l(1)$, line l is said to be *closed.*

Note that a closed line plus its endpoint is homeomorphic to the standard unitary circle $S_1 = \{p \in \mathbb{R}^2 \mid ||p|| = 1\}$. A closed line plus its endpoint is also called a *loop.*

[1]More precisely, we should talk about the *relative interior* (instead of interior) of a point p in $\mathbb{R}^2$, because the interior of p is empty if p is regarded as a subset of $\mathbb{R}^2$ with the usual topology. However, in the following, if no ambiguity arises, we will talk about the interior of a point to mean its relative interior. The same remark holds for a line.

By Jordan theorem, a loop separates $\mathbb{R}^2$ into two open sets: one set is bounded, called the *internal set*, and denoted $\mathrm{int}(l)$; the other set is unbounded, called the *external set*, and denoted $\mathrm{ext}(l)$.

A (simple) *chain* is a sequence of lines $c = (l_0, \ldots, l_{k-1})$ such that $\forall i = 1, \ldots, k-1$, $l_{i-1}(1) = l_i(0)$ and $\cup_{i=0}^{k-1} l_i$ is a (simple) line. A chain is said *open* or *closed* if it is open or closed as a line, respectively. With abuse of notation, points $l_0(0)$ and $l_{k-1}(1)$ are called *endpoints* of c; all other endpoints of lines forming c are called *joints* of c.

A chain c is said a *free chain* if for any joint p of c, the only lines having p as an endpoint are those belonging to c. In other words, c meets no other lines, except possibly at its endpoints.

Definition 2.1.3 *A region r is characterised by a simple closed line (or chain) l_0, plus possibly a set $\{l_1, \ldots, l_k\}$ of simple closed lines (or chains), such that: (i) no two such lines intersect; (ii) for $i = 1, \ldots, k$ line l_i is contained in $int(l_0)$ and in $ext(l_j)$, $\forall\ 0 \neq j \neq i$. Region r is the subset of $\mathbb{R}^2$ defined as*

$$\cap_{i=1}^{k} ext(l_i) \cap int(l_0).$$

Line l_0 is called the *outer border*, lines $l_1, \ldots, l_k$ are called the *inner borders*; the union, as point sets, of lines $l_0, l_1, \ldots, l_k$ defines the Euclidean border of r (i.e., the border of r regarded as a subset of $\mathbb{R}^2$ with the standard topology), denoted $b(r)$.

Thus a region is the set of points internal to the outer border, and external to all inner borders. Inner borders define *holes* in the region. The number $\chi(r) = k + 1$ of closed lines defining a region r is called the *characteristic* of r. A region r without holes has $\chi(r) = 1$, and it is said *simply connected*, while a region with holes has $\chi(r) > 1$, and it is said *multiply connected*. The *interior* and *closure* of a region r correspond to the standard interior and closure of r regarded as a set of $\mathbb{R}^2$ with the Euclidean topology. Note that regions are open sets in $\mathbb{R}^2$ with the standard topology.

2.2 Spatial Relations

As we have seen in Chapter 1, spatial databases must store spatial properties of data. Among others, topological relations seem to be fundamental when querying a spatial information system.

Topological relations between a pair of simply connected regions are usually classified according to the so-called *4-intersection relations* [67], which are obtained combinatorially by considering the mutual intersections between interiors and borders. Only eight relations are possible: *disjoint, meet, contains, covers, equal, overlap, inside,* and *coveredBy*. For regions with holes the classification is obtained by combining the 4-intersections among the simply connected regions interior to the outer borders (i.e., those obtained by eliminating the holes), and the simply connected regions corresponding to the holes [74]. Topological relations can be extended easily to pairs of atomic entities, including also lines and points: in this case, also relations *bounds* and *boundedBy* are possible, between entities of different dimensions, and such that one is contained in the border of the other [54].

Note that, for each pair of entities, the total number of configurations can be very high. However, the actual number of existing relations may be reduced by applying simple topological considerations (also depending on constraints imposed on the set of entities). In particular, in Section 2.6 we will see that the number of relations between entities in a map is highly reduced.

2.3 Generic vs Structured Sets of Entities

As already stressed, in geographic information systems, some topological relations are usually stored explicitly for reasons of efficiency and consistency [103]. Topology is an important criterion for determining the correctness of a database state: neighbouring countries must have a common border, seas and lakes do not overlap, etc. Recomputing topology from geometric information each time a consistency test must be performed would yield an unacceptable overhead. However, it is important to note that if topological relations are required to be explicitly maintained, then all their parts (interior, border and exterior) must be stored as well.

In a generic collection of entities (with no constraint on possible intersections) it is not possible to establish criteria to sort topological relations out. On the contrary, if, for example, only neighbouring relations are allowed between regions of the set, it is possible to sort the list of regions sharing a border line with a given region r. If topological relations cannot be ordered, no efficient way for storing/retrieving them (as well as for answering topological queries) is applicable. Also, the number of topological relations identified among a generic set of entities is often too large for practical use (it can be quadratic in the number of entities).

For the above reasons, sets of entities having a particular geometrical/topological structure (i.e., those composing plane maps) are often dealt with in GIS, which are obtained from generic collections by means of some regularization and/or overlay operations (see Section 2.5).

An overlay of maps consists of producing a new map that is obtained by merging information (i.e., intersecting entities) contained in the input maps. In other words, the resulting entities inherit the thematic attributes of the input maps entities.

A map overlay process consists of two basic steps: (a) computation of the intersection points (and segmentation of the lines into parts remaining on either side of the intersection points); (b) reconstruction of the topology of the resulting map.

Overlay operations are required when a query involving entities whose thematic information are spread over two or more different maps, depending on their semantics, is performed. For instance, given a map describing the hydrography of Europe (in which atomic entities will be assigned semantic values like *lake* or *river*) and a map describing the countries of Europe, we may want to perform queries such as "which countries are traversed by a given river" [99]. More generally speaking, modern applications demand vertical integration of several thematic descriptions of a single region. Other examples are: the population of a country, the soil type of a terrain, the land cover/use of a geographic area, etc. This requires the interaction or overlay involving entities of multiple layers.

Map overlay is a complicated operation and it is expensive from the computational point of view. Thus, it seems more convenient to compute it only once, instead of having to recompute it each time such a query is being processed. Obviously, explicitly maintaining the map obtained through the overlay computation allows to efficiently solving such queries. On the other hand, this results in a greater amount of information to be stored, and in a greater segmentation of entities (i.e., original entities are subdivided into smaller pieces, and they must be reconstructed each time one wants to access them as a whole). Also adding an entity to such a map would involve overlay operations. As usual, a trade-off between storage and computation must be taken into account. Here, the problem is whether to prefer to efficiently inquiring maps or to efficiently building/updating them.

When entities to be handled do not intersect each other, the possible relations between pairs of them are highly simplified. This eases processing queries and, if the information on the original entities are preserved, their semantic meaning is not changed, but specified. Let us consider a road that crosses a river through a bridge: extra boundary points and lines delimiting the intersection area between the two strips will be introduced, thus subdividing the road into "road to the left of the river", "road on the bridge" and "road to the right of the bridge"; not only this is semantically correct, it also provides the chance to associate more precise information to the map entities.

2.4 Plane Maps

In this Section, we define a plane map and some related notions. In the following we denote by C a generic collection of spatial entities, each belonging to one of three sets: P, the set of points, L, the set of lines, and R, the set of regions.

Definition 2.4.1 *The subset of* $\mathbb{R}^2$ *obtained as the union of all entities of* C*, regarded as pointsets,*

$$D_C = (\bigcup_{p \in P} p) \cup (\bigcup_{l \in L} l) \cup (\bigcup_{r \in R} r)$$

is called the domain *of* C*, and* C *is said a* covering *of* D_C*.*

Definition 2.4.2 *A* disjoint covering *of a portion* D *of the plane* $\mathbb{R}^2$*, (called a* domain*) is a set* C *of entities such that the interiors of any two such entities never intersect (thus* C *forms a partition of* D*).*

In other words, if C is a disjoint covering, for each point $p \in D$, p is contained in exactly one of the entities of C.

We consider sets of entities satisfying a weaker constraint: isolated/dangling lines (with their endpoints) and isolated points can be completely contained in a region r: such lines and points are called *features* of r (the set composed of the features of r is denoted $ft(r)$), and are different from the remaining lines and points of the map in being not part of the border of any region, and being not endpoints of any line, respectively (see also [54]). More formally, we provide the following definitions.

Definition 2.4.3 *A line* l *is called a* lineal feature *of a region* r *if and only if* $i(l) \subset i(r)$ $(l \subset r)$*. A point* p *is called a* point feature *of* r *if and only if* $p \in i(r)$ $(p \in r)$ *and there is no line* l *such that* $p \in b(l)$*.*

Note that features are defined only for regions; we do not admit point features for lines since no point can intersect the relative interior of a line.

We define the *boundary*[2] of a region as the union (as pointsets) of the border and the features of the region. For a line and a point the boundary and the border coincide.

Definition 2.4.4 *Let* r *be a region and* $b(r)$ *its border. The* boundary *of* r *is defined as follows*

$$bd(r) = b(r) \cup \{p \mid p \in l,\ l\ line\ feature\ of\ r\} \cup \{p' \mid p's \in b(l),\ l\ line\ feature\ of\ r\}$$
$$\cup \{p'' \mid p''\ point\ feature\ of\ r\}$$

Figure 2.1 shows the difference between the border and the boundary of a region.

[2]Note that the term boundary is used to denote the union of border and features of a region and not as a synonymous of border. We choose this terminology because in the formalism we will adopt in Chapter 3, a relation is defined on a set of cells which is called *bounding relation* and also includes the features of a cell.

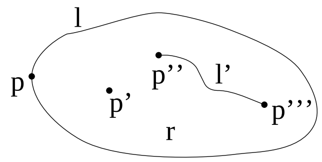


Figure 2.1: In this example, $b(r) = \{p\} \cup l$ and $bd(r) = b(r) \cup \{p'\} \cup \{p'', p'''\} \cup l'$.

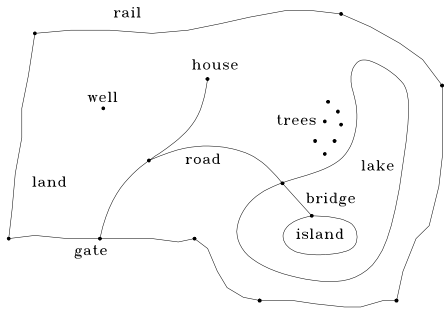


Figure 2.2: An example of a weakly disjoint covering: *land*, *lake*, and *island* are its regions; *roads*, *trees*, *house*, and *well* are features of *land*; *bridge* is a feature of *lake*; *rail* and *gate* form the border of *land*.

Definition 2.4.5 *A covering C of a domain D is called a* weakly disjoint covering *if and only if the interior of no region intersects the interior of a distinct region; the interior of no line intersects the interior of a different line; no point coincides with a different point, or intersects the relative interior of a line.*

An example of a weakly disjoint covering of a domain is shown in Figure 2.2. Note that in this case, for each point $p \in D$, p is contained either in a single entity, or in two entities, namely a region and a lower-dimensional entity.

A weakly disjoint covering C of a domain D can be represented as a triple (P, L, R), where sets P, L, and R contain the points, lines, and regions of C, respectively.

We also define the *free interior* of an entity as the interior of the entity without its features (regarded as pointsets).

Definition 2.4.6 *The* free interior *of a point or a line coincides with its interior. The* free interior *of a region r, denoted $i_f(r)$, is defined by $i_f(r) = i(r) \setminus (\cup_{x \in ft(r)} \overline{x})$.*

Note that the set composed of all points, of all (relative interiors of) lines and of all free interiors of regions of a weakly disjoint covering of a domain D form a disjoint covering (subdivision) of D.

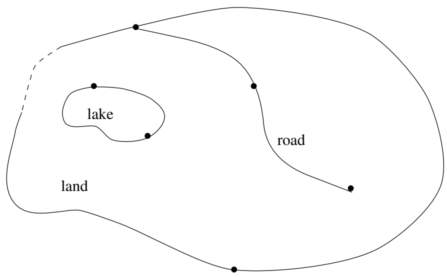


Figure 2.3: A non-regular set of entities.

Definition 2.4.7 *Let C be a set of entities and D_C its domain. C is said to be* regular *if, for each $e \in C$, $r(e) \subseteq D_C$ ($r(e)$ denotes the* regularization *of e).*

In other words, in a regular set C, for each region of C, a set of disjoint lines (with their endpoints) covering its border belong to C, and for each open line of C also its endpoints are entities of C. This fact implies that the domain of a regular set is a closed set in $\mathbb{R}^2$ with the standard topology. For example, the set of entities in Figure 2.2 is regular, while the set of entities in Figure 2.3 is not regular because some of the lines and points covering the border of *land* do not belong to it.

Definition 2.4.8 *A* plane map *(also* geographic map*) $M = (P, L, R)$ is a weakly disjoint covering such that $P \cup L \cup R$ is a regular set and regions in R correspond to interiors of pointsets obtained by regularizing all maximal portions of the plane such that any two points in their interior can be connected by a line that does not intersect any line of L.*

The definition of a region in a map M also includes the unbounded portion of plane surrounding the domain D covered by all other entities of M. Such region, called the *infinite region* [109] (denoted r_∞), is such that $r_\infty = \mathbb{R}^2 \setminus \mathrm{D}$. Note that, since we assume the existence of the infinite region, each line in a map either belongs to the border of exactly two regions or is a lineal feature.

Although M is a weakly disjoint covering of the whole plane, we will call D the domain of M. In the following, we will consider domains D such that $D = r(D)$ to avoid dangling and isolated entities in r_∞.

Also, from the definition of regions in a plane map (which actually are subsets of $\mathbb{R}^2$ conforming to the definition of region given in Section 2.1) it follows that all connected components of D are simply connected subsets of $\mathbb{R}^2$; in other words, all lines corresponding to holes of regions in D are filled up with regions (i.e., they do not bound the map). This implies the connectivity of r_∞. However, if D has k connected components, r_∞ has characteristic k and $b(r_\infty) = b(D)$.

Note that, according to the above properties of its set R of regions, a map M is completely characterised by its sets P and L of points and lines. The motivation for considering regions as obtained from regularizations of some pointsets, is due to the existence of point/lineal features. Furthermore, in this way, we do not allow a region to be subdivided into two disjoint portions by a lineal feature. Indeed, we consider such two portions as different regions. This, for example, means that all closed lines/chains in the map are intended as borders of regions (i.e., no closed line can be a lineal feature of a region). On the other hand, we allow a lineal feature to have both its endpoints on the border of the region r containing it, if this does not disconnect r (see, for example, feature *bridge* in Figure 2.2).

We do not allow point features on a line l for the same reason: they would disconnect l.

2.5 Regularization and Overlay

In this Section, we show how a plane map can be obtained from a generic collection of entities by applying regularization and overlay operations. Regularization is necessary to get a regular set of entities. Overlay permits to obtain a disjoint covering.

Given a generic collection C of entities, we first apply a *regularization* operation in order to obtain a regular set:

- for each region $r \in R$, all lines composing $b(r)$, which are not already included in L, are added to L;
- for each line $l \in L$, all endpoints of l, which are not already included in P, are added to P;
- all regions filling holes as well as r_∞ (if they are missing) are added to R;
- all dangling/isolated points and lines inside r_∞ are eliminated from P and L.

A map can be obtained from a regular set through an *overlay* process involving only lines and points. Several map overlay algorithms have been developed in the computational geometry literature for the case of plane subdivisions (i.e, partitions of the plane into polygonal faces, with no dangling/isolated entities). Such algorithms essentially exploit methods for reporting segment intersections, enriched with additional information for reconstructing the topology of the output subdivision [9, 21, 38, 39, 40, 44, 90, 107, 108, 119, 129, 132]. Some methods proposed in the GIS literature mainly consist of heuristic "filters", which try to reduce the number of pairs to be tested if a brute force approach is adopted [84, 92, 130, 152]. Although the filter are not guaranteed to be effective, these techniques are commonly used since they show a good practical performance.

In the following, we describe a simple overlay method for obtaining a plane map from a regular set (P, L, R): no efficiency requirements are taken into account.

The following steps must be performed:

1. for each $l, l' \in L$, if (the interior of) l and l' intersect, then they are replaced in L by all lines composing the three following sets: the connected components of $l \setminus l'$, the connected components of $l' \setminus l$, and the connected components of $l \cap l'$ (such lines are deprived of their endpoints); furthermore, each point p which is a connected component of $l \cap l'$ is added to P (and it becomes an endpoint of a line in the connected components of $l \setminus l'$ and of a line in the connected components of $l' \setminus l$); also, for each line l'' in the connected components of $l \cap l'$, its endpoints are added to P (and each of them becomes an endpoint of a line in the connected components of $l \setminus l'$ and of a line in the connected components of $l' \setminus l$);

2. for each $l \in L$, $p \in P$, if p is contained in (the interior of) l, then l is subdivided into two open lines l_1 and l_2, both sharing p as an endpoint (i.e., $p \in b(l_1)$ and $p \in b(l_2)$);

3. for each $p \in P$, $p' \in P$, if p and p' coincide, then only one of the two points is maintained, while the other is eliminated from P.

At the end of this process, the set L and P of lines and points have already been updated. The initial set R of regions is replaced by a set containing all maximal portions r of the plane such that any two points interior to r can be connected by a line that does not touch any line of L (such regions are regularized). Basically, what we obtain is a partition of the plane having the structure of plane graph (see Appendix C).

As already pointed out, the new spatial entities inherit the semantic/thematic attributes associated to the initial (input) entities they are derived from. For instance, each region of the resulting set may be contained in one or more of the input regions, thus it inherits attributes of such regions.

2.6 Relations in a Plane Map

The number of possible topological relations that can occur between entities in a weakly disjoint covering is highly reduced with respect to the case of a generic collection of spatial entities.

More precisely, we can only have the following situations (summarized in Table 2.1):

- for each $p, p' \in P$, p and p' are *disjoint* (since no two points can coincide);

- for each $p \in P$, $l \in L$, either p and l are *disjoint* or p *bounds* (i.e., it is an endpoint of) l;
- for each $p \in P$, $r \in R$, we can have one of the following cases (which correspond to all possible relations between a point and a region since no constraint has been imposed in this case):
  - p and r are *disjoint*;
  - p *bounds* r (i.e., it is endpoint of some line on the border of r);
  - p is *inside* r (i.e., p is either a point feature of r or an endpoint of a lineal feature of r);
- for each $l, l' \in L$, they are either *disjoint*, or they *meet* at a common endpoint (since no line can intersect another line);
- for each $l \in L$, $r \in R$, if l is open, we can have one of the following cases:
  - l and r are *disjoint*;
  - l and r *meet* at a single point p, such that p *bounds* both l and r;
  - l *bounds* r;
  - l is *inside* r (i.e., l is a lineal feature of r and no endpoint of l *bounds* r);
  - l is *coveredBy* r (i.e., l is a lineal feature of r and at least one endpoint of l *bounds* r).

  Note that if l is closed, then it cannot be a feature of r.
- for each $r, r' \in R$, r and r' are either *disjoint* or they *meet* at a common border line or point (i.e., the dimension of the intersection can be either 0 or 1).

For each non-symmetric relation listed above, the *converse* relation is verified too (e.g., l is *coveredBy* r if and only if r *covers* l).

2.7 Combinatorial Concepts in a Plane Map

We give the concepts of *combinatorial border*, *combinatorial boundary* and *combinatorial coboundary* (or *star*) of entities in a map M, where each spatial entity is regarded as an atom. The combinatorial border differs from the topological border defined before in being a collection of entities, rather than a subset of $\mathbb{R}^2$. The combinatorial boundary includes also point and lineal features.

	point	**line**	**region**
point	*disjoint*	*disjoint* *bounds*	*disjoint* *bounds* *inside*
line	*disjoint* *boundedBy*	*disjoint* *meet*	*disjoint* *bounds* *meet* *inside* *coveredBy*
region	*disjoint* *boundedBy* *contains*	*disjoint* *boundedBy* *meet* *contains* *covers*	*disjoint* *meet*

Table 2.1: Topological relations in a plane map.

The star of a point is formed by all lines and regions having it on their boundary (it is the region containing the point in its interior, for an isolated point) plus the point itself; the star of a line is formed by the (at most two) regions having it as part of their boundary (it is the region containing the line in its interior, for a lineal feature) plus the line itself; the star of a region is the region itself.

Furthermore, the set $ft(r)$ of features of a region r defined in Section 4 is a combinatorial concept.

Definition 2.7.1 *We define the* combinatorial border, combinatorial boundary *and* star *of an entity e, denoted $\beta(e)$, $\partial(e)$ and $*e$, respectively, as follows:*

$$\beta(e) = \begin{cases} b(e) & if\ e \in P \cup L \\ \{l \in L \mid l \subseteq b(e)\} \cup \{p \in P \mid p \in b(e)\} & if\ e \in R \end{cases}$$

$$\partial(e) = \begin{cases} \beta(e) & if\ e \in P \cup L \\ \beta(e) \cup ft(e) & if\ e \in R \end{cases}$$

$$*(e) = \begin{cases} \{l \in L \mid e \in b(l)\} \cup \{r \in R \mid e \in b(r)\ or\ e \in ft(r)\} \cup \{e\} & if\ e \in P \\ \{r \in R \mid e \subseteq b(r)\ or\ e \in ft(r)\} \cup \{e\} & if\ e \in L \\ e & if\ e \in R \end{cases}$$

In the sequel, we will always omit the adjective combinatorial, whenever no ambiguity arises.

We underline again that, since we have assumed the existence of the infinite region, each line belonging to the border of some region has exactly two regions in its star; the only lines having only one region in their star are lineal features.

Remarks

In this Chapter, we have defined the concept of plane map and related notions such as topological relations in a map, combinatorial border and boundary, and star. In Chapter 3, on the basis of the concept of star, we make an abstraction over a map M, by considering only its combinatorial structure, while completely disregarding its metric structure. This fact will allow us to formally define and study purely combinatorial aspects separately from purely metric ones.

Chapter 3

The Combinatorial Structure of Geographic Maps

In this Chapter, we use formal tools to provide a rigorous characterization of the combinatorial structure of geographic maps and to exploit theoretical results to show their properties.

In [141], *abstract cell complexes (ACCs)* (i.e., concepts defined in algebraic topology [166]) were exploited for modeling the whole combinatorial structure of geographic maps, independently of their geometry. We adopt such formalization and elaborate on it. In the following, we first introduce abstract cell complexes, and show how they capture the combinatorial structure of maps. ACCs are basically sets of cells on which an ordering relation is defined. Thus, they can be seen as topological spaces with the topology induced by a preorder (see Appendix B).

The main results obtained by using such formalization are the following. *Combinatorial map transformation*, i.e., an operation that relates two consistent maps at different detail, can be expressed through a function between the ACCs representing the maps. In this Chapter, a set of functions between abstract cell complexes is detected, which corresponds to the set of consistent map transformations modeling the combinatorial aspect of some generalization mappings between geographic maps. On this basis, an innovative categorical characterization of ACCs is obtained.

However, not all ACCs represent geographic maps. Thus, we refine the category of ACCs into a subcategory that captures specific characteristics of geographic maps. Within such a category, minimal morphisms are defined. The set of minimal morphisms includes all morphisms that cannot be obtained as composition of other morphisms of the category. Conversely, each non-minimal morphism of the category can be obtained as finite composition of minimal morphisms, i.e., they are sufficient to generate all morphisms of the category (which represent the consistent map transformations). The set of minimal morphisms is partitioned into classes: morphisms within a single class are characterized by an action that transforms the map locally, while leaving the rest of it unchanged. Each such class is called a *minimal operator*. Minimal operators of the category of GACCs will be characterized in Chapter 6.

We also define an ordering relation on the set of objects of the category, i.e., we define a poset of GACCs. More precisely, such a relation is defined on the set of all ACCs and thus induced on the subset of GACCs. In this way, we can evaluate the "diversity" between two (comparable) GACCs as their minimum distance in the poset. Such a relation is defined in the following way: two objects a and b are in relation if there exists a morphism f in the category mapping a onto b. Thus, by exploiting minimal operators that generate f, we have a way to measure the distance between a and b in the poset (see Chapter 6).

The structure of the Chapter is organized as follows. In Section 1, we introduce the concept of abstract cell complex and related notions. In Section 2, we show how ACCs can represent the combinatorial structure of maps. Section 3 is dedicated to describing how the topology induced by a preorder provides an interpretation of an ACC as a topological space. In Section 4, the set of functions between ACCs formalizing the combinatorial/topological aspects of consistent transformations between geographic maps is described. The category of ACCs is defined in Section 5, while Section 6 introduces the subcategory of GACCs an related concepts.

3.1 Abstract Cell Complexes

In this Section, by following the formalization proposed in [141], we define abstract cell complexes and related notions, that will be used in the following sections [106, 118].

Definition 3.1.1 *Let C be a finite set, called the* set of cells. *An* abstract cell complex (ACC) $\Gamma = (C, \preceq, \dim)$ *with cells in C is defined as follows:*

- $\preceq$ *is a partial ordering on the elements of C (i.e., $\preceq$ is a reflexive, antisymmetric, and transitive binary relation) called the* bounding relation*;*
- $\dim : C \to \mathbb{N}$*, called the* dimension function*, is such that*

$$\gamma' \preceq \gamma'' \wedge \gamma' \neq \gamma'' \Rightarrow dim(\gamma') < dim(\gamma'').$$

A cell γ for which $\dim(\gamma) = k$ is called a *k-cell*. It is not restrictive to have dim such that there always exist some cells of dimension zero. An abstract cell complex is called *d-dimensional* complex or a *d-complex* if $\max_{\gamma \in C}(\dim(\gamma)) = d$.

Note that the bounding relation defines an ordering on the set of cells of an ACC. Thus, basically a d-dimensional ACC is a poset with a dimension function $\dim : C \to \{0, \ldots, d\}$, i.e., a poset partitioned into at most $d+1$ sets (one for each possible dimension) and with chains of length at most $d+1$.

The *boundary* of a cell γ of Γ is defined as $\partial(\gamma) = \{\gamma' \in C \mid \gamma' \preceq \gamma\} \setminus \{\gamma\}$.

The *star* of a cell γ of Γ is defined as $*\gamma = \{\xi \in C \mid \gamma \preceq \xi\}$.

The overloading of terms *boundary* and *star* - that were already defined in the context of maps - is intentional here.

Note that for each $\xi \in *\gamma$, $\dim(\gamma) \leq \dim(\xi)$ and for each $\gamma' \in \partial(\gamma)$, $\dim(\gamma) > \dim(\gamma')$.

A *subcomplex* $\Gamma' = (C', \preceq', \dim')$ of a given complex $\Gamma = (C, \preceq, \dim)$ is a complex whose set C' is a subset of C, and relation $\preceq'$ and function $\dim'$ are restrictions of $\preceq$ and $\dim$ to C', respectively. Thus, a subcomplex is uniquely determined by the associated subset of cells: we will denote a subcomplex Γ' of Γ either by $\Gamma' \subset \Gamma$ or by $C' \subset C$.

A subcomplex Γ' of Γ is *open* if for every cell γ of Γ' all cells of the star of γ in Γ are also cells of Γ'. A subcomplex Γ' of Γ is *closed* if for every cell γ of Γ' all cells of the boundary of γ in Γ are also cells of Γ'. A subcomplex Γ' of Γ is *regular* if each cell γ of Γ' belongs to the boundary of some cell γ' of Γ' that has maximal dimension (possibly with $\gamma' = \gamma$).

Let $\Gamma = (C, \preceq, \dim)$ be an ACC. We say that Γ is *connected* if and only if, by closing $\preceq$ first for symmetry and then for transitivity, it becomes a total relation on C. Whenever no ambiguity arises, we will say that C (instead of Γ) is connected.

3.2 Maps as Abstract Cell Complexes

As stated by Puppo and Dettori in [141], it is possible to define an abstract cell complex on a map, which retains the combinatorial structure of M, while disregarding all its metric aspects. The proof of the following proposition is trivial.

Proposition 3.2.1 *Let $M = (P, L, R)$ be a map. Let us define $C = P \cup L \cup R$. Let $\preceq$ be a relation on C defined as follows:*

$$\forall\, \gamma, \gamma' \in C, \quad \gamma \preceq \gamma' \Leftrightarrow \gamma \subseteq \overline{\gamma'},$$

where the symbol $\subseteq$ denotes containment between sets of $\mathbb{R}^2$. Let $\dim$ be a function defined as follows:

$$\dim(\gamma) = \begin{cases} 0 & \text{if } \gamma \in P \\ 1 & \text{if } \gamma \in L \\ 2 & \text{if } \gamma \in R \end{cases}$$

Then, $\Gamma = (C, \preceq, \dim)$ *is a 2-dimensional abstract cell complex.*

Given a 2-complex Γ corresponding to a map M, we define the set of *features* of a 2-cell γ of Γ as follows:

$$\xi \in ft(\gamma) \Leftrightarrow \{\xi' \in (*\xi \setminus \{\xi\}),\ dim(\xi') = 2\} \equiv \{\gamma\}$$

Note that $ft(\gamma) \subset \partial(\gamma)$. The set of features of both a 0- and a 1- cell is empty. The *border* of a cell γ of Γ is defined as

$$\beta(\gamma) = \partial(\gamma) \setminus ft(\gamma)$$

In the following, we will always consider 2-complexes, thus we will omit the dimension. Also, from now on, when dealing with purely combinatorial aspects of maps, we will use interchangeably a map M, and its associated cell complex Γ, whenever no ambiguity arises. Also, we will speak of a *submap* of M by meaning its associated subcomplex $\Gamma' \subset \Gamma$. Therefore, we will speak of points, lines, and regions of a complex, by meaning its cells of dimension zero, one, and two, respectively. Also, all figures illustrating examples of ACCs will be drawn interpreting them as maps (although, as we will see, not all ACCs correspond to geographic maps).

Note that purely geometric concepts defined on spatial entities, such as simple line, relative interior, portion, internal and external set, outer and inner border, hole, interior, and closure, have no meaning in the context of an abstract complex. Nevertheless, some concepts like endpoints, open and closed line, chain, characteristic of a region, and, thus, simply and multiply connected region, have a straightforward translation in the context of an abstract complex. For example, a cell γ is an open line if and only if $\dim(\gamma) = 1$, and there exist two cells σ, σ' such that $\sigma \neq \sigma'$, $\dim(\sigma) = \dim(\sigma') = 0$ and $\beta(\gamma) = \{\sigma, \sigma'\}$ (conversely, it is closed if $\beta(\gamma) = \{\sigma\}$).

It is interesting to notice that in abstract cell complexes we are still able to say whether a region has holes or not, although we cannot decide which adjacent regions lie inside the holes, and which in the outer space: there is indeed no concept of outer space (unless we define the infinite region). Since in a map M we have assumed the existence of the infinite region, we label the 2-cell corresponding to r_∞ in the complex Γ describing M in order to distinguish it from all other 2-cells. This allows us to distinguish between concepts like interior, exterior, etc. (see also [109], for a discussion on the importance of defining the infinite region in order to obtain lossless representations). The concept of characteristic is extended to a regular subcomplex (and its relative submap) by counting the number of closed chains forming the boundary of the subcomplex, as defined before.

3.3 Abstract Cell Complexes as Topological Spaces

An abstract cell complex Γ is a set with an ordering relation (i.e., a reflexive, antisymmetric and transitive relation); thus, we can see it as a topological space with the topology induced by a preorder (see Appendix B for related notions).

It is easy to see that for any cell $\gamma \in \Gamma$, its star $*\gamma$ is the smallest open neighbourhood of γ (i.e., it corresponds to U_γ); hence, all cells with maximum dimension are open sets. Also, stars of cells in the complex are a basis of open sets for this topology.

The fact that Γ is a topological space allows the exploitation of results of topology - in particular, results concerning functions between topological spaces and homeomorphisms - to study the combinatorial structure of abstract cell complexes.

By means of functions and homeomorphisms between cell complexes (as topological spaces) we can compare spatial maps. Let $\Gamma = (C, \preceq, \dim)$ and $\Gamma' = (C', \preceq', \dim')$ be two cell complexes. We call *mapping*[1] between Γ and Γ' an application $f : C \rightarrow C'$. With abuse of notation, we will indicate $f : \Gamma \rightarrow \Gamma'$.

A mapping f defined as above is said *continuous* if for each U open set of Γ' the inverse image $f^{-1}(U)$ is open in Γ. In particular, f is continuous in $\gamma \in \Gamma$ if $f^{-1}(*f(\gamma))$ is an open neighborhood of γ. As shown in Appendix B, this is equivalent to saying that f preserves the ordering, i.e., it is *monotonic*, in the terminology of functions between posets. More formally, $\gamma \preceq \xi$ and $\gamma \neq \xi \Rightarrow f(\gamma) \preceq' f(\xi)$.

Continuous (monotonic) mappings have nice properties such as preserving connectedness. Thus, if γ, ξ *meet* in Γ, and f is continuous, then $f(\gamma)$ and $f(\xi)$ cannot be *disjoint* in Γ'.

A one-to-one correspondence $f : \Gamma \rightarrow \Gamma'$ for which both f and the inverse function f^{-1} are continuous is a homeomorphism of topological spaces: in this case Γ and Γ' are said *isomorphic* (as complexes) [118]. The isomorphism between two complexes guarantees that their associated maps are combinatorially equivalent, i.e., that both the entities composing the maps, and their spatial relationships are in one-to-one correspondence.

In [75], *object homeomorphisms* and *relation homeomorphisms* between sets of spatial entities were defined. The isomorphism between cell complexes incorporates both the object homeomorphism, and the relation homeomorphism when spatial entities are considered in the context of a map.

[1] In algebraic topology, an application between cell complexes is called a *map*. Here, we use the term *mapping* to avoid confusion with geographic maps, which are semantic objects whose structure is represented by cell complexes.

3.4 Consistent Map Transformations

We recall that the bounding relation $\preceq$ defined on the cells of an ACC defines a partial order on the set of cells, i.e., ACCs are posets. Definitions and concepts related to posets and functions on posets can be found in Appendix B.

In this Section, we identify a set of functions between ACCs (as posets) that characterize the combinatorial/topological aspects of consistent transformations between geographic maps we intend to model. Our aim is to formalize the process of obtaining a less detailed map from a given map by means of functions between the corresponding ACCs.

Different maps that correspond to non-isomorphic ACCs are not topologically equivalent. However, we are interested in studying maps that can be converted into each other by means of reciprocal abstraction/refinement operations, in which details are either discarded or introduced while maintaining consistent the overall structure.

We try to find properties of the set of functions we are defining on the basis of requirements of generalization of geographic maps described in the literature. As outlined in the Introduction, generalization can be used as a tool for obtaining multiple representations. Although it involves changes from different points of view (e.g., metric, semantic, topological), here we only consider topological aspects. Thus, we must informally understand what topological changes can involve transforming a given map into a less detailed map. A first approach to analyze how generalization operations interact with the topological and metric structure of maps was proposed in [59]. As already pointed out, when two maps M and M' differ in detail, then some objects can be represented only in the map at higher resolution (let it be M), and some objects that appear in both maps can be represented by entities of lower complexity and/or dimension in the map at lower resolution (let it be M'). However, any object represented by an entity in the map at lower resolution must also be represented by either an entity or a group of entities in the map at higher resolution. This is the so-called *monotonicity assumption*. This assumption guarantees that for each entity in map M' there must exist a corresponding entity in M; moreover, since each entity of M must correspond to something either simpler or equivalent in M', then a single entity of M can correspond to at most one entity in M'.

On the basis of these simple observations we can conclude that the map-to-map correspondence describing a transformation of M into M' can be formalized through a surjective mapping $F : \Gamma \rightarrow \Gamma'$, where Γ and Γ' are the cell complexes describing M and M', respectively.

Let us consider now a generic entity e of a map M. We outline three possible basic transformations of e into an entity of M':

- *preservation*: the object represented by e appears also in M' with the same dimension, and possibly with a simplified structure (e.g., if e is a region, its corresponding entity in M' will be also a region, possibly with a simplified boundary, or a smaller set of features, or a smaller characteristic); note that preservation of a whole map corresponds to an isomorphism since it implies no topological changes (but maybe some metric change);

- *reduction*: the object represented by e appears also in M', but with a lower dimension (e.g., if e is a region, its corresponding entity in M' could be a single point);
- *immersion*: the object represented by e disappears in M', i.e., it is immersed into some larger object.

Obviously, only reductions and immersions correspond to a decrease in topological detail. If we translate the previous cases in terms of mappings on ACCs we obtain:

- *preservation*: $\dim(F(\gamma)) = \dim(\gamma) \quad \wedge \quad \forall \tau \in \partial\gamma, \dim(F(\tau)) < \dim(\gamma)$;
- *reduction*: $\dim(F(\gamma)) < \dim(\gamma)$;
- *immersion*: $\dim(F(\gamma)) > \dim(\gamma) \quad \vee \quad (\dim(F(\gamma)) = \dim(\gamma) \quad \wedge$
  $\exists \tau \in \partial\gamma : \dim(F(\tau)) = \dim(\gamma))$.

Note that when the image of a cell γ maintains the same dimension as γ itself, it may be either that γ is maintained in the transformation, or that it is immersed into a "larger" cell. Indeed, when a whole submap of M is mapped into a unique region r' of M', any cell γ corresponding to a region of such submap must have as image γ' corresponding to region r'. The discrimination between preservation and immersion is obtained through the condition on the boundary: indeed, if region r is immersed into region r', also one portion of the boundary of r must be immersed inside m'.

Further constraints on transformations are concerning the consistency of topological relationships in the domain and codomain. For instance, objects that *meet* cannot possibly be transformed into *disjoint* objects; also, a point that is *disjoint* from a region cannot jump *inside* the region in the less detailed map (unless both the point and the region are immersed into something bigger); on the other hand, it is possible that two *disjoint* regions r and r' will *meet* after transformation, if, for instance, a third region r'' separating r from r' is reduced to a line. An example is provided in Figure 3.1.

A complete analysis of all possible cases arising from the transformation of pairs of entities is possible, on the basis of the admissible relationships between pairs of entities in a map (listed in Chapter 2).

From the above considerations, it follows that functions of ACCs, which are of interest for map generalization include all surjective, monotonic functions preserving connected sets

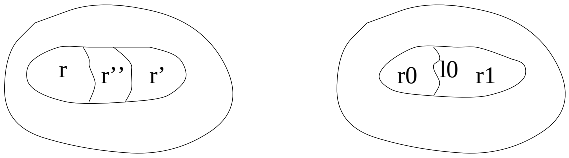


Figure 3.1: Regions r and r' are mapped onto r_0 and r_1, respectively; region r'', in between r and r', and connecting them is mapped onto a line l_0 shared by r_0 and r_1.

by inverse image (preservation of connectivity is implied by monotonicity). Formally, if $f : A \rightarrow B$ is a function of ACCs, monotonicity is expressed by

$$\gamma \preceq \xi,\ \gamma \neq \xi \Rightarrow f(\gamma) \preceq f(\xi),$$

while the preservation of connected sets by inverse image is expressed by

$$\forall B' \subseteq B, B' \textit{ connected} \Rightarrow f^{-1}(B') \textit{ connected}.$$

Monotonicity guarantees that two objects that are not disjoint in the domain will not be disjoint in the codomain either, while the preservation of connected sets by inverse image does not permit to map sets of completely disjoint entities onto connected sets.

From now on, we will refer to such functions as *consistent combinatorial transformations* (or, sometimes, for short *map transformations*).

In the following Sections, such functions will be used to give a categorical characterization of geographic maps and their generalization.

3.5 The category of ACCs

In this Section, we use notions from category theory as tools to provide a formal and rigorous characterization of ACCs and their consistent transformations. As we will see later, this characterization will also allow us to define a set of minimal operators performing atomic steps, and sufficient to generate all consistent transformations on maps (see Chapter 4).

Definition 3.5.1 *Let $\mathcal{O}$ be the (infinite) set of all (finite) ACCs. Let $\mathcal{A}$ be the set of all possible functions (also morphisms, or arrows) between objects in $\mathcal{O}$ such that they are surjective, monotonic with respect to $\preceq$ and that preserve connected sets by inverse image. The pair $\mathcal{C} = (\mathcal{O}, \mathcal{A})$ defines a category called the* category of ACCs.

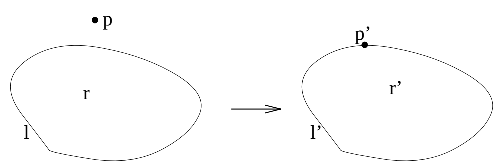


Figure 3.2: In this example, $p' = f(p)$ and $l' = f(l)$ are related while p and l are not related.

Note that $\mathcal{C} = (\mathcal{O}, \mathcal{A})$ is a category because both the identity functions and all compositions of functions of $\mathcal{A}$ are in $\mathcal{A}$ (the composition of surjective monotonic functions preserving connected sets by inverse image is a surjective monotonic function that preserves connected sets by inverse image).

Remark: We assume that two objects A and B in $\mathcal{O}$ are identical (and we denote $A = B$) if and only if they are isomorphic as ACCs; also, two morphisms f and g in $\mathcal{A}$ are identical (and we denote $f = g$) if and only if there exists an isomorphism h in $\mathcal{A}$ such that $f = h \circ g$ or $f = g \circ h$. Thus, A and B, as well as f and g are the same (respectively) up to isomorphism.

In the following, we show an important property of injective functions in the category of ACCs: all injective functions f preserve $\preceq$ in both directions (i.e., both f and f^{-1} are monotonic), so they are isomorphisms.

Proposition 3.5.2 *Any injective function in the category of ACCs is an isomorphism.*

Proof. Note that, since all functions in ACC are surjective, any injective function in the category of ACCs is a bijection. Given a bijection $f : X \rightarrow Y$ in the category of ACCs, the only possible situation when f is not an isomorphism is when its inverse function f^{-1} is not monotonic. i.e., $\exists y, y' \in Y$ such that $y \preceq y'$ and $f^{-1}(y') \preceq f^{-1}(y)$ or $f^{-1}(y)$ and $f^{-1}(y')$ are not related. Since f is surjective, $\exists x, x' \in X$ such that $f(x) = y$ and $f(x') = y'$; also, $x = f^{-1}(y)$ and $x' = f^{-1}(y')$ being f injective. If we have $f(x) \preceq f(x')$ ($x \neq x'$), we can't have $x' \preceq x$ (because otherwise we would have $f(x') \preceq f(x)$ but $\preceq$ is antisymmetric). Thus, the only possibility is to have that x and x' are not related. In this case, f would not preserve connected sets by inverse image. Indeed, $f^{-1}(\{y, y'\}) = \{x, x'\}$ would not be connected (see Figure 3.2 for an example). □

3.5.1 Ordering relation between objects in the category of ACCs

In this Section, we define a relation on the set of objects of the category of ACCs, and we show that it is an ordering relation. Thus, the set of ACCs with this relation is a poset and we can measure the minimum distance between two elements of such poset. In particular, two objects are related if and only if there exists a morphism of the category mapping one onto the other.

Definition 3.5.3 *Given two elements A and B in $\mathcal{O}$, we say that $B \leq A$ if and only if there exists a function (morphism) f in $\mathcal{A}$ such that $f: A \rightarrow B$.*

Fact 3.5.4 *$\leq$ is a partial ordering relation (i.e., it is reflexive, antisymmetric and transitive).*

Proof. Reflexivity and transitivity are obvious. Antisymmetry: we have to show that, if $B \leq A$ and $A \leq B$, then $A = B$ (for the remark made in Section 3.5, this means A and B are isomorphic, i.e., there exists an isomorphism f' such that $f' : A \rightarrow B$).

By hypothesis, there exist f, g in $\mathcal{A}$ (i.e., surjective and monotonic) such that

$$\begin{tikzcd} A \arrow[r, shift left, "f"] & B \arrow[l, shift left, "g"] \end{tikzcd}$$

Since A and B are finite sets, then they must have the same number of elements, thus, both f and g are bijective (i.e., there exist f^{-1} and g^{-1} such that $f \circ f^{-1} = id_B$, $f^{-1} \circ f = id_A$, and $g \circ g^{-1} = id_A$, $g^{-1} \circ g = id_B$) and monotonic.

It follows from Proposition 3.5.2 that f and g are isomorphisms; thus, A and B are isomorphic. $\square$

3.5.2 Minimal morphisms in the category

In this Section, we define minimal morphisms of the category and show that every composition of such morphism generating a non-minimal morphism is finite.

A morphism in $\mathcal{C}$ is called *minimal* if it cannot be obtained as composition of two other non-identical morphisms in $\mathcal{C}$. More formally, we provide the following definition.

Definition 3.5.5 *A morphism f in $\mathcal{C}$ is* minimal *if, whenever there exist two morphisms g and h in $\mathcal{C}$ such that $f = g \circ h$, then either g or h is the identity morphism.*

Notice that there are an infinite number of minimal morphisms. By definition, non-minimal morphisms can be obtained as compositions of minimal morphisms. The following proposition shows that such compositions are finite.

Proposition 3.5.6 *Every non-minimal morphism is obtained as finite composition of minimal morphisms.*

Proof. Every non-minimal morphism corresponds to at least one sequence of minimal morphisms. Also, every morphism $f : A \to B$ corresponds to a relation $B \leq A$ between objects A and B. Thus, a sequence of minimal morphism which, composed, generates a given morphism corresponds to a chain of objects in the poset of ACCs. We must show that each such chain is finite. Let $A_1, \ldots, A_n, \ldots$ be a decreasing chain such that for each i, there exists a minimal morphism $f_i : A_i \to A_{i+1}$. Note that $A_i < A_{i+1}$ because no minimal morphism is an isomorphism (i.e., the chain is strictly decreasing). Also, being f_{i+1} strictly non-injective, $|A_{i+1}| < |A_i|$ (where $|A|$ denotes the finite number of cells of ACC A). Thus each sequence $A_1, \ldots, A_n, \ldots$ of ACCs corresponds to a strictly decreasing chain of the cardinalities of their sets of cells. Such chain is finite because all the sets A_i $(1 \leq i \leq n)$ of cells are finite. $\square$

3.5.3 Minimal operators in the category

Minimal operators are defined as classes of minimal morphisms which "operate in a given way". An operator is given by its description, i.e., by the way it maps a cell (or a set of cells) of a complex to a cell (or a set of cells) of the other complex.

We define minimal operators through "restrictions" of minimal morphisms to the subset of their domain in which they are strictly non-injective.

Definition 3.5.7 *Given a morphism $f : A \to B$, its* non-injective restriction *is defined by $f_{|_{A'}} : A' \to B'$ such that $B' = \{b \in B$ such that $f^{-1}(b)$ contains more than one element$\}$ and $A' = f^{-1}(B')$.*

We say that f is *strictly non-injective* on A' because, by definition, $\forall a'' \in A'$, $\exists\ a''' \in A', a''' \neq a''$ such that $f(a'') = f(a''')$.

The proof of the following fact is straightforward.

Fact 3.5.8 *The non-injective restriction $f_{|_{A'}}$ of a morphism f of the category is a morphism of the category.*

Also, by definition, A' is the maximal subset of A on which f is strictly non-injective. Furthermore, being B' unique, A' is also unique. This means that the non-injective restriction of a morphism is uniquely determined.

In the following, we show that applying the non-injective restriction to a morphism corresponds to "subtracting" an isomorphism from it (see Figure 3.3).

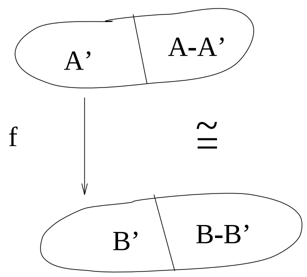


Figure 3.3: The non-injective restriction of a morphism.

Fact 3.5.9 *Let $f : A \to B$ be a morphism and let $f_{|_{A'}}$ be its non-injective restriction. Then:*

- $f^{-1}(B \setminus B') = A \setminus A'$;
- $f_{|_{A \setminus A'}} : A \setminus A' \to B \setminus B'$ *is an isomorphism.*

Proof. We show that $f^{-1}(B \setminus B') \subseteq A \setminus A'$ and $A \setminus A' \subseteq f^{-1}(B \setminus B')$. Let $a \in f^{-1}(B \setminus B')$; then $f(a) \in B \setminus B'$. If we assume by contradiction that $a \in A'$ (i.e., $a \notin A \setminus A'$), then, being $A' = f^{-1}(B')$, we would have $f(a) \in B'$. Thus it must be $a \in A \setminus A'$. Conversely, if $a \in A \setminus A'$, then let us assume by contradiction that $a \notin f^{-1}(B \setminus B')$, then $f(a) \notin B \setminus B'$. But being f surjective $f(a) \in B$, thus it will be $f(a) \in B'$, i.e., $a \in A'$ which is absurd. Thus, $f_{|_{A \setminus A'}}$ is surjective. Also, since $f^{-1}(B \setminus B') = A \setminus A'$ and f preserves connected sets by inverse image, then $f_{|_{A \setminus A'}}$ preserves connected sets by inverse image. We can conclude that $f_{|_{A \setminus A'}}$ is a morphism in the category because it is also monotonic (being a restriction of f). Furthermore, $\forall a, a' \in A \setminus A'$, $f(a) = f(a') \Rightarrow a = a'$, i.e., f is injective on $A \setminus A'$. Thus $f_{|_{A \setminus A'}}$ is an isomorphism. $\square$

On the basis of the concept of non-injective restriction, we now define *minimal operators* in the category.

First, we define an equivalence relation on minimal morphisms of the category: two such morphisms are equivalent if they have the same non-injective restriction (up to isomorphism). In other words, they behave in the same way on the subset of their domain in which they are strictly non-injective. Obviously, the two morphisms do not necessarily have same domain/codomain.

Definition 3.5.10 *Let $f : A \to B$, $g : C \to D$ be two minimal morphisms. Let $A' \subseteq A$ and $C' \subseteq C$ be the maximal subsets on which f and g are strictly non-injective, respectively. The relation $\sim$ is defined as follows: $f \sim g$ if and only if there exist two isomorphisms p and q such that the following diagram commutes.*

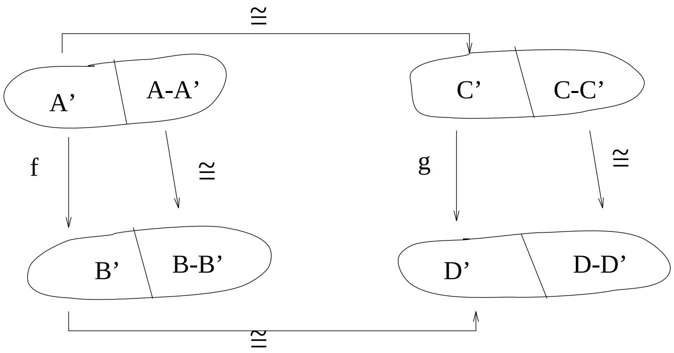


Figure 3.4: Equivalence relation between pairs of minimal morphisms.

$$\begin{tikzcd} A' \arrow[r, "f"] \arrow[d, "p"'] & B' \arrow[d, "q"] \\ C' \arrow[r, "g"] & D' \end{tikzcd}$$

Informally speaking, f and g perform the same transformation on isomorphic sets of entities, while they do not affect the rest of their domain (see Figure 3.4).

Proposition 3.5.11 $\sim$ *is an equivalence relation.*

Proof. Obviously, $\sim$ is reflexive

$$\begin{tikzcd} A' \arrow[r, "f"] \arrow[d, "id_{A'}"'] & B' \arrow[d, "id_{B'}"] \\ A' \arrow[r, "f"] & B' \end{tikzcd}$$

and symmetric

$$\begin{tikzcd} C' \arrow[r, "g"] \arrow[d, "p^{-1}"'] & D' \arrow[d, "q^{-1}"] \\ A' \arrow[r, "f"] & B'. \end{tikzcd}$$

Also, if $f \sim g$ and $g \sim h$ (with $h : E \to F$), then $f \sim h$. Indeed, since C' is unique, there exist two isomorphisms $p'' = p' \circ p$ and $q'' = q' \circ q$ such that the following diagram commutes:

$$\begin{tikzcd}
A' \arrow[r, "f"] \arrow[d, "p"'] & B' \arrow[d, "q"] \\
C' \arrow[r, "g"] \arrow[d, "p'"'] & D' \arrow[d, "q'"] \\
E' \arrow[r, "f"] & F'
\end{tikzcd}$$

□

This relation partitions the set of minimal morphisms into classes. Indeed, for each minimal morphism, its non-injective restriction is uniquely defined (i.e. each minimal morphism belongs to one and only one such class). The canonical representative of each class is the non-injective restriction of each morphism of the class (it is the same for all, up to isomorphism). A minimal operator is each such class: we identify it with its canonical representative. Thus, in the following, we will refer interchangeably to the operator (a class of morphisms) and to its canonical representative morphism.

It is obvious that, among all morphisms in a given class the representative morphism is the one having the smallest domain (because the non-injective restriction is unique). Also, the set of minimal operators coincide with the set of minimal morphisms which are strictly non-injective on their domain.

3.5.4 Conservative extension

Given a morphism, defined on a "small" domain, we can consider its "contextualizations", or in other terms, its extensions to larger contexts (domains). We are interested in what we call *conservative* extensions. The morphisms we start from are strictly non-injective. Thus, extending them corresponds to the inverse operation of the non-injective restriction of a morphism.

Definition 3.5.12 *Let A, B, C, D be ACCs, with $C \subseteq A$, $D \subseteq B$; let $f : A \to B$, $f_0 : C \to D$ be morphisms in the category of ACCs, with f_0 strictly non-injective. We*

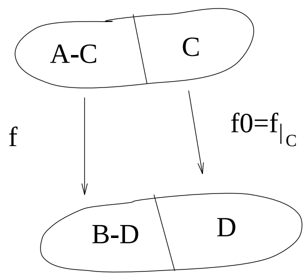


Figure 3.5: Conservative extension of a morphism.

say that f is a conservative extension *of f_0 if and only if f_0 is the non-injective restriction of f.*

In other words, a morphism f is a *conservative extension* of a morphism f_0 if f is obtained by extending f_0 through an isomorphism defined on the rest of the domain of f (see Figure 3.5).

In particular, we are interested in morphisms which are conservative extension of minimal operators, in the sense that they act on the same entities (i.e., they identify entities affected by such operators) and preserve the others (i.e., they maintain them as distinct). Conservative extensions of minimal operators are minimal morphisms.

Each equivalence class in the relation $\sim$ defined before is composed of a minimal strictly non-injective morphism and all its conservative extensions.

Thus, conservative extensions of minimal operators generate all and only minimal morphisms. In other words, minimal operators are sufficient to generate all morphisms of the category.

By exploiting the notion of conservative extension, we now give a property of the codomain of minimal morphisms.

Lemma 3.5.13 *For each $f : A \to B$, with f a minimal morphism in the category of ACCs, B is connected (this also implies A is connected, since f preserves connected sets by inverse image).*

Proof. In case $\#B = 1$, the property holds. If $\#B > 1$, let us assume by contradiction that B is not connected. This means that there exist two non-empty subsets B' and B'' of B such that $B' \cup B'' = B$, $B' \cap B'' = \emptyset$ and $\forall b' \in B', b'' \in B''$, b' and b'' are disjoint i.e., neither $b' \preceq b''$ nor $b'' \preceq b'$ (in this case, we say that B' and B'' are not related). Since f is surjective, every element in B has an inverse image. Let $A' = f^{-1}(B')$ and $A'' = f^{-1}(B'')$; $A = A' \cup A''$, A' and A'' are non-empty and $A' \cap A'' = \emptyset$. Also, if there was

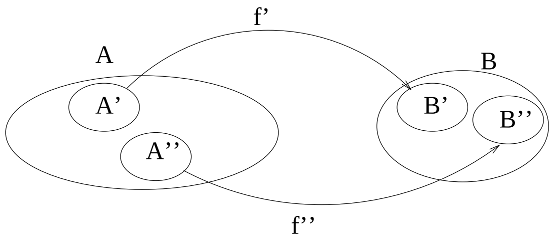


Figure 3.6: The codomain of a minimal morphism is always connected.

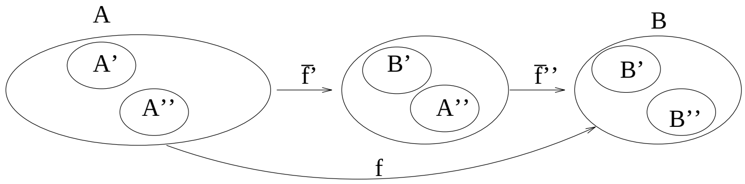


Figure 3.7: Illustration of the proof of Lemma 3.5.13.

a pair of elements $a' \in A'$ and $a'' \in A''$ such that $a' \preceq a''$ (or $a'' \preceq a'$), then we would have $b' \in B'$, $b'' \in B''$ such that $b' \preceq b''$ (with $b' = f(a')$, $b'' = f(a'')$) for the monotonicity of f. Thus, A must be non-connected as well (see Figure 3.6).

Let $f' = f_{|_{A'}} : A' \to B'$ and $f'' = f_{|_{A''}} : A'' \to B''$; both f' and f'' are morphisms in the category of ACCs. Let $\overline{f'}$ be conservative extension of f' to $A' \cup A''$ and let $\overline{f''}$ be conservative extension of f'' to $A'' \cup B'$ (more precisely, we should consider function $\overline{f''}$ as defined on $\overline{f'}(A'') \cup B'$; but being $\overline{f'}$ an isomorphism on A'', we have that $\overline{f'}(A'') = A''$). Being A' and A'' as well as A'' and B' not related, both $\overline{f'}$ and $\overline{f''}$ are well defined and are morphisms in the category of ACCs and $f = \overline{f''} \circ \overline{f'}$ (see Figure 3.7).

The same result is obtained by first extending f'' and then f'. Thus, f is not minimal. $\square$

3.6 The category of GACCs

In this section, we refine the category of ACCs into a subcategory that characterizes the set of ACCs representing geographic maps. Such a subcategory is composed of all ACCs that have the structure of a planar graph, i.e., they satisfy Euler's formula.

We first describe properties of geographic maps and then define the subcategory. Then, we show that topological relations between entities in a map can be translated in terms of cells of the corresponding complex. We also provide a definition of minimal morphisms and operators of the subcategory. Minimal operators will be formally characterized in Chapter 6.

3.6.1 Properties of geographic maps

As already stressed, an abstract cell complex is essentially a poset in which the set of cells is partitioned into subsets, each characterized by a dimension.

However, not all abstract cell complexes correspond to geographic maps. First of all, in an abstract cell complex representing a geographic map (called *Geographic Abstract Cell Complex, GACC*, in the following), cells can only have dimension 0, 1, or 2 (2-dimensional ACC). Furthermore, geographic maps satisfy some additional properties. For example, a 1-cell cannot have more than two bounding 0-cells, and it cannot have more than two 2-cells in its star; also, a 1-cell with only one 0-cell in its boundary (corresponding to a closed line) must always have two 2-cells in its star, etc.

Also, as we have seen in the definition of a geographic map, if an entity belongs to the map, also its boundary belongs to it. For an abstract cell complex, representing the combinatorial structure of a geographic map, we impose that each entity of the complex has a boundary in the complex. For example, each line always has either one (closed line) or two endpoints (open line), and these belong to the complex.

The following property characterizes a GACC.

Definition 3.6.1 *Let v be the number of 0-cells, e the number of 1-cells, f the number of 2-cells (including the 2-cell corresponding to the infinite region), and c the number of connected components of the graph formed by all 0-cells and 1-cells of an ACC A. Then A is a GACC if and only if the following formula (Euler formula) is satisfied*

$$v - e + f = c + 1.$$

This obviously implies that the set of cells of a GACC corresponds to a regular set of entities, i.e., for each entity in the complex also entities of its boundary are cells of the complex.

In the following, we list some properties of GACCs that are consequences of the above property.

1. For every 0-cell:
    a) there is at least one 2-cell in its star (due to the presence of the infinite region);
    b) the number of 2-cells in its star is less than or equal to the number of 1-cells plus one.
2. For every 1-cell:

a) its boundary is composed of either one or two 0-cells (closed line and open line, respectively): this follows from the definition of edge in a graph;

b) its star can be composed of either one or two regions (line feature and boundary line, respectively): this follows from the planarity of the graph; in particular, every closed line is a boundary line and is the only line on the boundary of both regions in its star; on the other hand, an open boundary line cannot be the only line on the boundary of the two regions in its star, i.e., there exists at least another open line bounding the same two regions; this property follows from the definition of face in a planar graph.

3. For every 2-cell:

   a) its boundary contains at least one 1-cell, with the exception of a geographic map composed of only the infinite region, or only isolated points and the infinite region: this follows from the definition of face in a planar graph;

   b) its boundary contains at least one 0-cell, with the exception of a map composed of only the infinite region (as a consequence of a);

   c) the number of 1-cells in its boundary is less than or equal to the number of 0-cells; this follows again from the definition of face.

Properties 2a) and 2b) imply that every 1-cell l always belongs to (at least) one complete chain $p \preceq l \preceq r$. Similarly, from properties 3a) and 3b) it follows that, for every 2-cell r other than the infinite region, there always exists at least a complete chain containing r.

In the following we define the category of GACCs.

Definition 3.6.2 *The* category of GACCs *is defined as a full subcategory*[2] *of the category of ACCs whose objects are all ACCs satisfying Euler's formula.*

Note that the ordering relation on the set of ACCs is inherited by the set of GACCs.

3.6.2 Topological Relations in a GACC

In this Section, we show that all topological relations defined between entities of a map (as described in Chapter 2) can be translated into relations between cells of the corresponding complex. Such relations are expressed by means of constraint on boundary and star of

[2]A full subcategory S of a category $\mathcal{C}$ is determined by giving only the set of its objects, and taking all morphisms between any two of such objects s, s'.

cells. Indeed, according to the definition of $\preceq$ in a cell complex representing a plane map, $\gamma \preceq \gamma'$ (i.e., $\gamma \in \partial(\gamma')$) if and only if γ *inside*, *bounds* or *coveredBy* γ' (remember that $\gamma \preceq \gamma'$ and $\gamma \neq \gamma \Rightarrow \dim(\gamma) < \dim(\gamma')$).

Given a GACC $(C, \preceq, \dim)$, for each $\gamma, \gamma' \in C$ $(\gamma \neq \gamma')$, only one of the following situations may arise:

1. if $dim(\gamma) = dim(\gamma') = 0$, then
   - γ *disjoint* γ';
2. if $dim(\gamma) = 0$, $dim(\gamma') = 1$, then
   - γ *disjoint* γ' iff $\gamma \notin \partial(\gamma')$;
   - γ *bounds* γ' iff $\gamma \in \partial(\gamma')$;
3. if $dim(\gamma) = 0$, $dim(\gamma') = 2$, then
   - γ *disjoint* γ' iff $\gamma \notin \partial(\gamma')$;
   - γ *bounds* γ' iff $\gamma \in \beta(\gamma')$;
   - γ *inside* γ' iff $\gamma \in ft(\gamma')$;
4. if $dim(\gamma) = dim(\gamma') = 1$, then
   - γ *disjoint* γ' iff $\beta(\gamma) \cap \beta(\gamma') = \emptyset$;
   - γ *meets* γ' iff $\beta(\gamma) \cap \beta(\gamma') \neq \emptyset$
     (this is enough because in a map interiors of two different regions are disjoint);
5. if $dim(\gamma) = 1$, $dim(\gamma') = 2$, then
   - γ *disjoint* γ' iff $\partial(\gamma) \cap \partial(\gamma') = \emptyset$;
   - γ *meets* γ' iff $\beta(\gamma) \cap \beta(\gamma') \neq \emptyset$ and $\gamma \notin \partial(\gamma')$;
   - γ *bounds* γ' iff $\gamma \in \beta(\gamma')$;
   - γ *coveredBy* γ' iff $\gamma \in ft(\gamma')$ and $\beta(\gamma) \cap \beta(\gamma') \neq \emptyset$;
   - γ *inside* γ' iff $\gamma \in ft(\gamma')$ and $\beta(\gamma) \cap \beta(\gamma') = \emptyset$;
6. if $dim(\gamma) = dim(\gamma') = 2$, then
   - γ *disjoint* γ' iff $\beta(\gamma) \cap \beta(\gamma') = \emptyset$
     (note that within a map this condition is equivalent to $\partial(\gamma) \cap \partial(\gamma') = \emptyset$, for γ and γ' regions);
   - γ *meets* γ' iff $\beta(\gamma) \cap \beta(\gamma') \neq \emptyset$.

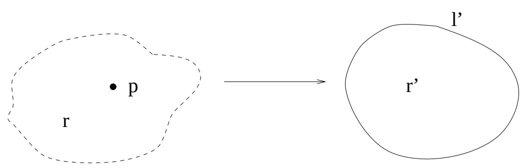


Figure 3.8: Mapping of a non-complete chain (composed of a region r and a point p) through an isomorphism that does not fulfill the condition on dimension.

3.6.3 Isomorphisms of GACCs

In the following, we show an important result on isomorphisms between GACCs: they map an entity to an entity of the same dimension. This condition follows from the constraints imposed on GACCs. Indeed, it does not hold for generic ACCs.

Proposition 3.6.3 *Let A and B be GACCs and let $f : A \to B$ be an isomorphism in the category of GACCs. Then, $\forall a \in A$, we have $dim(f(a)) = dim(a)$.*

Proof. By definition, in a GACC we can only have chains of strict inequalities having at most length three (indeed we cannot have $x \preceq x'$, with $dim(x) = dim(x')$ and $x \neq x'$). In other words, the dimension function is strictly increasing. Note that an isomorphism maps entities belonging to chains of length three (that we call *complete* chains) to entities of the same dimension (because it preserves $\preceq$).

The only case in which we can have problems (i.e., an entity which is mapped to an entity of different dimension by an isomorphisms between ACCs) is when we consider mappings involving non-complete chains. The only possible non-complete chains in two-dimensional ACCs are: $p \preceq l$, $p \preceq r$, $l \preceq r$. However, the only possible case for GACCs is a non-complete chain $p \preceq r$, where r is the infinite region and p is an isolated point inside r, because, as we have already seen, in a geographic map each line belongs to a complete chain.

The only possible mapping of the $p - r$ chain through isomorphism that does not fulfill the condition on dimension is

- $f(p) = l'$, $f(r) = r'$ with $l' \preceq r'$

An example is shown in Figure 3.8. However, this case is not possible either, because any line has at least one endpoint so l' would have an endpoint p' thus forming a complete chain $p' \preceq l' \preceq r'$. $\square$

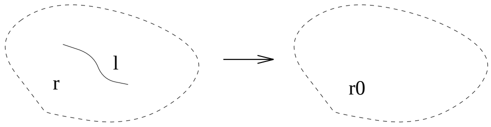


Figure 3.9: Example of minimal operator in the category of ACCs which is not a morphism in the category of GACCs.

3.6.4 Minimal morphisms and operators

Minimal morphisms in the category of GACCs are morphisms that cannot be obtained as compositions of other non-identical morphisms in the category. A minimal morphism in the category of GACCs may not be minimal in the category of ACCs.

We again define minimal operators in the category of GACCs as equivalence classes of minimal morphisms. The difference with respect to minimal operators in the category of ACCs is the fact that there is no canonical representative (the non-injective restriction of a minimal morphism is not necessarily a morphism in the category of GACCs).

Note that minimal operators in the category of ACCs may be defined on sets of entities which are not necessarily geographic maps. An example of minimal operator in the category of ACCs is depicted in Figure 3.9. In this example, the morphism maps r and l to r_0 and it is minimal in the category of ACCs, but it is not a morphism in the category of GACCs (since l has no endpoint).

Definition 3.6.4 *Let $f : A \to B$, $g : C \to D$ be two minimal morphisms in the category of GACCs. Let A' be the maximal subset of both A and C on which f and g are strictly non-injective. The relation $\sim$ is defined as follows: $f \sim g$ if and only if $f_{|_{A'}} = g_{|_{A'}}$.*

Similarly to what we have done for ACCs, it is easy to show that this is an equivalence relation. This relation is well-defined and it partitions the set of minimal morphisms in the category of GACCs into classes. A minimal operator is each such class. We identify the class with the non-injective restriction of each morphism of the class (it is a morphism in the category of ACCs).

Also, the definition of conservative extension holds in the category of GACCs as well. In the following, we state a property of the codomain of a minimal morphism.

Lemma 3.6.5 *For each $f : A \to B$, with f a minimal morphism in the category of GACCs, B is connected (this also implies A is connected, since f preserves connected sets by inverse image).*

Proof. The proof is the same as for the case of ACCs (see Lemma 3.5.13), i.e., it is made by contradiction. We assume that B is not connected; thus, there exist two subsets B_1 and B_2 of B such that $B_1 \neq \emptyset$, $B_2 \neq \emptyset$, $B_1 \cup B_2 = B$, $B_1 \cap B_2 = \emptyset$, B_1 and B_2 are not related. Let $A_1 = f^{-1}(B_1)$ and $A_2 = f^{-1}(B_2)$. Then $A_1 \neq \emptyset$, $A_2 \neq \emptyset$, $A_1 \cup A_2 = A$, $A_1 \cap A_2 = \emptyset$, and A_1 and A_2 are not related. Thus A is also non-connected.

Let $f_1 = f_{|_{A_1}} : A_1 \rightarrow B_1$ and $f_2 = f_{|_{A_2}} : A_2 \rightarrow B_2$. Let $\overline{f_1}$ be conservative extension of f_1 to $A_1 \cup A_2$ and let $\overline{f_2}$ be conservative extension of f_2 to $A_2 \cup B_1$. Being A_1 and A_2 as well as A_2 and B_1 not related, both $\overline{f_1}$ and $\overline{f_2}$ are well defined. We must show that $A_2 \cup B_1$ is a GACC. A_1, A_2, B_1, are all not related with each other B_2. Thus, in order for A and B to be GACCs, such subsets must also be GACCs. Thus $A_2 \cup B_1$ is a GACC (the union of GACCs, intended as union of the sets of cells, with their respective bounding relation, and without any relation between the different sets of cells, returns a GACC). Thus, $\overline{f_1}$ and $\overline{f_2}$ are morphisms in the category of GACCs and $f = \overline{f_2} \circ \overline{f_1}$ (i.e., non-minimal). $\square$

Similarly, it can be shown that the codomain of the non-injective restriction of any minimal morphism is connected.

Lemma 3.6.6 *For each $f : A \rightarrow B$, with f a minimal morphism in the category of GACCs, let B' be the codomain of the non-injective restriction of f. B' is connected.*

Proof. We assume by contradiction that B' is not connected. Then there exist B'_1, B'_2, A'_1, and A'_2 as in the proof of Lemma 3.6.5. Also, let $f' : A' \rightarrow B'$, $f'' : A'' \rightarrow B''$ ($f'' = f_{A''}$ isomorphism, with $B = B' \cup B''$ and $A = A' \cup A''$), $f'_1 = f'_{|_{A'_1}} : A'_1 \rightarrow B'_1$ and $f'_2 = f'_{|_{A'_2}} : A'_2 \rightarrow B'_2$. Let $\overline{f'_1}$ be conservative extension of f'_1 to $A'_1 \cup A'_2$ and let $\overline{f'_2}$ be conservative extension of f'_2 to $A'_2 \cup B'_1$. Being A'_1 and A'_2 as well as A'_2 and B'_1 not related, both $\overline{f'_1}$ and $\overline{f'_2}$ are well defined. Let $\hat{f}'_1 : A' \cup A'' \rightarrow B'_1 \cup A'_2 \cup A''$ be the conservative extension of $\overline{f'_1}$ to A'' and let $\hat{f}'_2 : B'_1 \cup A'_2 \cup A'' \rightarrow B' \cup A''$ be the conservative extension of $\overline{f'_2}$ to A''. $\hat{f}'_1$ and $\hat{f}'_2$ are well defined because B'_1 and A'_2 are not related and it is possible to define the conservative extension to A'' (being f the conservative extension of f' to A''); thus, entities and relations can be preserved. We can make similar considerations as in the proof of Lemma 3.6.5. In particular, since A'_1 and A'_2 are not related, transformations on them can be applied independently. Although A'' can be related to entities in A'_1 and A'_2, tranformations on each of them do not affect neither entities in A'' nor relations involving such entities. Thus, we can conclude that $\hat{f}'_1$ and $\hat{f}'_2$ are morphisms in the category of GACCs such that $f = \hat{f}'_2 \circ \hat{f}'_1$ (f non-minimal). $\square$

Chapter 4

Combinatorial Transformations of Geographic Maps

In the previous Chapter, we have provided a formalization of combinatorial transformations of geographic maps through mappings between GACCs. We have also defined minimal operators in the corresponding category without providing a method to explicitly detect the set of such operators. In this Chapter, we list a set of morphisms in the category of ACCs, that we call *atomic operators*, and we prove that they correspond to the set of minimal operators in the category of GACCs, i.e., they partition the set of minimal morphisms of such category into classes. Thus, from the representative of each class (which is a strictly non-injective morphism in the category of ACCs), we obtain, through conservative extension to wider domains, all minimal morphisms of the category of GACCs. In other words, they are sufficient to generate, through extension and composition, all consistent map transformations. Thus, consistency is implicit in multiple representation models built on the basis of such operators.

Atomic operators provide rules to define a map transformation that performs gradual changes. Indeed, so far we have described which map transformation are to be considered as consistent, but we have not given any constraint to guarantee the graduality of changes in detail during such transformations. As we have seen, surjectivity, monotonicity and preservation of connected sets by inverse image guarantee consistency but they are not sufficient to guarantee graduality: for instance, mapping an arbitrarily complex map into a single point is a consistent transformation but it is not of much interest! On the contrary, we wish to have transformations that do not modify maps too abruptly skipping meaningful intermediate representations.

We will exploit atomic operators as units to evaluate the diversity between two geographic maps; two alternative definitions are provided in this Chapter.

The rest of the Chapter is organized as follows: in Section 1 we define atomic operators, in Section 2 we show that they are morphisms in the category of ACCs, while in Section 3 we show that every conservative extension of atomic operators in the category of GACCs is a minimal morphism in the category of GACCs. In Section 4, the sufficiency of atomic operators is shown. Finally, Section 5 defines the concept of *diversity* between maps.

4.1 Atomic Operators

In this Section, we define a set of functions $f : A \to B$ (with A and B ACCs) and we show that such functions are morphisms in the category of ACCs. Basically, they make immersions or reductions of an entity e and, possibly, reductions/immersions of elements of $\beta(e)$ and $*e$.

Since we are interested in applying these morphisms to GACCs, we will consider conservative extensions which are morphisms in the category of GACCs. This often means to add/complete the border of entities.

Also, by extending conservatively, we are guaranteed that we reduce/immerse a single entity without transforming entities of higher dimension in its star: in other words, entities in $*e$ that are involved in the mapping do not change dimension.

Each function $f : A \to B$ we define fulfills the following property: it maps either two or three entities (cell complex A) onto one entity (cell complex B). For this reason, we call it an *atomic operator*.

In the following, we list the set of atomic operators:

- **line-to-point contraction** (lp_contr): let l be an open line[1] and let p and p' ($p \neq p'$) be its endpoints; l as well as p and p' reduce to a point p_0 (Figure 4.1). In synthesis:

  $$lp\_contr : A \to B$$

  where $A = \{p, l, p'\}$ with $p \preceq l,\ p' \preceq l$, and $B = \{p_0\}$ such that

  $$lp\_contr(l) = lp\_contr(p) = lp\_contr(p') = p_0.$$

- **region-to-point contraction** (rp_contr): a simply connected region r and such that $\beta(r) = \{l, p\}$ with l a closed line and $\beta(l) = \{p\}$ reduces to a point p_0 (Figure 4.2); in this case, elements of the border of r are also reduced to p_0. In synthesis:

  $$rp\_contr : A \to B$$

[1] The case of a closed line is included in the *region-to-point contraction* operator.

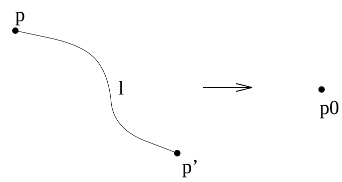


Figure 4.1: Contraction of an open line to a point.

where $A = \{p, l, r\}$ with $p \preceq l \preceq r$, and $B = \{p_0\}$ such that

$$rp\_contr(p) = rp\_contr(l) = rp\_contr(r) = p_0.$$

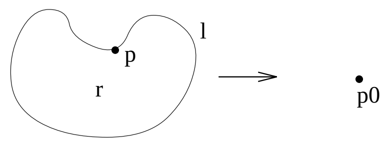


Figure 4.2: Contraction of a region to a point.

- **region-to-line contraction** (rl_contr): a simply connected region r such that $\beta(r) = \{l, l'\}$ reduces to a line l_0 (Figure 4.3); in this case l and l' reduce to l_0. In synthesis we have:

$$rl\_contr : A \rightarrow B$$

where $A = \{l, l', r\}$ with $l \preceq r, l' \preceq r$, and $B = \{l_0\}$ such that

$$rl\_contr(l) = rl\_contr(l') = rl\_contr(r) = l_0.$$

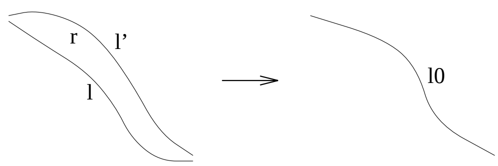


Figure 4.3: Contraction of a region to a line.

- **line merge** (l_merge): let l, l', $l \neq l'$ (the case $l = l'$ has no meaning for a GACC because we cannot eliminate the endpoint from a closed line), be two lines with a common endpoint p; p is immersed into a line l_0 including also l and l' (see Figure 4.4). In synthesis:

$$l\_merge : A \rightarrow B$$

where $A = \{l, p, l'\}$ with $p \preceq l, p \preceq l'$, and $B = \{l_0\}$ such that

$$l\_merge(p) = l\_merge(l) = l\_merge(l') = l_0.$$

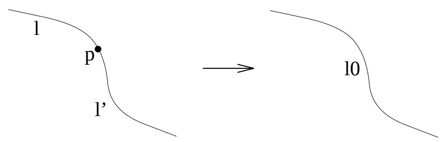


Figure 4.4: Merging two lines into one.

- **region merge** (r_merge): let r, r', $r \neq r'$ (if $r = r'$, the operator reduces to *line abstraction*), be two regions having a common bounding line l; l is immersed into a region r_0 including also r and r' (see 4.5). In synthesis:

$$r\_merge : A \rightarrow B$$

where $A = \{l, r, r'\}$ with $l \preceq r, l \preceq r'$, and $B = \{r_0\}$ such that

$$r\_merge(l) = r\_merge(r) = r\_merge(r') = r_0.$$

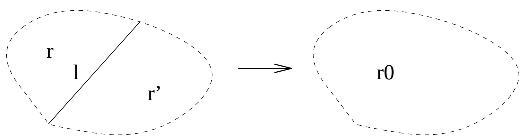


Figure 4.5: Merging two regions into one.

- **point abstraction** (p_abs): a point p is immersed into the region r containing it (i.e., p is a point feature of r, see Figure 4.6). In synthesis:

$$p\_abs : A \rightarrow B$$

where $A = \{r, p\}$ with $p \preceq r$, and $B = \{r_0\}$ such that

$$p\_abs(p) = p\_abs(r) = r_0.$$

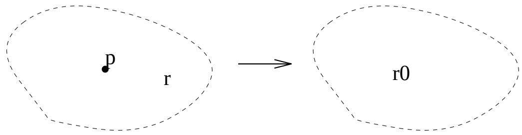


Figure 4.6: Immersion of a point in the region containing it.

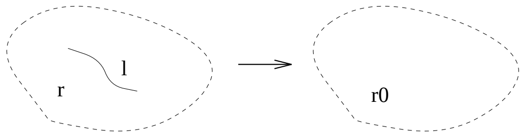


Figure 4.7: Immersion of a line in the region containing it.

- **line abstraction** (l_abs): a line l is immersed into the region r containing it (i.e., l is a line feature of r, see Figure 4.7). In synthesis:

$$l\_abs : A \rightarrow B$$

where $A = \{r, l\}$ with $l \preceq r$, and $B = \{r_0\}$ such that

$$l\_abs(l) = l\_abs(r) = r_0.$$

Remark

Note that, according to the classification of basic transformations of an entity into preservations, reductions and immersions (made in Section 3.4), *contraction* operators correspond to reductions, while *merge* and *abstraction* operators correspond to immersions.

4.2 Atomic operators as morphisms in the category of ACCs

In this Section, we show that the set of operators we have defined are functions between ACCs which are surjective, monotonic, and preserve connected sets by inverse image, i.e., they are morphisms in the category of ACCs. Also, we show that they are strictly non-injective on their domain.

- **surjectivity and strict non-injectivity**: each operator we have defined is surjective and strictly non-injective; indeed, the codomain B is always composed of one entity b, while the domain A consists of 2 or 3 entities and $\forall a \in A$, $a \in f^{-1}(b)$;

- **monotonicity**: since $\forall a \in A,\ f(a) = b$, the condition $\forall a, a' \in A,\ a \preceq a',\ a \neq a' \Rightarrow f(a) \preceq f(a')$ is always satisfied;
- **preserving connected sets by inverse image**: for each operator, the inverse image A of b is a connected set:
  - *line-to-point contraction*: $A = \{l, p_0, p_1\}$, with $p_0 \preceq l, p_1 \preceq l$;
  - *region-to-point contraction*: $A = \{p, l, r\}$ with $p \preceq l \preceq r$;
  - *region-to-line contraction*: $A = \{l, l', r\}$ with $l \preceq r, l' \preceq r$;
  - *line merge*: $A = \{p, l, l'\}$ with $p \preceq l, p \preceq l'$;
  - *region merge*: $A = \{l, r, r'\}$ with $l \preceq r, l \preceq r'$;
  - *point abstraction*: $A = \{p, r\}$ with $p \preceq r$;
  - *line abstraction*: $A = \{l, r\}$ with $l \preceq r$.

4.3 Atomic Operators as Minimal Morphisms

In this Section, we show that each conservative extensions of the operators we have defined is a minimal morphism in the category of GACCs. On the other hand, not all such operators are minimal morphisms in the category of ACCs.

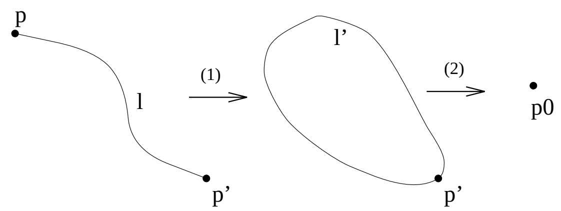
In order to show that conservative extensions of our operators are minimal morphisms in the category of GACCs, we will show that they can only be obtained as composition of two morphisms in the category of ACCs that, even extended conservatively, are not morphisms in the category of GACCs. On the other hand, it is easy to show that such morphisms are minimal in the category of ACCs (since their non-injective restriction maps two entities into one).

Also, it is easy to see that these are the only possible compositions because we can only play on mappings of entities which belong to the non-injective restriction of morphisms, while preserved entities must as well be preserved in intermediate steps. In other words, let *op* be one of our operators, for each conservative extension F of *op* such that F is a morphism in the category of GACCs, all entities that are added to the domain of *op* must also be present in its codomain as well as in all intermediate domains of morphisms composed to obtain F. Furthermore, morphisms in the category of ACCs are monotonic; thus, also constraints on monotonicity restrict the possible compositions.

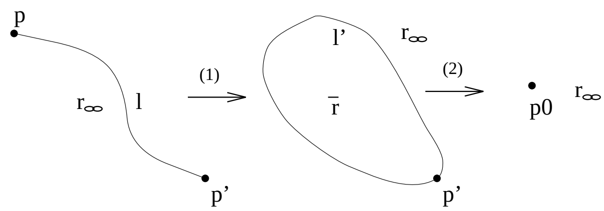
- **Line-to-point contraction**: for each morphism f in the category of ACCs, due to monotonicity constraints (i.e., it must be $f(p') \preceq f(l)$, $f(p) \preceq f(l)$), $f(l)$ can only be a line or a region. In the following we consider both cases.

a) *l mapped to a line*: $lp\_contr = h \circ g$, where g and h are the following two morphisms (which are minimal in the category of ACCs):

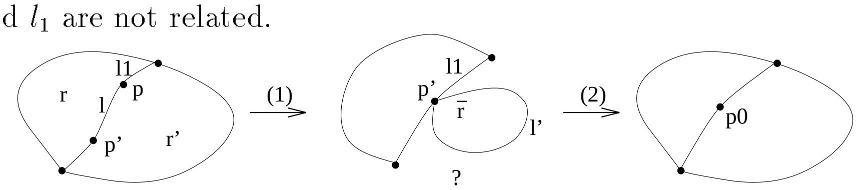
1. $g(p) = g(l) = l'$, $g(p') = p'$, with $p' \preceq l'$;
2. $h(p') = h(l') = p_0$.



Note that each conservative extension F of lp_contr such that F is a morphism in the category of GACCs should include either one or two regions in the star of open line l (l line feature and l bounding line, respectively). Line l' must be a closed line. Indeed, if it was open, then it would have two endpoints p' and p'', then p'' would be a preserved point. In this case, $F^{-1}(\{p'', l'\}) = \{p'', p, l\}$ is not connected (indeed, $p'' \neq p$ is not an endpoint of l). Let us consider the case when l is a line feature of a region r (possibly, $r = r_\infty$). Since l' is closed, then it is a bounding line for two regions r and $\overline{r}$. But in this case $F^{-1}(\{l', \overline{r}\})$ is not connected.

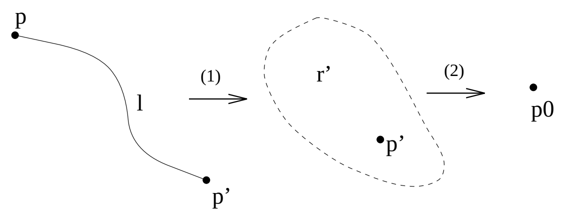


On the other hand, if l is a bounding (open) line, let r and r' be the regions bounded by l; l cannot be the only bounding line for r and r'; thus, there must exist a line l_1 incident at p which is mapped onto itself by a conservative extension of g. In this case, monotonicity is not preserved because $p \preceq l_1$ but l' and l_1 are not related.



b) *l mapped to a region*: $lp\_contr = h \circ g$, where g and h are the following two morphisms (which are minimal in the category of ACCs):

1. $g(p) = g(l) = r'$, $g(p') = p'$, with $p' \preceq r'$;
2. $h(p') = h(r') = p_0$.



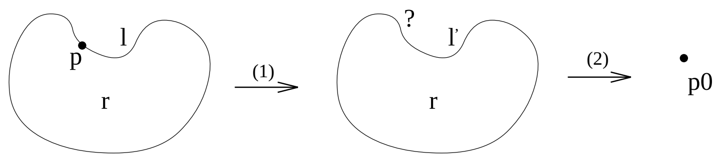
Each conservative extension F of h, such that F is a morphism in the category of GACCs, contains a further line l' such that $l' \preceq r'$. Since $F(r') = p_0$, then to preserve monotonicity, it must be $F(l') = p_0$, i.e., F is not a conservative extension of h.

In both cases a) and b), the same is obtained by first identifying p' and l.

- **Region-to-point contraction**: for each morphism f in the category of ACCs, due to monotonicity constraints (i.e., it must be $f(l) \preceq f(r)$, $f(p) \preceq f(l)$), p and l can only be mapped to a line or a point (if $f(r) = r$), while l and r can only be mapped to a region or a line (if $f(p) = p$). In the following we consider all four cases.

  a) *p and l mapped to a line*: $rp\_contr = h \circ g$, where g and h are the following two morphisms (which are minimal in the category of ACCs):

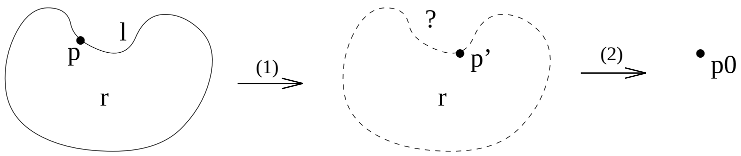
     1. $g(p) = g(l) = l'$, $g(r) = r$, with $l' \preceq r$;
     2. $h(r) = h(l') = p_0$.



  All conservative extensions F of g and F' of h, such that F and F' are morphisms in the category of GACCs, contain a further point p' such that $p' \preceq l'$, in their codomain and domain, respectively. Being F a conservative extension, $F(p') = p'$. Thus, $F^{-1}(\{p', l'\}) = \{p, p', l\}$. In order for this set of entities to be connected, p' must be an endpoint of l. In this case, l is an open line, and thus, there exists another line l_1 bounding r. But if we extend the domain of rp_contr with another line bounding r such line must also be mapped onto p_0 (thus, F' is not a conservative extension). This means l must be closed and, since there exists a point p' such that $p' \preceq l'$, we have that $F^{-1}(\{p', l'\})$ is not connected.

  b) *p and l mapped to a point*: $rp\_contr = h \circ g$, where g and h are the following two morphisms (which are minimal in the category of ACCs):

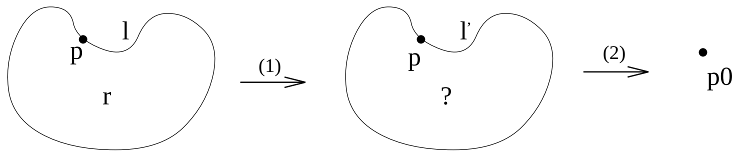
     1. $g(p) = g(l) = p'$, $g(r) = r$, with $p' \preceq r$;
     2. $h(r) = h(p') = p_0$.



Each conservative extension F of h, such that F is a morphism in the category of GACCs, contains a further line l' such that $l' \preceq r$. Since $F(r) = p_0$, then to preserve monotonicity, it must be $F(l') = p_0$, i.e., F is not a conservative extension of h.

c) *r and l mapped to a line*: $rp\_contr = h \circ g$, where g and h are the following two morphisms (which are minimal in the category of ACCs):

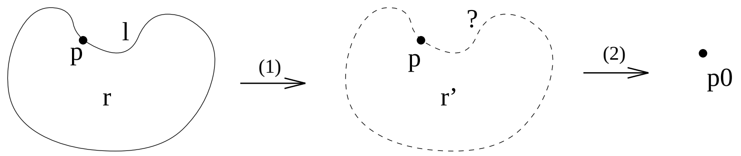
1. $g(r) = g(l) = l'$, $g(p) = p$, with $p \preceq l'$;
2. $h(p) = h(l') = p_0$.



Let us consider a conservative extension F of h, such that $F$is a morphism in the category of GACCs. Line l' must be a closed line. Indeed, if it was open, let p and p' be its endpoints. Since $h(l') = p_0$, to preserve monotonicity, it must be $F(p') = p_0$, i.e., F is not conservative. If l' is closed, then it externally bounds a region that must as well be mapped onto p_0 by any monotonic morphism. Thus F is not conservative.

d) *r and l mapped to a region*: $rp\_contr = h \circ g$, where g and h are the following two morphisms (which are minimal in the category of ACCs):

1. $g(r) = g(l) = r'$, $g(p) = p$, with $p \preceq r'$;
2. $h(p) = h(r') = p_0$.



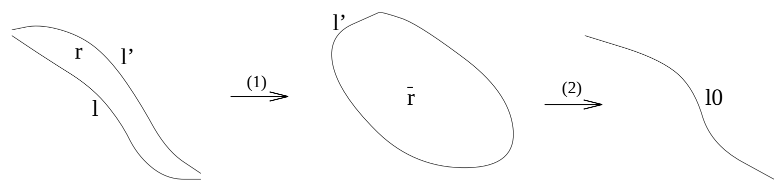
We can conclude as in case b).

- **Region-to-line contraction**: for each morphism f in the category of ACCs, due to monotonicity constraints (i.e., it must be $f(l') \preceq f(r)$, $f(l) \preceq f(r)$), $f(r)$ can only be a region.

  Thus $rl\_contr = h \circ g$, where g and h are the following two morphisms (which are minimal in the category of ACCs):

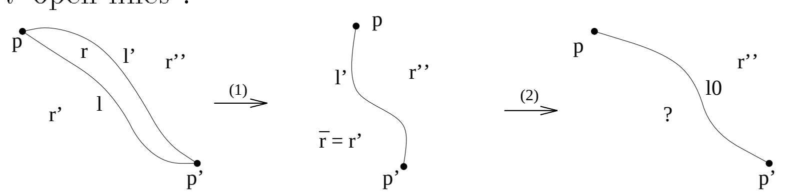
  1. $g(r) = g(l) = \overline{r}$, $g(l') = l'$, with $l' \preceq \overline{r}$;

2. $h(\overline{r}) = h(l') = l_0$.



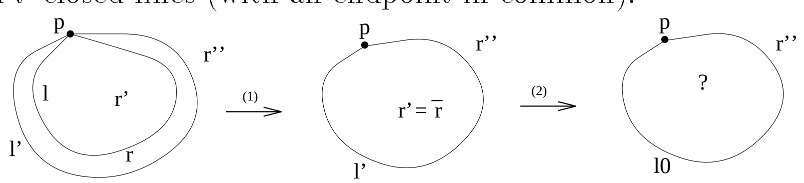
Note that in order for the domain of rl_contr to become a GACC, we must add endpoints of l and l' (either two or one) as well as other regions bounded by l and l' (let them be r' and r'', respectively, possibly with $r' = r''$); the following two situations are possible[2]:

– l and l' open lines[3]:

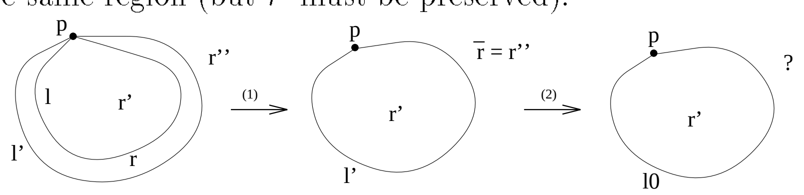


in this case, if we map l and r to a region $\overline{r}$, since $l \preceq r'$, also r' must be mapped to the same region (but r' must be preserved by a conservative extension of rl_contr).

– l and l' closed lines (with an endpoint in common):



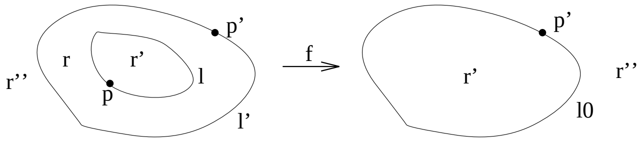
in this case, if we map l and r to a region $\overline{r}$, since $l \preceq r'$, also r' must be mapped to the same region (but r' must be preserved).



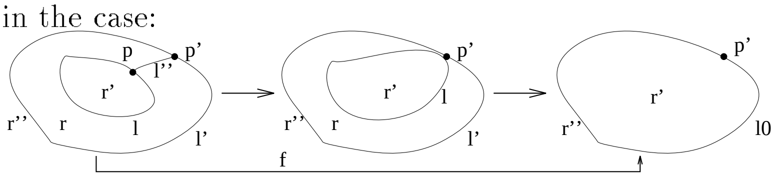
Note that l and l' must have the same endpoint. In this example:

[2]It is easy to show that in the case of an open and a closed line the defined mapping does not preserve connected sets by inverse image.

[3]Note that if we had more than two lines on the border of r, we might first merge all lines and reduce to having only two lines.



we would have a mapping that does not preserve connected sets by inverse image ($f(p) = f(l) = f(l') = f(r) = l_0$ and $f^{-1}(\{p', r'\}) = \{p', r'\}$ is not connected), while in the case:



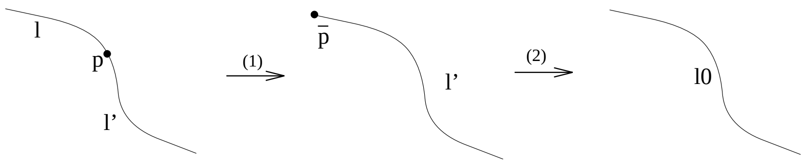
we can first reduce to the case when l and l' have the same endpoint (through lp_contr).

The same is obtained by first identifying l' and r.

- **Line merge**: for each morphism f in the category of ACCs, due to monotonicity constraints (i.e., it must be $f(p) \preceq f(l)$, $f(p) \preceq f(l')$), p and l can only be mapped to a point.

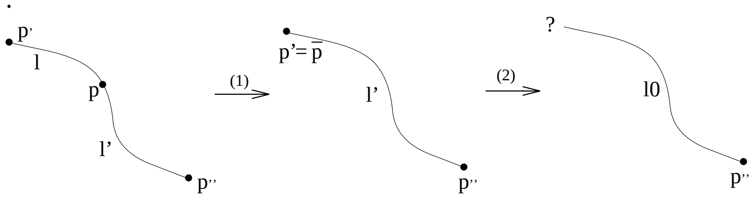
  Thus $l\_merge = h \circ g$, where g and h are the following two morphisms (which are minimal in the category of ACCs):

  1. $g(l) = g(p) = \overline{p}$, $g(l') = l'$, with $\overline{p} \preceq l'$;
  2. $h(\overline{p}) = h(l') = l_0$;

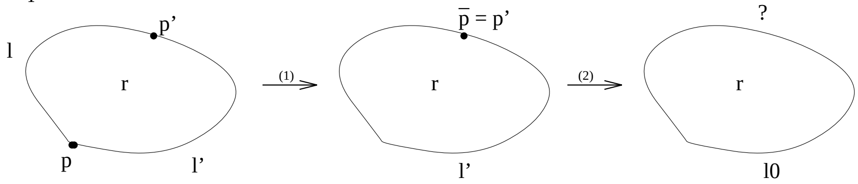
  Note that in order for the domain of l_merge to be a GACC, we must add the second endpoint of l and l' (p' and p'', respectively); the following two situations are possible:

  – $p' \neq p''$:

  in order for the first mapping to preserve the connectivity of $\{p, l, p'\}$, also p' should be mapped to $\overline{p}$ (but p' must be preserved).

– $p' = p''$:



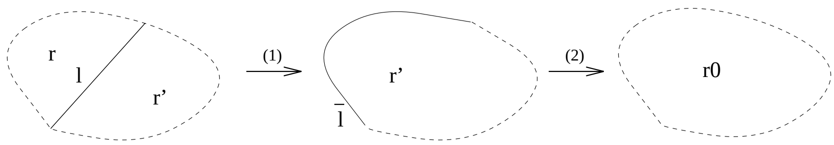
again, to preserve connectivity, the first mapping should as well map p' to $\overline{p}$ (but p' must be preserved).

The same is obtained by first identifying l' and p.

- **Region merge**: for each morphism f in the category of ACCs, due to monotonicity constraints (i.e., it must be $f(l) \preceq f(r)$, $f(l) \preceq f(r')$), if $r \neq r_\infty$[4], r and l can only be mapped to a line or a point (if $f(r') = r'$). In the following we consider both cases.

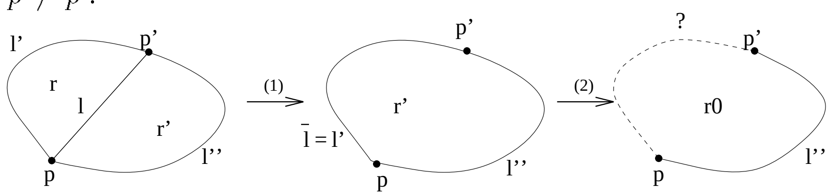
  a) *r and l mapped to a line*: $r\_merge = h \circ g$, where g and h are the following two morphisms (which are minimal in the category of ACCs):

     1. $g(l) = g(r) = \overline{l}$, $g(r') = r'$, with $\overline{l} \preceq r'$;
     2. $h(r') = h(\overline{l}) = r_0$;

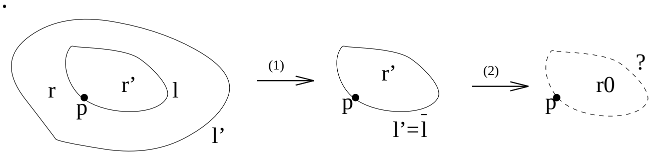
     In order for the domain of r_merge to be a GACC, it must include endpoints p and p' of l and lines bounding r and r'; the following two situations are possible:

     * $p \neq p'$:

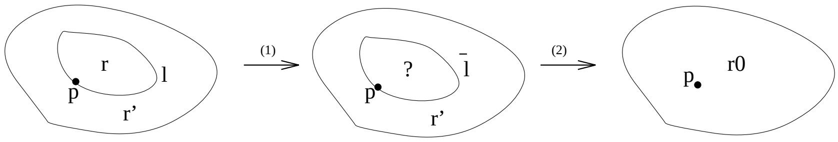
       

       in this case, there must exist another line l' bounding r (because $r \neq r_\infty$, $r \neq r'$ and l open). Thus, if we map l and r to a line $\overline{l}$, for each l' such that $l' \preceq r$, also l' must be mapped to $\overline{l}$ (but lines l' must be preserved).

     * $p = p'$:

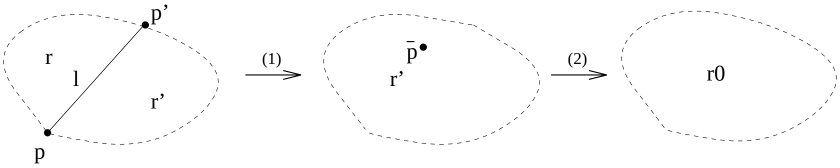
[4] If $r = r_\infty$, it cannot be mapped to a line or a point.



in this case, if we map l and r to a line $\overline{l}$, for each line l', $l' \neq l$ such that $l' \preceq r$ (there must be some such line, being $r \neq r_\infty$), l' must as well be mapped to $\overline{l}$ (with $\overline{l} \preceq r'$, being $l \preceq r'$); but such lines must be preserved.

b) *r and l mapped to a point*: $r\_merge = h \circ g$, where g and h are the following two morphisms (which are minimal in the category of ACCs):

1. $g(r) = g(l) = \overline{p}$, $g(r') = r'$, with $\overline{p} \preceq r'$;
2. $h(r') = h(\overline{p}) = r_0$.



In this case, the endpoints p and p' of l (possibly with $p = p'$) will have to be mapped to $\overline{p}$ by any monotonic extension F of g. Thus, F is not a conservative extension.

- **Point abstraction**: minimal morphism in the category of ACCs;
- **Line abstraction**: minimal morphism in the category of ACCs.

Remark: we have seen that every conservative extension of atomic operators is a minimal morphism in the category of GACCs. Furthermore, they are all necessary: they are all minimal and different (i.e., they are not equivalent to each other).

4.4 Sufficiency of Atomic Operators

We have defined a set of atomic operators (reductions and immersions) with the property of being surjective monotonic functions preserving connected sets by inverse image, and of not implying changes of dimension of entities belonging to the star of the reduced/immersed entity.

In this Section, we show that they are sufficient to describe all consistent combinatorial transformations between geographic maps, i.e., each such transformation can be obtained through a sequence (composition) of atomic operations.

In other words, we will show that the set of operators we have defined corresponds to the set of minimal operators in the category of GACCs. This means that such morphisms are sufficient to generate all morphisms in the category.

In the following Lemmas, we assume that $f : A \to B$ is a minimal morphism in the category of GACCs. We will denote by A' and B' the domain and codomain of the non-injective restriction of f. Also, we will use p, l, and r (with both superscripts and subscripts) to denote a point, a line, and a region, respectively; e will denote a generic entity.

In particular, Lemmas 4.4.12, 4.4.13 and 4.4.14 characterize inverse image of entities in the codomain of f. More specifically, such Lemmas show that the inverse image of any entity through a minimal morphism corresponds to the domain of one of our operators. In other words, each entity in the more generalized geographic map is obtained through one of our operators. The proof is made by contradiction, and shows that when the inverse image of an entity through the considered morphism is larger than all domains of our operators, it is always possible to reduce to one such domain by applying our operators (in which case, the given morphism would not be minimal). Furthermore, we show that the codomain of each minimal operator (i.e., of the morphism in the category of ACCs, corresponding to the representative of the class) contains only one entity (Lemma 4.4.15). From such Lemma and Lemmas characterizing the inverse image of entities, we obtain the sufficiency proof (Theorem 4.4.16)

Lemma 4.4.1 states a property of a connected GACC. The proof follows trivially from the definition.

Lemma 4.4.1 *Let M be a connected GACC[5]. For each $r_i \in M$ ($r_i \neq r_\infty$), there is a region r_j such that either r_i and r_j share some line (or some chain of lines and points) or they are connected through a point or a chain of lines and points.*

Figure 4.8 illustrates Lemma 4.4.1.

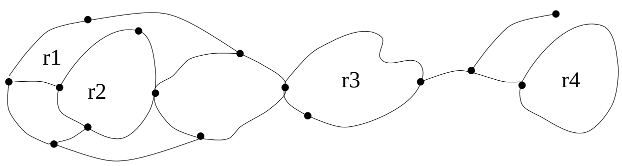


Figure 4.8: Regions r_1 and r_2 share a chain of lines and points, while r_3 and r_4 are connected through a chain of lines and points.

The following Lemma states a property of $*e$, for each e in the inverse image of a region and is illustrated in Figure 4.9.

[5]We assume that M is connected also if deprived of r_∞.

Lemma 4.4.2 *For each $r_0 \in B'$, for each $e \in f^{-1}(r_0)$, all entities in $*e$ belong to $f^{-1}(r_0)$.*

Proof. Since there cannot be any entity e' such that $r \preceq e'$, for the monotonicity of f entities in $*e$ are also mapped to r_0. From a topological point of view, this corresponds to saying that $f^{-1}(r_0)$ is an open set. $\square$

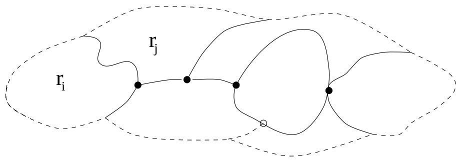


Figure 4.9: Regions r_i and r_j must belong to $f^{-1}(r_0)$ because they are in the star of entities of $f^{-1}(r_0)$ (Lemma 4.4.2).

Lemma 4.4.3 states a property of $\partial(e)$, for each e in the inverse image of a point p_0.

Lemma 4.4.3 *For each $p_0 \in B'$, for each $e \in f^{-1}(p_0)$, all entities in $\partial(e)$ belong to $f^{-1}(p_0)$.*

Proof. Since there cannot be any entity e' such that $e' \preceq p_0$, for the monotonicity of f entities in $\partial(e)$ are also mapped to r_0. From a topological point of view, this corresponds to saying that $f^{-1}(p_0)$ is a closed set. $\square$

Lemma 4.4.4 states a property of the boundary of entities in the inverse image of a line l_0.

Lemma 4.4.4 *For each $l_0 \in B'$, let p_0 and p_1 (possibly, with $p_0 = p_1$) be the endpoints of l_0 in B. Let $\partial(f^{-1}(l_0))$ be the set of entities of A bounding $f^{-1}(l_0)$. More formally, $\partial(f^{-1}(l_0)) = \{e \in A \mid \exists e' \in f^{-1}(l_0),\ e \preceq e',\ and\ \exists e'' \in A \setminus f^{-1}(l_0),\ e'' \in *e\}$. Then $\partial(f^{-1}(l_0))$ is partitioned into four connected sets X_1, X_2, X_3, X_4 such that:*

- $f(X_1) = f(X_2) = l_0$;
- $f(X_3) = p_0$;
- $f(X_4) = p_1$.

Also, X_1 and X_2, as well as X_3 and X_4 are not connected to each other.

Proof. Part of $\partial(f^{-1}(l_0))$ is mapped to endpoints (either one or two) of l_0 (as a consequence of the monotonicity of f). Thus, a part of $\partial(f^{-1}(l_0))$ is mapped to p_0, a part to p_1 and a part to l_0 as well. Actually, the only possibility is to have four connected parts (i.e., chains of lines and points), two (X_3 and X_4) mapped to the endpoints, and the other two (X_1 and X_2) mapped to l_0 (since the two parts mapped to endpoints of l_0 cannot be connected to each other, due to the monotonicity of f). If l_0 is closed (i.e., $p_0 = p_1$), then there will be only three such parts.

Note that X_3 and X_4 cannot contain features and internal boundaries of regions in $f^{-1}(l_0)$.

Indeed, if isolated points and lines inside regions of $f^{-1}(l_0)$ were mapped to endpoints of l_0, then connected sets would not be preserved by inverse image. For a dangling line feature (i.e., a line attached to the boundary of the region containing it through a point p, see Figure 4.10), if it is mapped to an endpoint, say p_0, of l_0, then it would be possible to previously map it to p and then p would be mapped to p_0 (both endpoints of such lines must be mapped to p_0) and, thus, f would not be minimal.

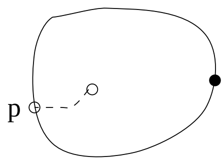


Figure 4.10: Dangling feature in $f^{-1}(l_0)$.

Furthermore, if an internal border of a multiply connected region r of $f^{-1}(l_0)$ were mapped onto endpoints of l_0, connected sets would not be preserved by inverse image (unless such border was connected to the external border through a point or a chain of points, in which case, however it would be possible to contract it to a point previously, see Figure 4.11).

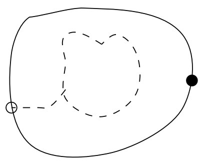


Figure 4.11: Internal border of a region in $f^{-1}(l_0)$.

Thus, the only possibility is that internal borders of regions in $f^{-1}(l_0)$ belong to $f^{-1}(l_0)$ too. $\square$

Lemma 4.4.5 shows that there can be no features in $f^{-1}(e)$, for each entity e in B.

Lemma 4.4.5 *For each $e \in B'$, $f^{-1}(e)$ can contain point and line features only in the following cases: $f^{-1}(e) = dom(p\_abs)$ and $f^{-1}(e) = dom(l\_abs)$, for $e = r_0$.*

Proof. If $f^{-1}(e)$ contains more than one region and also some of their features, then it is possible to first merge such features into the region r containing them (by composing *p_abs* applied to isolated points, and *lp_contr*, *l_merge* and *l_abs* applied to line features, see Figure 4.12) and then map r to e without changing the effect of the morphism. Indeed, isolated features are not related to other entities (except the region r containing them); in the case of a dangling line connected to the boundary through a point p, we can contract the line to p (as in Figure 4.10).

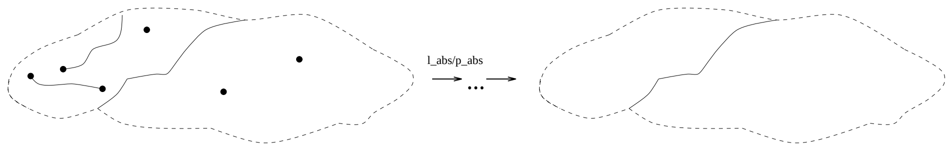


Figure 4.12: Merging features into the region they are contained.

Thus, f would not be minimal, except for the two cases: $f^{-1}(r_0) = \{r, p\}$, with $p \preceq r$, and $f^{-1}(r_0) = \{r, l\}$, with $l \preceq r$, corresponding to $dom(p\_abs)$ and $dom(l\_abs)$, respectively. In both cases $f^{-1}(r_0)$ contains only one region. Note that if there was more than one feature or the line features where composed of chains inside such region, then it would be possible to reduce (by contractions and abstractions) to the case of only one point/line feature (f would not be minimal). $\Box$

Lemmas 4.4.6 and 4.4.7 state that there cannot be chains of lines and points with more than three entities in $f^{-1}(e)$.

Lemma 4.4.6 *For each $e \in B'$, $f^{-1}(e)$ does not contain free chains lines and points, with more than one line, that bound regions also belonging to $f^{-1}(e)$.*

Proof. Let us assume by contradiction that there are free chains of bounding lines and points with more than one line. Such chains cannot be line features (by Lemma 4.4.5). If such chains were shared by two regions, then it would be possible to contract/merge lines in such chains in order to have only one line shared by the two regions that can then be merged (f non minimal, see Figure 4.13).

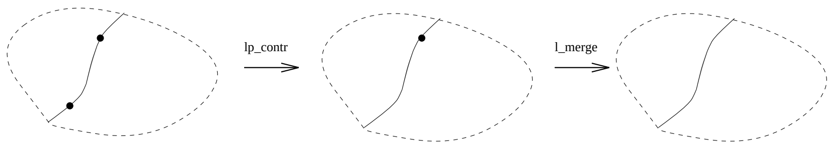


Figure 4.13: Contracting/merging chains of lines shared by two regions.

If such chains bound only one region r in $f^{-1}(e)$ (impossible in the case of $e = r_0$, for Lemma 4.4.2), then again it would be possible to reduce such chains to having only one line. More precisely, in the case of $e = p_0$, we would reduce to having only one closed line plus its endpoint (because the border of r is all contained in $f^{-1}(p_0)$ for Lemma 4.4.3, see Figure 4.14). Thus, f is not minimal except when $f^{-1}(p_0) = dom(rp\_contr)$. Similarly, if

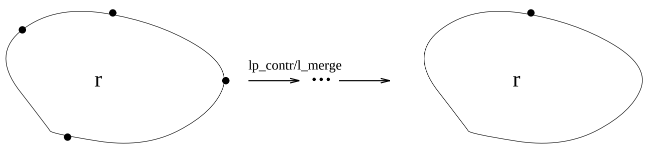


Figure 4.14: Contracting/merging chains of lines on the border of a region in $f^{-1}(p_0)$.

$e = l_0$, it is possible to reduce each such chain to having only one line (f non-minimal, see Figure 4.15 for an example). □

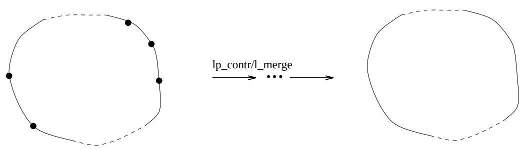


Figure 4.15: Contracting/merging chains of lines on the border of a region in $f^{-1}(l_0)$.

Lemma 4.4.7 *For each $e \in B'$, if $f^{-1}(e)$ contains open chains of lines and points that are not features and do not bound any regions of $f^{-1})(e)$, then the only possibilities are:*

- *$f^{-1}(e) = dom(lp\_contr)$, with $e = p_0$;*
- *$f^{-1}(e) = dom(l\_merge)$, with $e = l_0$;*

Proof. If $e = r_0$, the property holds for Lemma 4.4.2 and for the fact that in a GACC, any line/point has at least a region in its star. If $e = p_0$, then each line also contains its endpoints (Lemma 4.4.3), then we can contract (*lp_contr*, see Figure 4.16) each such line (f non-minimal, except when $f^{-1}(p_0) = dom(lp\_contr)$).

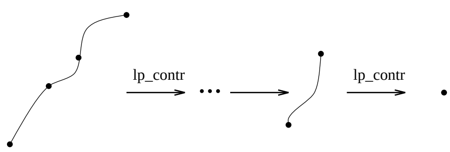


Figure 4.16: Contracting/merging chains of lines in $f^{-1}(p_0)$ not bounding any region.

If $e = l_0$, let us assume that there are chains of lines and points not bounding any region. Then each line in one of such chains can either have both endpoints or lack one of them. However only two lines can lack an endpoint (the two parts mapped onto endpoints of l_0, Lemma 4.4.4). Thus, we can reduce to the case $f^{-1}(l_0) = dom(l\_merge)$ (f non-minimal, see Figure 4.17). □

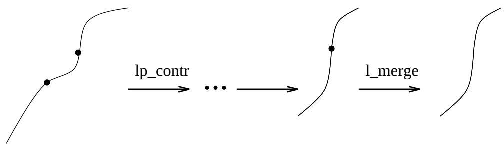


Figure 4.17: Contracting/merging chains of lines in $f^{-1}(l_0)$ not bounding any region.

The following Lemma excludes the presence of more than two adjacent regions in the inverse image of entities of B.

Lemma 4.4.8 *For each $e \in B'$, if e is either a point or a line, $f^{-1}(e)$ does not contain regions sharing lines (*adjacent *regions). If e is a region, then there can be at most two adjacent regions and such regions, only share one line.*

Proof. For Lemma 4.4.5 and Lemma 4.4.6 regions in $f^{-1}(e)$ do not have features and they cannot share chains of lines and points.

Pairs of adjacent regions could be previously merged (by applying *r_merge*) and reduce to the case when $f^{-1}(e)$ contains only two regions r and r' sharing only one line (see Figure 4.18).

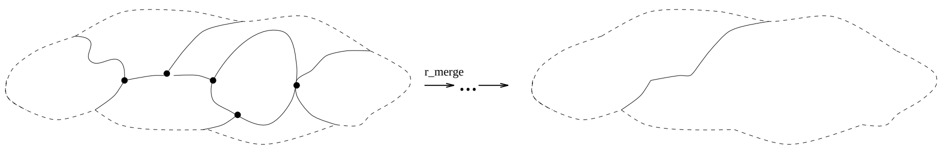


Figure 4.18: Merging adjacent regions.

This includes the case of multiply connected regions: if also (part of) internal boundaries of regions are contained in $f^{-1}(e)$, then the multiply connected region and the regions filling its holes can be merged (such regions would also belong to $f^{-1}(e)$ if e is a region r_0 being on the star of lines in $f^{-1}(r_0)$ for Lemma 4.4.2). In any of the above described cases, f would not be minimal.

Furthermore, if regions shared more than one line as in Figure 4.19, we could merge them eliminating one of such lines and then merge the others as features of the single resulting region (f non-minimal).

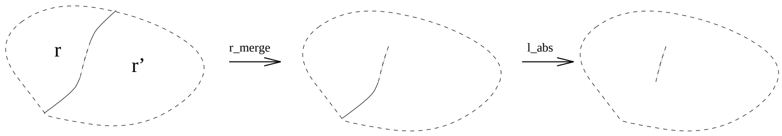


Figure 4.19: Regions in $f^{-1}(e)$ sharing more than one line.

For $e = p_0$, also the boundary of the two regions belongs to $f^{-1}(p_0)$ (Lemma 4.4.3). Thus, we can merge also such regions and reduce to one region plus its border (one line and its endpoint, no chains for Lemma 4.4.6), i.e., to $dom(rp\_contr)$ (see Figure 4.20, for an example). Thus f is not minimal.

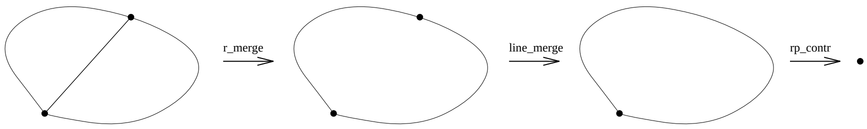


Figure 4.20: Merging two adjacent regions belonging to $f^{-1}(p_0)$ into one.

For $e = l_0$, part of the border of regions is mapped to endpoints of l_0. Again, the two regions can be merged into one and reduced to $dom(rl\_contr)$ (see Figure 4.21).

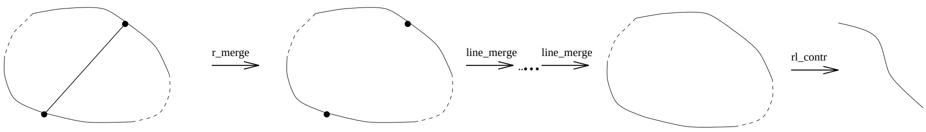


Figure 4.21: Merging two adjacent regions belonging to $f^{-1}(l_0)$ into one.

So there can be at most two adjacent regions in $f^{-1}(r_0)$ and they can only share one line. □

Lemma 4.4.9 excludes the presence of cycles of lines and points (closed chains) in $f^{-1}(e)$ such that the "enclosed" region does not belong to $f^{-1}(e)$.

Lemma 4.4.9 *For each $e \in B'$, there is no closed chain of lines and points in $f^{-1}(e)$, unless the region having such chain as its border belongs to $f^{-1}(e)$ as well.*

Proof. Let us assume that such chains are in $f^{-1}(e)$. Examples are provided by two regions sharing more than one point as in Figure 4.22 and by cycles of regions as in Figure 4.23. In both Figures, filled regions belong to $f^{-1}(e)$ while r'' (in Figure 4.22) and r (in Figure 4.23) are not included.

This is impossible if $e = r_0$ for Lemma 4.4.2. If $e = p_0$, then in order to preserve monotonicity and connected sets by inverse image, also the enclosed regions (r'' in Figure 4.22 and r

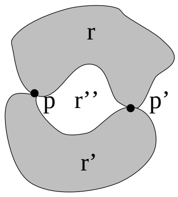


Figure 4.22: Two regions sharing more than one point.

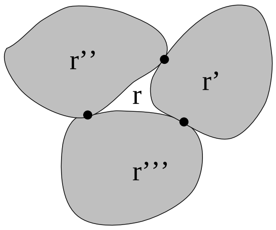


Figure 4.23: A cycle of regions.

in Figure 4.23) would belong to $f^{-1}(p_0)$. Thus we would have adjacent regions (impossible for Lemma 4.4.8).

The same holds for $e = l_0$ since internal boundaries of regions in $f^{-1}(l_0)$ also belong to $f^{-1}(l_0)$ (they cannot be mapped to endpoints of l_0 for Lemma 4.4.4). In this case, also the enclosed region must belong to $f^{-1}(l_0)$, resulting in adjacent regions (impossible for Lemma 4.4.8). □

We now define a *tree of regions*, a concept that will be used in Lemma 4.4.11 (see Figure 4.24 for an example).

Definition 4.4.10 *We define a* tree of regions *as a tree in which nodes are simply connected regions and there exists an arc between two nodes if and only if the corresponding regions share only one bounding point.*

Lemma 4.4.11 *For each $e \in B'$, the set of all regions in $f^{-1}(e)$ cannot form a tree of regions (with more than one node).*

Proof. If e is a region r_0, the property holds because, for Lemma 4.4.2, for each point shared by two regions (i.e., corresponding to an arc of the tree) also the other regions in its star would belong to $f^{-1}(e)$. If e is a point p_0, let us assume that regions $f^{-1}(p_0)$ form a tree of regions with more than one node. For Lemma 4.4.3 all such regions have

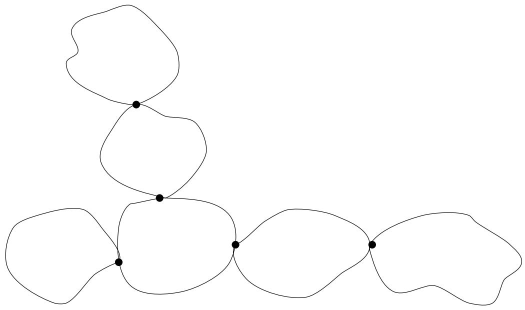


Figure 4.24: A tree of regions.

all their boundary in $f^{-1}(p_0)$. Thus we can contract all regions (but one) in $f^{-1}(p_0)$ to a point through *rp_contr* and at the same time reduce their borders to only one line plus its endpoint (*lp_contr*), see Figure 4.25 for an example.

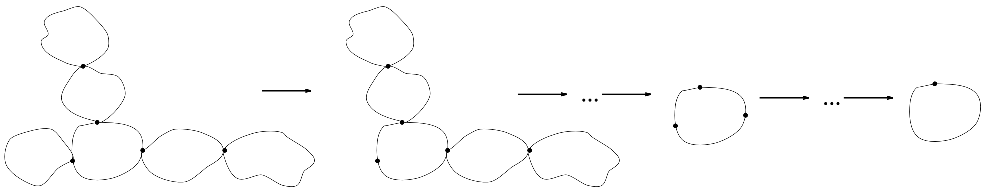


Figure 4.25: Contracting a tree of regions.

Thus f is not minimal. The only possible case is $f^{-1}(p_0) = \{r, l, p\}$, corresponding to *rp_contr*.

If e is a line l_0, let us assume that regions in $f^{-1}(l_0)$ form a tree of regions with more than one node. For Lemma 4.4.4, two connected parts of the boundary of entities in $f^{-1}(l_0)$ are mapped to endpoints of l_0. Each such part belongs to the boundary of a single region in $f^{-1}(l_0)$ (or we would disconnect the tree of regions, see Figure 4.26) and it must be a chain including endpoints (each line contracted to a point induces also its endpoints to be mapped to the same point), see Figure 4.27 for an example.

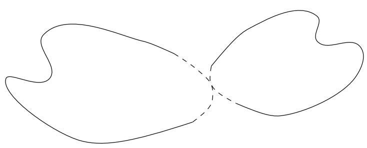


Figure 4.26: A non-connected tree of regions.

In other words, at most two regions in $f^{-1}(l_0)$ can have part of their boundary which is not mapped to l_0. Thus, all the other regions can be contracted (*rp_contr*). Also in this case, f would not be minimal. Thus, we can have at most two regions in $f^{-1}(l_0)$ if they

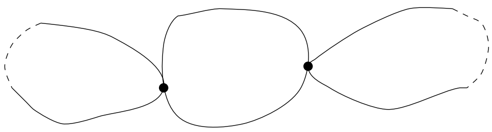


Figure 4.27: Parts mapped to endpoints of l_0 are chains including endpoints.

form a tree of regions. If we have two regions, the only possibility is that they share a point p and part of the boundary of both is not contained in $f^{-1}(l_0)$ (see Figure 4.28).

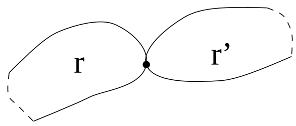


Figure 4.28: Two regions sharing one point in $f^{-1}(l_0)$.

In this case, however, it is possible to apply separately *rl_contr* to each region (since its boundary is composed of two lines[6] sharing p) and then *l_merge* to the resulting configuration. Thus, f would not be minimal (see Figure 4.29). □

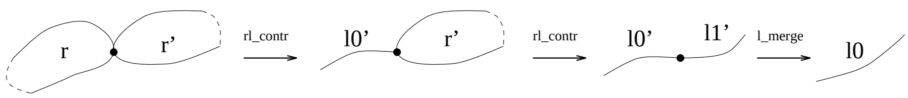


Figure 4.29: An example where f is obtained as composition of *rl_contr* and *l_merge*.

Lemma 4.4.12 characterizes the inverse image of a region r_0.

Lemma 4.4.12 *For each $r_0 \in B'$, $f^{-1}(r_0)$ can only be equal to one of the following ACCs:*

1. *$\{r, r', l\}$, with $l \preceq r$ and $l \preceq r'$ (i.e., dom(r_merge));*
2. *$\{r, p\}$, with $p \preceq r$ (i.e., dom(p_abs));*
3. *$\{r, l\}$, with $l \preceq r$ (i.e., dom(l_abs)).*

Proof. For Lemma 4.4.2, $f^{-1}(r_0)$ must contain at least one region (because each line and point in a geographic map has at least one region in its star). Let us assume by contradiction that there are more than two regions in $f^{-1}(r_0)$, let them be $r_1, r_2, \ldots, r_n$, $n > 2$.

[6]Note that such lines must exist since the part of boundary mapped to the endpoint of l'_0 must include endpoints but cannot include the point shared by r and r'.

Being $f^{-1}(r_0)$ connected, Lemma 4.4.1 holds. However, it is not possible that two regions are connected through a point or a chain of lines and points (as r_3 and r_4 in Figure 4.30), because in this case also regions in the star of those lines and points would belong to $f^{-1}(r_0)$ (for Lemma 4.4.2). Furthermore, for Lemma 4.4.8, there can be at most two adjacent regions r and r', and they share only one line. In this case, there cannot be other

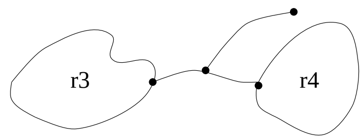


Figure 4.30: Regions connected through chains of lines and points.

line/points in $f^{-1}(r_0)$. Indeed, for Lemma 4.4.5, there cannot be features; other points/line on the border of either r or r' (except the shared line) would cause other regions in their star to be included in $f^{-1}(r_0)$ too (Lemma 4.4.2). Thus, we can only have $f^{-1}(r_0) = \{r, r', l\}$, with $l \preceq r$ and $l \preceq r'$ (i.e. $dom(r\_merge)$).

If $f^{-1}(r_0)$ contains only one region (no chains for Lemma 4.4.6), the only possibilities are: $f^{-1}(r_0) = \{r, p\}$, with $p \preceq r$, and $f^{-1}(r_0) = \{r, l\}$, with $l \preceq r$, corresponding to $dom(p\_abs)$ and $dom(l\_abs)$, respectively (since points and lines on the border of r would cause other regions on their star to belong to $f^{-1}(r_0)$ too, for Lemma 4.4.2). $\square$

Lemma 4.4.13 characterizes the inverse image of p_0.

Lemma 4.4.13 *For each $p_0 \in B'$, $f^{-1}(p_0)$ can only be equal to one of the following ACCs:*

1. *$\{r, l, p\}$ with $p \preceq l \preceq r$ (i.e., $dom(rp\_contr)$);*
2. *$\{l, p, p'\}$, with $p \preceq l$ and $p' \preceq l$ (i.e., $dom(lp\_contr)$).*

Proof. Let us assume by contradiction that there is more than one region in $f^{-1}(p_0)$, let them be $r_1, r_2, \ldots, r_n$, $n > 1$. Such regions cannot have features for Lemma 4.4.5. Being $f^{-1}(p_0)$ connected, Lemma 4.4.1 holds.

For Lemma 4.4.8, there cannot be adjacent regions. For Lemma 4.4.7, regions cannot be connected by chains of lines and points (not bounding regions in $f^{-1}(p_0)$), so the only possibility is that all lines in $f^{-1}(p_0)$ bound some region and regions are all connected through shared points. Also, there cannot be cycles of regions (Lemma 4.4.9). For Lemma 4.4.11, $f^{-1}(p_0)$ cannot be a tree of regions with more than one region (in which case it would correspond to $dom(rp\_contr)$). If there is no region in $f^{-1}(p_0)$ (i.e., lines in $f^{-1}(p_0)$ bound no region), then the only possibility is $f^{-1}(p_0) = \{p, p', l\} = dom(lp\_contr)$ (Lemma

4.4.7). □

Lemma 4.4.14 characterizes the inverse image of a line l_0.

Lemma 4.4.14 *For each $l_0 \in B'$, $f^{-1}(l_0)$ can only be equal to one of the following ACCs:*

1. *$\{p, l, l'\}$, with $p \preceq l$ and $p \preceq l'$ (i.e., dom(l_merge));*
2. *$\{l, l', r\}$, with $l \preceq r$ and $l' \preceq r$ (i.e., dom(rl_contr)).*

Proof. For Lemma 4.4.4, $\partial(f^{-1}(l_0))$ is partitioned into four connected sets. Also, each part mapped onto an endpoint of l_0 is either a chain including endpoints (each line contracted to a point induces also its endpoints to be mapped to the same point) belonging to only one region, or is a missing endpoint of a dangling line.

Let us assume by contradiction that there is more than one region in $f^{-1}(l_0)$, let them be $r_1, r_2, \ldots, r_n$, $n > 1$. For Lemma 4.4.5, there are no features inside regions in $f^{-1}(l_0)$. Also, being $f^{-1}(l_0)$ connected, Lemma 4.4.1 holds. For Lemma 4.4.8, there cannot be adjacent regions, while for Lemma 4.4.6, regions cannot be connected by chains of lines and points.

Thus, the only lines in $f^{-1}(l_0)$ bound some region or are dangling lines (with an endpoint missing[7]) attached to the boundary of some region and regions are all connected through shared points (see Figure 4.31).

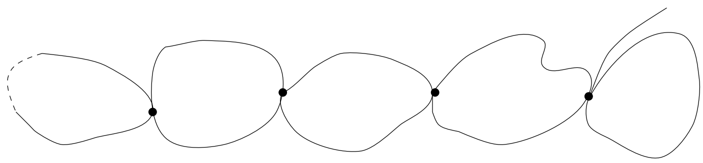


Figure 4.31: Entities in $f^{-1}(l_0)$.

However, for Lemma 4.4.9 there cannot be cycles of regions, while, for Lemma 4.4.11, regions cannot form a tree with more than one node.

This means that $f^{-1}(l_0)$ can contain at most one region r. Let $f^{-1}(l_0)$ contain one region. In this case, we have two possibilities:

- the only lines belonging to $f^{-1}(l_0)$ bound r as in Figure 4.32 (note that there cannot be bounding chains for Lemma 4.4.6).

  Thus, we have $f^{-1}(l_0) = \{l, l', r\}$ with $l \preceq r$, $l' \preceq r$ (corresponding to *dom(rl_contr)*).

[7]If an endpoint was not missing, such lines could be contracted to a point.

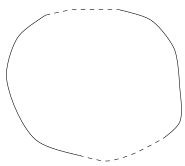


Figure 4.32: Lines in $f^{-1}(l_0)$.

- some of the lines belonging to $f^{-1}(l_0)$ bound r, and there is an additional dangling line attached to the connection point p lacking the other endpoint (see Figure 4.33).

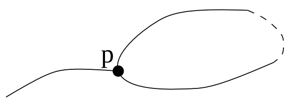


Figure 4.33: An example of $f^{-1}(l_0)$.

As in the case of Figure 4.29, it is possible to apply *rl_contr* to r (again its boundary is composed of two lines without their endpoints sharing p) and then *l_merge* to the resulting configuration (i.e., f is not minimal, see Figure 4.34).

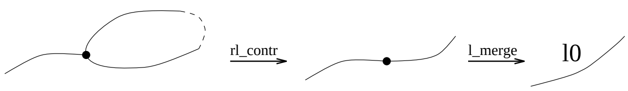


Figure 4.34: f is obtained as composition of *rl_contr* and *l_merge*.

When $f^{-1}(l_0)$ does not contain any region, then for Lemma 4.4.7, the only possibility is $f^{-1}(l_0) = \{p, l, l'\}$ with $p \preceq l$ and $p \preceq l'$ (i.e., $dom(l\_merge)$). $\square$

The following Lemma shows that the codomain of the non-injective restriction of a minimal morphism in the category of GACCs is composed of only one entity.

Lemma 4.4.15 *For each minimal morphism $f : A \to B$, $|B'| = 1$.*

Proof. Let us assume by contradiction that $|B'| > 1$. Then there exist two entities $b, b' \in B'$ such that $b \preceq b'$ (since B' is connected for Lemma 3.6.6). Since f preserves connected sets by inverse image, $f^{-1}(\{b, b'\})$ must be connected, i.e., there exist some entity $a \in f^{-1}(b)$ and some entity $a' \in f^{-1}(b')$ such that $a \preceq a'$.

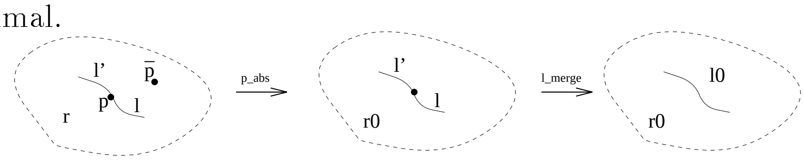
Being $b \preceq b'$, b cannot be a region and b' cannot be a point. Three cases are possible for b and b':

1. b is a line and b' is a region;
2. b is a point and b' is a line;
3. b is a point and b' is a region.

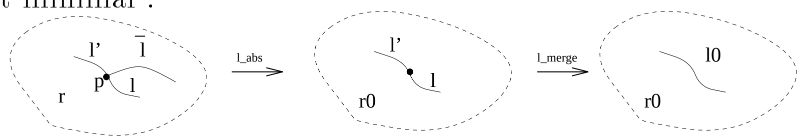
For each such case, we consider the possible inverse images of b and b' according to Lemmas 4.4.12, 4.4.13, 4.4.14. Thus we have 6 possibilities for case 1), 4 possibilities for case 2), and 6 possibilities for case 3). Also, we consider all possible relations between entities in $f^{-1}(b)$ and $f^{-1}(b')$. We examine each case separately and show that in each of them we obtain a contradiction, i.e., f is not minimal.

1. let $b = l_0$, $b' = r_0$ with $l_0 \preceq r_0$;

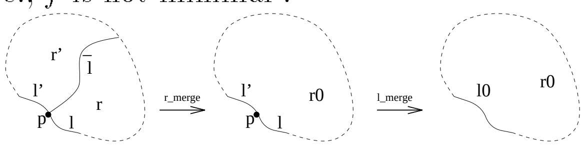
   a) $f^{-1}(l_0) = \{l, l', p\}$ (l_merge) and $f^{-1}(r_0) = \{\overline{p}, r\}$ (p_abs); the only possible relations between $f^{-1}(l_0)$ and $f^{-1}(r_0)$ are $l \preceq r$ and $l' \preceq r$ (and thus also $p \preceq r$); indeed it is not possible to have $\overline{p} \preceq l$ or $\overline{p} \preceq l'$ because it is $l_0 \preceq r_0$. Thus, we can apply l_merge to $\{l, l', p\}$ and p_abs to $\{\overline{p}, r\}$ independently, i.e., f is not minimal.

   b) $f^{-1}(l_0) = \{l, l', p\}$ (l_merge) and $f^{-1}(r_0) = \{\overline{l}, r\}$ (l_abs); the only possible relations between $f^{-1}(l_0)$ and $f^{-1}(r_0)$ are $p \preceq \overline{l}$, $l \preceq r$ and $l' \preceq r$ (and thus also $p \preceq r$). Thus, we can apply l_abs to $\{\overline{l}, r\}$ and then l_merge to $\{l, l', p\}$, i.e., f is not minimal[8].

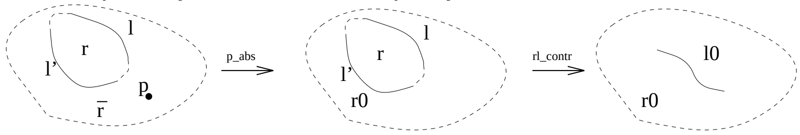
   

   c) $f^{-1}(l_0) = \{l, l', p\}$ (l_merge) and $f^{-1}(r_0) = \{\overline{l}, r, r'\}$ (r_merge); the only possible relations between $f^{-1}(l_0)$ and $f^{-1}(r_0)$ are $p \preceq \overline{l}$, $l \preceq r$, $l' \preceq r$, $l \preceq r'$, and $l' \preceq r'$ (and thus also $p \preceq r$). Thus, we can apply r_merge to $\{\overline{l}, r, r'\}$ and then l_merge to $\{l, l', p\}$, i.e., f is not minimal[9].

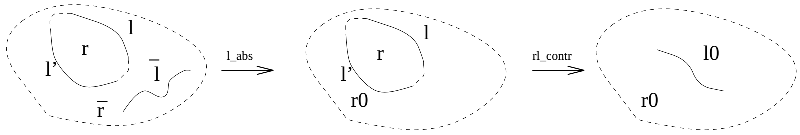
   

[8]Note that if $p \preceq \overline{l}$, the order in which we apply the two operators matters.
[9]Note that if $p \preceq \overline{l}$, the order in which we apply the two operators matters.

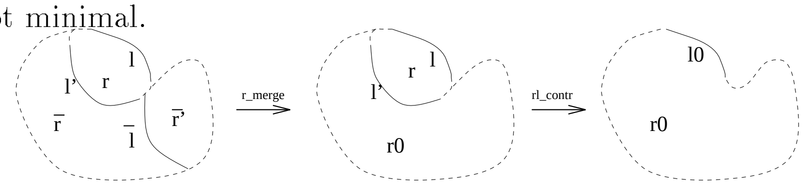
d) $f^{-1}(l_0) = \{r, l, l'\}$ (*rl_contr*) and $f^{-1}(r_0) = \{p, \overline{r}\}$ (*p_abs*); the only possible relations between $f^{-1}(l_0)$ and $f^{-1}(r_0)$ are $l \preceq \overline{r}$ and $l' \preceq \overline{r}$. Thus, we can apply *rl_contr* to $\{r, l, l'\}$ and *p_abs* to $\{p, \overline{r}\}$ independently, i.e., f is not minimal.



e) $f^{-1}(l_0) = \{r, l, l'\}$ (*rl_contr*) and $f^{-1}(r_0) = \{\overline{l}, \overline{r}\}$ (*l_abs*); the only possible relations between $f^{-1}(l_0)$ and $f^{-1}(r_0)$ are $l \preceq \overline{r}$ and $l' \preceq \overline{r}$. Thus, we can apply *rl_contr* to $\{r, l, l'\}$ and *l_abs* to $\{\overline{l}, \overline{r}\}$ independently, i.e., f is not minimal.

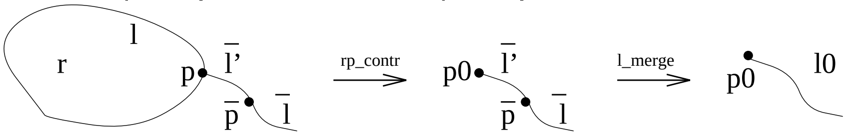


f) $f^{-1}(l_0) = \{r, l, l'\}$ (*rl_contr*) and $f^{-1}(r_0) = \{\overline{l}, \overline{r}, \overline{r'}\}$ (*r_merge*); the only possible relations between $f^{-1}(l_0)$ and $f^{-1}(r_0)$ are $l \preceq \overline{r}$, $l' \preceq \overline{r}$, $l \preceq \overline{r'}$ and $l' \preceq \overline{r'}$. Thus, we can apply *rl_contr* to $\{r, l, l'\}$ and *r_merge* to $\{\overline{l}, \overline{r}, \overline{r'}\}$ independently, i.e., f is not minimal.

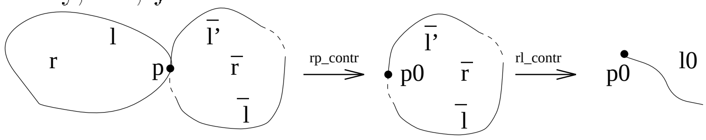


2. let $b = p_0$, $b' = l_0$ with $p_0 \preceq l_0$;

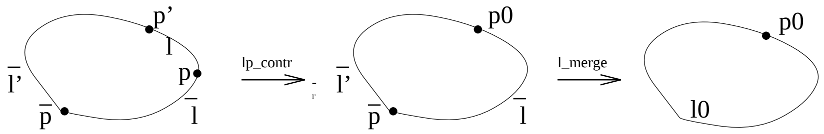
a) $f^{-1}(p_0) = \{r, l, p\}$ (*rp_contr*) and $f^{-1}(l_0) = \{\overline{l}, \overline{l'}, \overline{p}\}$ (*l_merge*); the only possible relations between $f^{-1}(p_0)$ and $f^{-1}(l_0)$ are $p \preceq \overline{l}$, $p \preceq \overline{l'}$. Thus, we can apply *rp_contr* to $\{r, l, p\}$ and *l_merge* to $\{\overline{l}, \overline{l'}, \overline{p}\}$ independently, i.e., f is not minimal.



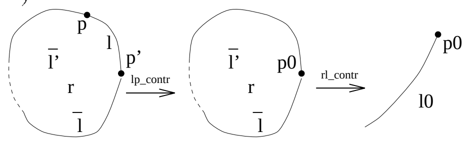
b) $f^{-1}(p_0) = \{r, l, p\}$ (*rp_contr*) and $f^{-1}(l_0) = \{\overline{l}, \overline{l'}, \overline{r}\}$ ($rl_c ontr$); the only possible relations between $f^{-1}(p_0)$ and $f^{-1}(l_0)$ are $p \preceq \overline{l}$, $p \preceq \overline{l'}$ (and thus, $p \preceq \overline{r}$). Indeed, it cannot be $l \preceq \overline{r}$ because otherwise, l, p, and r should be mapped to l_0 instead of p_0. Thus, we can apply *rp_contr* to $\{r, l, p\}$ and *rl_contr* to $\{\overline{l}, \overline{l'}, \overline{r}\}$ independently, i.e., f is not minimal.



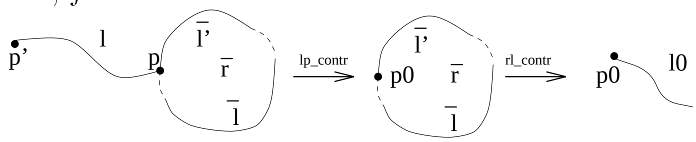
c) $f^{-1}(p_0) = \{p, p', l\}$ (*lp_contr*) and $f^{-1}(l_0) = \{\overline{l}, \overline{l'}, \overline{p}\}$ (*l_merge*); the only possible relations between $f^{-1}(p_0)$ and $f^{-1}(l_0)$ are $p \preceq \overline{l}$, $p \preceq \overline{l'}$, $p' \preceq \overline{l}$, and $p' \preceq \overline{l'}$ $(p, p' \neq \overline{p})$. Thus, we can apply *lp_contr* to $\{p, p', l\}$ and *l_merge* to $\{\overline{l}, \overline{l'}, \overline{p}\}$ independently, i.e., f is not minimal.



d) $f^{-1}(p_0) = \{p, p', l\}$ (*lp_contr*) and $f^{-1}(l_0) = \{\overline{l}, \overline{l'}, r\}$ (*rl_contr*); the only possible relations between $f^{-1}(p_0)$ and $f^{-1}(l_0)$ are $p \preceq \overline{l}$, $p \preceq \overline{l'}$, $p' \preceq \overline{l}$, $p' \preceq \overline{l'}$, and $l \preceq r$ (however, if $l \preceq r$, p and p' cannot contemporarily be endpoints of both $\overline{l}$ and $\overline{l'}$, see the example).

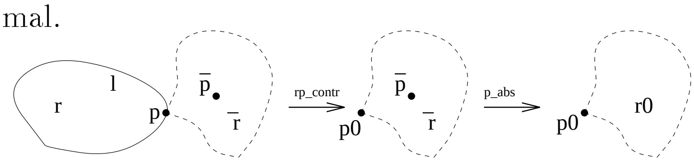


Thus, we can apply *lp_contr* to $\{p, p', l\}$ and *l_merge* to $\{\overline{l}, \overline{l'}, \overline{p}\}$ (order matters if $l \preceq r$), i.e., f is not minimal.

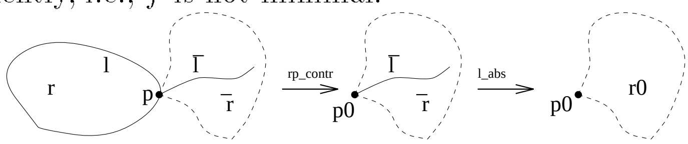


3. let $b = p_0$, $b' = r_0$ with $p_0 \preceq r_0$;

a) $f^{-1}(p_0) = \{r, l, p\}$ (*rp_contr*) and $f^{-1}(r_0) = \{\overline{p}, \overline{r}\}$ (*p_abs*); the only possible relations between $f^{-1}(p_0)$ and $f^{-1}(r_0)$ are $p \preceq \overline{r}$, $l \preceq \overline{r}$ (i.e., $p \preceq l \preceq \overline{r}$. Thus, we can apply *rp_contr* to $\{r, l, p\}$ and *p_abs* to $\{\overline{p}, \overline{r}\}$ independently, i.e., f is not minimal.

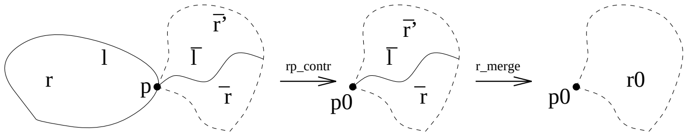


b) $f^{-1}(p_0) = \{r, l, p\}$ (*rp_contr*) and $f^{-1}(r_0) = \{\overline{l}, \overline{r}\}$ (*l_abs*); the only possible relations between $f^{-1}(p_0)$ and $f^{-1}(r_0)$ are $p \preceq \overline{l}$ (and thus $p \preceq \overline{r}$), $p \preceq \overline{r}$ (with $p \not\preceq \overline{l}$), and $l \preceq \overline{r}$. Thus, we can apply *rp_contr* to $\{r, l, p\}$ and *l_abs* to $\{\overline{l}, \overline{r}\}$ independently, i.e., f is not minimal.

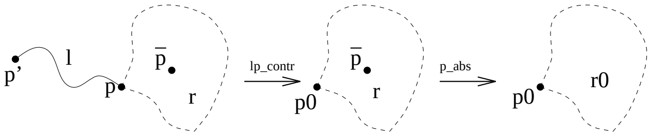


c) $f^{-1}(p_0) = \{r, l, p\}$ (*lp_contr*) and $f^{-1}(r_0) = \{\overline{p}, \overline{r}, \overline{r'}\}$ (*r_merge*); the only possible relations between $f^{-1}(p_0)$ and $f^{-1}(r_0)$ are $p \preceq \overline{l}$, $p \preceq \overline{r}$, $p \preceq \overline{r'}$, $l \preceq \overline{r}$, $l \preceq \overline{r'}$.

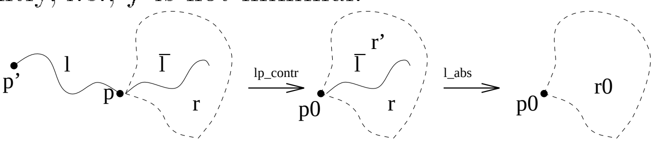
Thus, we can apply *rp_contr* to $\{r, l, p\}$ and *r_merge* to $\{\overline{l}, \overline{r}, \overline{r'}\}$ independently, i.e., f is not minimal.



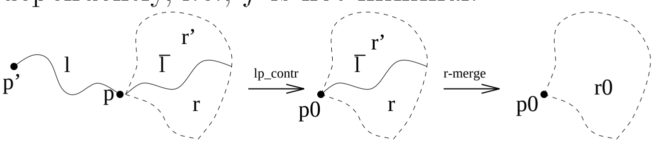
d) $f^{-1}(p_0) = \{l, p, p'\}$ (*lp_contr*) and $f^{-1}(r_0) = \{\overline{p}, r\}$ (*p_abs*); the only possible relations between $f^{-1}(p_0)$ and $f^{-1}(r_0)$ are $p \preceq r$, $p' \preceq r$, $l \preceq r$. Thus, we can apply *lp_contr* to $\{l, p, p'\}$ and *p_abs* to $\{\overline{p}, r\}$ independently, i.e., f is not minimal.



e) $f^{-1}(p_0) = \{l, p, p'\}$ (*lp_contr*) and $f^{-1}(r_0) = \{\overline{l}, r\}$ (*l_abs*); the only possible relations between $f^{-1}(p_0)$ and $f^{-1}(r_0)$ are $p \preceq r$, $p' \preceq r$, $p \preceq \overline{l}$, $p' \preceq \overline{l}$, $l \preceq r$ (actually, since $\overline{l}$ must be a line feature of r, it cannot have both p and p' as endpoints). Thus, we can apply *lp_contr* to $\{l, p, p'\}$ and *l_abs* to $\{\overline{l}, r\}$ independently, i.e., f is not minimal.



f) $f^{-1}(p_0) = \{l, p, p'\}$ (*lp_contr*) and $f^{-1}(r_0) = \{\overline{l}, r, r'\}$ (*r_merge*); the only possible relations between $f^{-1}(p_0)$ and $f^{-1}(r_0)$ are $p \preceq \overline{l}$, $p' \preceq \overline{l}$, $l \preceq r$, $l \preceq r'$, $p \preceq r$, $p \preceq r'$, $p' \preceq r$, $p' \preceq r'$. Thus, we can apply *lp_contr* to $\{l, p, p'\}$ and *r_merge* to $\{\overline{l}, r, r'\}$ independently, i.e., f is not minimal.



□

Theorem 4.4.16 *The set of minimal operators in the category of GACCs coincides with the set* $\{lp\_contr, rp\_contr, rl\_contr, l\_merge, r\_merge, p\_abs, l\_abs\}$.

Proof. It follows from Lemmas 4.4.13, 4.4.12, 4.4.14 and 4.4.15. □

4.5 Diversity between ACCs

In this Section, we provide a formalization of the concept of *diversity* between geographic maps, as a way to measure to what degree a map is the generalization of another one. We consider two possible definitions:

- *global diversity*: it corresponds to the total number of minimal operators applied to the first map in order to obtain the second one;
- *local diversity*: for each entity in the less detailed map, we consider the number of operators we must apply in order to obtain it from the group of entities in its inverse image, and then we get the maximum over all the entities.

An interesting problem is the following: given a non-minimal morphism f do all sequences of minimal (non-identical) morphisms such that their composition gives f have the same length?

In Figure 4.35, we show an example of a non-minimal morphism (Figure 4.35(a)) obtained as composition of minimal morphisms in two different ways (Figure 4.35(b)(1) and 4.35(b)(2)). Note that the two sequences have different lengths: in particular, in Figure 4.35(b)(1), *lp_contr*, *p_abs*, and *rp_contr* are applied in the order, while in Figure 4.35(b)(2), the sequence is *l_abs*, *p_abs*, *p_abs*, and *rp_contr*.

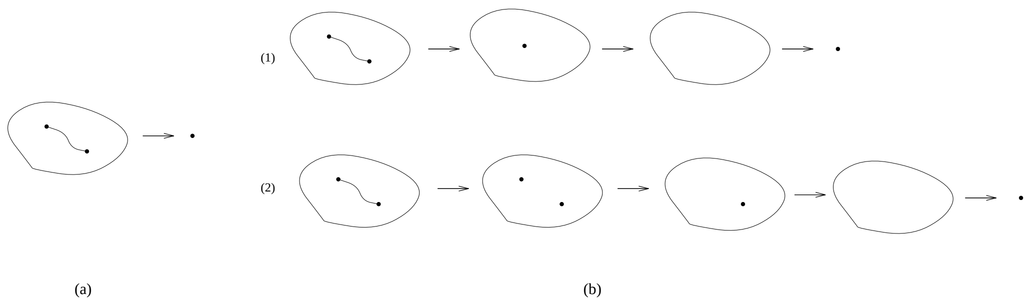


Figure 4.35: Sequences of different lengths.

In Figure 4.36 a similar example is shown where, on the contrary, the sequences have the same length.

Here we consider the sequence with the minimum length.

Definition 4.5.1 *Let $f : A \to B$ be a consistent map transformation. Let $f_1, \ldots, f_n$ be the minimum sequence of minimal operators applied to A in order to obtain B through f. We say that the* global diversity *between A and B is of degree n.*

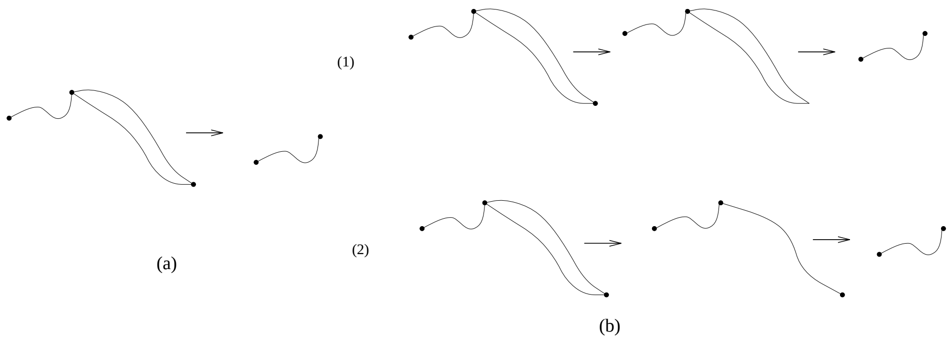


Figure 4.36: Sequences with same length.

This definition corresponds to the distance between two objects in a generalized metric space (defined as an augmented category on $\mathbb{R}_+$, see [112]). In the following, we provide an alternative definition of diversity.

Definition 4.5.2 *Let $f : A \rightarrow B$ be a consistent map transformation such that $|B| = m$. Let $f_{i,1}, \ldots, f_{i,p_i}$ be the minimum sequence of minimal operators applied to $f^{-1}(e_i)$ to obtain e_i $(1 \leq i \leq m)$ through f. Let $k = max_{i=1}^{m}\{p_i\}$. We say that the* local diversity *between A and B is of degree k.*

The above defined numbers (degrees of diversity) provide evaluations of the computational effort involved in the process of obtaining a less detailed map (each minimal operator is considered to have unit cost), and of the resulting reduction of information.

Chapter 5

From the Category to a Multiresolution Data Structure

The main topic of this thesis is the representation of geographic maps at different levels of detail. Multiple representations are usually maintained in a spatial database in order to avoid on-line computation when a representation less detailed than the stored one is required.

Within a multiple representation database, given a representation of a portion of a map or of a single entity, it is often useful to have a fast way to access other representations of the same portion/entity. In other words, for some applications, methods for "vertically" navigating through the levels are required. By enriching multiple representations of a geographic map through information on relations between different representations of the same entity at different levels, we obtain a *multiresolution model*. Different kinds of models are obtained depending both on how the model was built and on which relations are explicited.

In this Chapter, we consider a multiresolution model that was previously proposed in [141]. The model is composed of a sequence of representations (corresponding to decreasing levels of detail), each describing the same area. The model is built on the basis of a sequence of map transformations. We assume that such sequence is composed of consistent transformations, i.e., transformations obtained as compositions of atomic operators. The resulting model is implicitly consistent: no *a posteriori* check is required.

Only representations at consecutive levels of detail are related. Thus, if $M_0, \ldots, M_n$ is a sequence of map representations of the same area, an entity e of M_i is linked to the set of entities of M_{i+1} that are mapped onto e through the function transforming M_i into M_{i+1}.

We also define data structures for encoding and manipulating the model. A data structure for encoding a multiresolution model combines a data structure for each map in the model with information on relations between different maps. We provide data structures both for individual maps and for the multilevel model.

The content of this Chapter both completes the theoretical work developed in the thesis and, at the same time, provides a first step towards future devlopments: both designing an implementation of such work and modeling changes in accuracy in geographic maps. Indeed, the realization of a prototype system as well as combining changes in detail and accuracy within the same framework are major aims of our future work (see also Conclusions).

The Chapter is subdivided into the following three Sections: Section 1 defines multiresolution models, while Section 2 presents data structures for efficiently encoding the model. Finally, Section 3 describes a first attempt of formalization of metric changes in maps through the concept of line homotopy.

5.1 Multiresolution Model

In this Section, we describe the multiresolution model proposed in [141]. Such model is defined independently of the particular map transformations considered. Here, we consider an instantiation of the model as built on the basis of consistent transformations.

Let M_0 be a map at the maximum available resolution, and let M_1 be a map obtained by a combinatorial transformation f_1 of M_0. If we apply a transformation $f_2 : M_1 \to M_2$, and so on iteratively, we obtain a whole sequence of upward compatible maps,

$$M_0 \stackrel{f_1}{\to} M_1 \stackrel{f_2}{\to} \dots \stackrel{f_n}{\to} M_n,$$

corresponding to less and less detailed descriptions of the same area. Note that the composed transformation

$$f^i = f_1 \circ f_2 \circ \dots \circ f_i : M_0 \to M_i,$$

$\forall i = 1, \dots, n$, is itself consistent, and it describes a *macro-transformation* of M_0, modifying it abruptly from the full resolution to the i-th degree of detail. The sequence $M_0, \dots, M_n$ is called a *multiple representation sequence.*

As stated before, by adding information on relations between different representations to a multiple representation sequence $M_0, \dots, M_n$, we obtain a *multiresolution model.* Such relations are formalized by the sequence of transformations used to build $M_0, \dots, M_n$. More formally, we provide the following definition.

Definition 5.1.1 *Let $\mathcal{M} = \{M_i \mid i = 0, \dots, n\}$ be the family of maps of a multiple representation sequence, and let $\mathcal{F} = \{f_i \mid i = 1, \dots, n\}$ be the corresponding family of* *transformations, as defined above. The pair* $(\mathcal{M}, \mathcal{F})$ *is called a* layered multiresolution model.

A layered multiresolution model represents the map of a domain at different levels of resolution through different independent representations that cover the whole domain. Representations that are consecutive in the sequence are related through combinatorial transformations, which can be implemented as "vertical" links that allow browsing the model through the different levels of resolution, while "horizontal" browsing happens in the context of a single map (and is performed by means of links implementing topological relations between the entities).

5.2 Encoding a Multiresolution Model

A data structure for encoding a multiresolution model combines a data structure for each map in the model with information on relations between different maps.

Usually only relations between consecutive levels are explicitly encoded while relations between non-consecutive levels are obtained as composition of them.

In next sections, we present two different data structures for plane maps, which are extensions of well-known data structures in the computational geometry literature (described in Chapter 1): the DCEL structure and the symmetric structure. Finally, we show how to augment such structures to use them in a multiresolution context.

5.2.1 A Data Structure for a Plane Map

A plane map is described by a triple $M = (P, L, R)$, such that

- P is a set of points in the plane;
- lines in L are chains of straightline segments[1] (i.e., sequences of points $(p_0, p_1, \ldots, p_{n-1}, p_n)$) with endpoints (p_0, p_n) in P such that no two lines in L cross.
- R is a set of regions r which are maximal portions of the plane bounded by lines in L and such that any two points inside r can be connected by a line that does not intersect lines in L.

We define incidence relations between pairs of entities in a map in the following way:

[1]Here, for simplicity, we consider lines as polygonal chains.

- $PP(p)$ = endpoints of lines incident at p (none if p is a point feature);
- $PL(p)$ = lines incident at p (none if p is a point feature);
- $PR(p)$ = regions in the star of p (the region containing p if p is a point feature);
- $LP(l)$ = endpoints of l: 2 if the line is open, 1 if it is a self-loop with endpoints, 0 if it is closed;
- $LL(l)$ = lines sharing an endpoint with l;
- $LR(l)$ = regions in the star of l: 2 if the line is a bounding line, 1 if it is a feature;
- $RP(r)$ = points on the border of r plus point features of r;
- $RL(r)$ = lines on the border of r plus line features of r;
- $RR(r)$ = regions sharing some bounding line with r (at least one).

Since we allow the presence of features, some of the above described relations (namely, PR, LR, RP, RL) combine the usual incidence information (as relations defined for plane subdivisions [139]) with containment information (e.g., for an isolated point PR corresponds to the region containing it in its interior).

Note that for closed lines relation LR always contains 2 regions (a closed line is always the border of 2 regions, see Property 2b) in Chapter 3). For point features, only relation PR is defined.

We can also define partial relations in the same way as they are defined for plane subdivisions: given a relation $\mathcal{R}$ and an entity e, $\mathcal{R}^*(e)$ returns only a subset of $\mathcal{R}(e)$.

We consider a data structure for representing a plane map (which is essentially an extension of the DCEL structure, see Chapter 1) and that was proposed in [54]. The following information are stored:

- for each $p \in P$:
  - its coordinates;
  - if p is a point feature, the region containing it (PR);
  - otherwise, a line incident to it (PL^*);
- for each $l \in L$:
  - its geometry;

  - its endpoints (LP);
  - regions "bounded" by l (LR);
  - 4 lines sharing an endpoint with l (LL like in DCEL structure);

- for each $r \in R$:

  - list of lines bounding it (one for each connected component of the border: subset of RL);
  - list of lines composing linear features (one for each connected component of the set of linear features: subset of RL);
  - list of point features (subset of RP).

Note that we also include metric information in the structure (coordinates of the points, geometry of the lines). As already noted in Chapter 1, such information can be easily added/eliminated from an ACC-based structure, since all entities are stored.

It can be shown that the relations stored in this data structure allow to retrieve all other non-stored topological relations as well as all to complete all relations which are only partially stored. Basically, the retrieval of such relations is performed in the same way as in the classical DCEL structure (indeed, for some of extensions allowed additional information is explicitly encoded, e.g., for isolated points PR relation is maintained).

In the following, we list the rules for retrieving all relations:

- The complete RL *relation* is obtained by combining partial relation RL^* with LL relation; given a region r, for each connected component in its boundary, the first line l_1 is found using relation RL^*; then, using relation LL, we find the successor l_2 of l_1 (in counterclockwise order) on that boundary component: if l_1 is oriented in such a way that r lies on the left side of l_1, then l_2 is the first element of the lines associated with l_1 through LL; otherwise, l_2 is the second element. We apply the same method to find all the other lines on the boundary component until we reach l_1 again. The same method is applied for linear features but for each of them, we stop when no more lines belonging to that feature are left.

- In order to retrieve complete RP *relation* (i.e., in order to retrieve points bounding a region, since its features are explicitly encoded), we can combine relations RL and LP. For each line l_i incident at a point p (found through PL), the regions bounded by l_i are retrieved.

- RR *relation* is obtained by combining relations RL and LR. For each boundary line l_i of a region r (found through RL), the region adjacent to r along l_i is returned.

- Complete PL *relation* is obtained by combining partial relation PL^* with LL. Given a point p, the first line l_1 is found that is incident at p using relation PL^*. Then, through relation LL, the line l_2 following l_1 (in counterclockwise order) around p is obtained: if p is the first endpoint for l_1, then l_2 is the first among the lines associated with l through LL (in counterclockwise order); otherwise, l_2 is the second element. The same method can be applied to find all other lines incident at p (until l_1 is met again).
- PP *relation* can be obtained as a combination of relations PL and LP. For each line l_i incident at a point p (found through PL), the endpoint of l_i different from p is retrieved. PR relation (for non-isolated points) can be obtained as a combination of relations PL and LR. For each line l_i incident at a point p (found through PL), regions bounded by l_i are retrieved.

5.2.2 An Alternative Data Structure

An alternative data structure for representing a map is based on an extension of the symmetric structure for encoding plane subdivisions [139]. Such structure stores:

- for each $p \in P$:
  - its coordinates;
  - if p is a point feature, the region containing it (PR);
  - otherwise, all lines incident at p (PL);
- for each $l \in L$:
  - its geometry;
  - its endpoints (LP);
  - regions "bounded" by l (LR);
- for each $r \in R$:
  - list of lines bounding it: both external and internal borders (one for each connected component of the border: subset of relation RL);
  - list of lines composing linear features (one for each connected component of the set of linear features: subset of relation RL);
  - list of point features (subset of relation RP).

The basic difference with the classical symmetric structure is that explicit relations RP and PR must be encoded in the case of point features (in order to relate such points with other entities in the map).

This relations are sufficient to retrieve all non-stored relations. Indeed, relations RP, RR, PP and PR are found in the same way as described for the previous data structure. Relation LL is obtained by combining relations LP and PL (or, also, relations LR and RL): let p_1 and p_2 be the endpoints of a line l. The pair (l_1, l_2) associated with l by LL is obtained in this way: l_1 is the line following l in the list of lines associated with p_1 by PL, while l_2 follows l in the list of lines associated with p_2 by PL.

Note that this data structure directly maps the ordering relation (bounding relation) in the definition of ACC.

5.2.3 Remarks

For encoding the multiresolution model, we must augment a data structure for a map (encoding each level), with information about vertical links. A standard scheme consists of adding links between an entity and its representations both at the previous and at the subsequent levels of resolution.

More specifically, let $(\mathcal{M}, \mathcal{F})$ be a layered multiresolution model, such that $\mathcal{M} = \{M_i \mid i = 0, \ldots, n\}$. For each i $(1 \leq i \leq n-1)$, and for each entity $e \in M_i$, two pointers are added that link e to both $f_{i+1}(e)$ (which is an entity in M_{i+1}), and to $f_i^{-1}(e)$ (which is a subset of the collection of entities of M_{i-1}), respectively. For $i = 0$ only the first pointer is stored, while for $i = n$ only the second information is available.

Next important step consists of detecting which set of operations are efficiently performed on the basis of this data structure. In particular, constructing operations, vertical navigation operations, and geometric/topological queries will be analysed.

5.3 Modeling Changes in Accuracy

In this Section, we describe a first attempt towards the formalization of changes in accuracy in a plane map. Also, we try to combine changes in accuracy and changes in detail within the framework developed in the thesis.

As already mentioned in Section 1.3.2, a first structural multiresolution model for geographic maps was defined in [14, 141]. Such a model was designed to describe changes in accuracy between maps. In this case, the accuracy of a map is measured through the horizontal error in representing the lines in the map, formalised through the mathematical

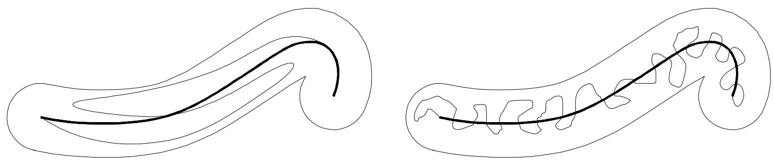


Figure 5.1: Approximations of lines (l is the thick line, l' is the thin line): (a) l' is not ε-homotopic to l; (b) l' is ε-homotopic to l.

concept of ε-homotopy.

Intuitively, a geometric object o is represented by an entity e at accuracy $\varepsilon \geq 0$ if and only if o is contained in a region obtained by "fattening" e of ε, i.e., in $r_{e,\varepsilon} = \{x \mid d(x, e) \leq \varepsilon\}$, where d denotes the Hausdorff distance in $\mathbb{R}^2$. Note that a "thin"(though possibly "long") region, such as a portion of road or river in a geographic map, can possibly be represented by a line; also, a region that is completely contained in a disc of radius ε can be approximated by a single point. Such idea of approximation is modeled by specifying a continuous transformation that maps the object o into the entity e representing it without leaving the fattened region $r_{e,\varepsilon}$.

Let $l, l' : I \to \mathbb{R}^2$ two continuous functions from the standard unit interval into the real plane. A *homotopy* between l and l' is a continuous function $H : I \times I \to \mathbb{R}^2$ such that $H(x, 0) = l(x)$ and $H(x, 1) = l'(x)$ [150]. Such homotopy H defines a continuous deformation from l to l' through all functions $H(\cdot, y)$, $y \in (0, 1)$. For $\varepsilon \geq 0$, a homotopy H is called an ε-homotopy if

$$\forall x \in I,\ \forall y \in I, \quad d(H(x, y), H(x, 0)) < \varepsilon.$$

If H is an ε-homotopy between functions l and l', then l and l' are said ε-homotopic. In particular, if l and l' are two lines, this means that we can approximate l by l' (or viceversa) with accuracy ε. Note that the ε-homotopy implies a condition much stronger than simply assuming l' inside the region fattening l: indeed, ε-homotopy requires that any portion of l' remains inside the fattened region of the corresponding portion of l; or, in other words, that two walkers synchronised to start together from $l(0)$ and $l'(0)$, and to arrive together at $l'(1)$ and $l(1)$ respectively, will always remain at a distance smaller or equal than ε. This fact excludes that l' could possibly be a line going back and forth an arbitrary number of times inside the "tube" surrounding l (see Figure 5.1(a)), while it is admitted, for instance, a fractal representation l' that has a length much higher than l, but remains locally near l (see Figure 5.1b)).

If l is a line and l' is a constant function, then the fact that H is a ε-homotopy means that l can be approximated by a single point at accuracy ε. Approximations of regions can be defined by considering ε-homotopies of the lines (chains) defining their boundaries. A region r can be approximated by a point p at accuracy ε if and only if the outer boundary of r is ε-homotopic to p (seen as a constant function). A region r can be approximated by a line l at accuracy ε if and only if the outer boundary of r can be subdivided into two chains such that each of them is ε-homotopic to l. A region r can be approximated by another region s if and only if (i) the outer boundary of r is ε-homotopic to the outer boundary of s, (ii) each inner boundary of s is ε-homotopic to an inner boundary of r, and (iii) any inner boundary of r that is not ε-homotopic to an inner boundary of s is ε-homotopic to a point. A regular submap $\bar{M}$ can be approximated by a region r at accuracy ε if and only if the region covered by $\bar{M}$ is ε-homotopic to r.

Based on the above definitions, we can introduce the concept of *ε-transformation* of a map. Given two maps M and M', a transformation $f : M \to M'$ is an *ε-transformation* if and only if

$$\forall e' \in M' \quad e' \text{ approximates } f^{-1}(e') \text{ at accuracy } \varepsilon.$$

Hence, combinatorial and metric map transformations, based respectively on variations of detail and of accuracy, are combined to obtain a map that is consistent both in maintaining compatible topological and metric relations.

Note that the level of accuracy does not give by itself any condition on map elements that survive, and elements that are immersed, when transforming M into M'. This is due to the fact that the degree of detail and the level of accuracy of a map are not necessarily related with each other, since there is no quantifiable dependence between the scale of a map and the fact that a given entity appears in it or not. Indeed, the relation between the relevance of an entity and the scale of a map can depend both on the size of the entity and on the semantics associated to the map. In actual implementations of systems for map generalization, combinatorial and metric criteria must be strictly interrelated, together with semantic criteria, and all such information must be integrated to guide map transformations. The detail reduction, whose consistence we have analysed by formal methods, must be actually driven by metric checks on the actual map entities, while reductions based on metric evaluations can be applied only as long as they do not violate structural consistency.

We can integrate changes in accuracy within the layered multiresolution model described in Section 5.1 as follows.

Let $\mathcal{M} = \{M_i \mid i = 0, \ldots, n\}$ be a family of maps of a multiple representation sequence, and $\mathcal{F} = \{f_i \mid i = 1, \ldots, n\}$ the corresponding family of transformations. If $\varepsilon_1 < \varepsilon_2 < \ldots < \varepsilon_n$ is an increasing sequence of tolerances such that the composed transformation f^i, $\forall i = 1, \ldots, n$ is a ε-transformation of M_0, then we have a sequence which is consistent both with detail and accuracy.

We remark that the work presented in this Section is only a first attempt towards the formalization of changes in accuracy and their integration with changes of detail in a map. This topic represents one of the main future developments of this thesis.

Conclusions

We have proposed a formalization for modeling changes of detail in a geographic map. The adopted approach exploits notions of category theory. After mathematically defining the concept of map, we have modeled its combinatorial structure by means of abstract cell complexes. Map transformations (i.e., operations reducing the level of detail in a map) have been formalized through functions between such complexes. Among all map transformations, we have detected transformations that we consider consistent. A basic set of operators has been defined, that permits to generate all such transformations. By applying such operators to a given map representation M, only representations consistent with M are generated. A multiresolution model, built by using consistent transformations, together with encoding data structures, has also been outlined.

In the following, we remark advantages and limits of the proposal, and we list some future developments.

Advantages of the Proposal

The contribution of the thesis is in providing a unified theoretical framework for multiresolution modeling of geographic maps.

The main advantages of the proposed formalism can be summarized as follows:

- it supports the representation of geographic maps having a generic combinatorial structure, including regions with holes, as well as dangling and isolated entities;
- it permits to separate topological aspects from metric aspects;
- it permits to characterize topological relationships between entities in a plane map in a completely combinatorial way;
- it is intrinsically consistent; thus, no further time-consuming check on models defined by using such a formalism is required; conversely, this problem usually arises both

when multiple representations of the same area are maintained in a database, and when classical generalization operations are applied for an on-line computation of a less detailed representation;

- a minimal and sufficient set of atomic operators to perform map transformations is provided: on the basis of such operators, a kernel of primitives for the manipulation of plane maps can be defined;
- locality and graduality of topological changes are guaranteed during the application of such operators; a meaningful method for evaluating the reduction of information between maps is also based on the application of atomic operators;
- it provides an effective support to map generalization: classical generalization operations implying changes of the topology of maps can be implemented as macro-operations obtained as compositions of minimal operators;
- not only a method for defining multiple representations, but also the possibility of relating different representations of the same entity at different details (i.e., of defining a multiresolution model), is implicit in the formalism: vertical links between two maps at consecutive levels correspond to the combinatorial transformation generating the less detailed map from the more detailed one;
- it naturally supports hierarchical spatial reasoning;
- metric and semantic aspects can be easily added, by treating them as attributes of the spatial entities in the topological representation.

Limits of the Proposal

In the following, we discuss limits of the proposed formalization. Three major classes of problems have been identified: the restriction to the two-dimensional case, the impossibility of describing situations of practical interest (e.g., aggregation of disjoint entities), and the applicability of the formalism.

1. *Dimensionality.* Both from a theoretical and from an applicative point of view, it would be very interesting to provide solutions to the problem of representing spatial entities in any dimension. However, we have focussed our attention on the two-dimensional case. Indeed, in two dimensions, we can exploit formalizations of the combinatorial structure of objects composed of non-simple parts (e.g., graph-based formalism), while there is a total lack of mathematical models suitable for describing three-dimensional structured objects. Also, from a more technical point of view, it

is difficult to define the concept of feature inside three-dimensional objects. Such theoretical difficulties have prevented us from providing extensions of the obtained results to higher dimensions.

2. *Expressive power.* Essentially, two issues must be discussed.

    - We impose a constraint on closed lines: they always have an endpoint; in other words, we do not allow the presence of self-loops without endpoints. The reason for imposing such constraint is related to the use of a graph-based formalism. Plane graphs do not have edges without endpoints. Results, as well as data structures and algorithms for plane maps could not be exploited if self-loops without endpoints were allowed. Furthermore, the set of atomic operators would have to be augmented with operators capturing transformations involving such entities.

    - Functions corresponding to consistent transformations must preserve connected sets by inverse image. This prevents our formalism from supporting the modeling of *aggregation* of disjoint entities (commonly used in map generalization). Aggregation consists of grouping (possibly disjoint) entities that, because of their size, would individually disappear, but, when considered as a whole, are still significant. Typical examples are groups of small lakes and islands (e.g., in countries like Canada or Finland). This operation cannot be easily formalized, since it is related to the semantics of the map, and not driven by exact rules. However, it is usually applied in practice by cartographers.

      In our work, the condition of preserving connected sets by inverse image was imposed both for semantic and for technical reasons.

      Semantic reasons are based on the fact that we want the same query, performed at different levels of detail, to return compatible results. For instance, in a region (i.e., a connected set), two internal points can always be joined by a line completely contained inside the region; if the region was the result of the aggregation of disjoint regions, the same query would not give a consistent result if the points were contained in different regions at a higher level of detail.

      Technical motivations relate to the fact that such condition allowed to define an ordering relation on the set of GACCs. If such constraint was not imposed, the relation defined on GACCs would only be a preorder, which is not antisymmetric. Furthermore, since the set of atomic operators we have defined have this property, they are only sufficient to generate, by composition, functions with the same characteristics.

      A tentative solution has been proposed in [59], where the semantic and topological inconsistency caused by aggregation operations is overcome by forcing an arbitrary subdivision of the region containing the aggregated entities into two parts: one part corresponds to the surrounding region, while the other forms a shell containing entities to be aggregated. However, being based on an arbitrary subdivision, the method should be revised. In [165], the authors deal with the assessment of topological consistency across multiple representations obtained by aggregating parts. Nevertheless, this operation still remains a difficult issue within a mathematical formalization.

3. *Applicability.* The proposed formalization is limited to modeling combinatorial aspects of maps and changes in detail, while metric aspects, as well as changes in accuracy have not been taken into account. In actual implementations for generating maps at lower resolutions, combinatorial/topological, metric, and semantic criteria must be integrated. An analysis of metric aspects and changes in accuracy in geographic maps was initially performed in [14], by using the concept of line homotopy, and it has been briefly discussed in Chapter 5. Some developments have also been proposed in [141], although a rigorous formalization, as well as an integration with the work developed in the thesis has not been extensively analysed yet. Semantic aspects of spatial information have been completely disregarded, although they play a fundamental role in the application of some classical generalization operations [104].

   As already stated in the thesis, we remark that the separation between different aspects, and their formalization through different mathematical tools is not to be considered as a limit of our approach, provided that the integration of other aspects is easily permitted in a modular way. This investigation represents one of our major future tasks.

   Although the multiresolution model we have described is independent of the technique adopted to build it, as well as of the specific application, we have considered only a particular instantiation targeted to applications within GISs. The applicability to different contexts has not been analysed yet.

   Finally, we recall that many applications of spatial information systems require a unified approach to both spatial and temporal information modeling (see [3, 161, 178] for a review on this topic). In this thesis, we did not treat temporal aspects of information. However, this is another interesting issue that we consider worthwhile to investigate.

Future Developments

We intend to concentrate our future work on three main aspects: theoretical developments for overcoming current limits of the formalism, application-related developments, and the realization of a prototype system implementing part of the obtained results.

Theoretical problems we plan to investigate include:

- the possibility of extending our formalism for treating also closed lines with no endpoint as well as map transformations implying aggregations of disjoint regions;
- the formalization of changes of accuracy in a map and its integration within the framework proposed in this thesis;
- a comparison of the two alternative methods for evaluating the diversity between maps;
- the possibility of extending the obtained results to higher dimensions (essentially, three dimensions) and to different application contexts (e.g., CAD systems).

From a more application-dependent point of view, we intend to provide algorithms for building and manipulating the proposed model, and to analyse the actual impact of our formalism as a support to map generalization. More specifically, the following issues correspond to our major aims:

- detection of meaningful criteria for triggering the application of minimal operators; for example, we can decide that an entity must be eliminated (i.e., immersed into some larger object) when it gets too small (i.e., on the basis of its size); more complicated techniques, based on the semantics associated to each entity should also be employed;
- development of algorithms for performing macro-operations (e.g., contraction of a submap to a point, or to a line, or to a region), obtained as composition of minimal operators; priorities must be assigned to operators that are applicable at the same time;
- the study of methods for obtaining a canonical sequence of minimal operators corresponding to a given map transformation;
- development of an algorithm for building the multiresolution model;
- development of primitives for "vertically" navigating the multilevel organization;
- development of algorithms for performing spatial queries on the proposed model, as well as an analysis of the performance of such queries;
- an attempt to interpret classical generalization operations in terms of minimal operators we have defined; in particular, as already remarked, topological transformations implied by generalization operations can be obtained as compositions of minimal operators; other aspects playing an important role in map generalization should also be analysed with the aim of detecting possible limits of our approach as a support to the generalization task.

As far as the implementation developments are concerned, we are currently planning the realization of a prototype system implementing part of the developed work and composed of a kernel, a visualization, and a query module, such that

- the kernel contains the data structure for encoding a model of a spatial map together with manipulating primitives, as well as algorithms for reducing the detail (based on atomic operators);
- the visualization module contains algorithms for displaying a representation of a map at a given resolution level, to be used through a graphic interface for interactive manipulation and visualization;
- the query module includes the possibility of performing queries on the map, such as location of a given entity, at different levels of detail.

The idea is to develop a prototype which is meaningful and, at the same time, easily extensible. Adding operators, functionalities, etc., as well as changing the generalization criterion should always be possible in an easy way.

The prototype should also be tested on meaningful datasets: experimental results showing the practical behaviour/performance of the developed algorithms will complete the work.

A further development of the planned implementation consists of supporting also changes in accuracy. This implies the implementation of geometric algorithms, e.g., for line simplification [61, 121].

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Appendix A

Point-set Topology Concepts

In the following, we provide basic concepts defined in point-set Topology (see [29, 41] for further details).

Definition A.1.1 *A* topology *on a set X is a family τ of subsets of X such that:*

- *X and $\emptyset$ belong to τ;*
- *every union of elements of τ belongs to τ;*
- *the intersection of every finite family of elements of τ belongs τ.*

Elements in τ are called open sets *of τ.*

Definition A.1.2 *A* topological space *is a set with a topology.*

Definition A.1.3 *A subset Y of a topological space X is said to be* closed *if its complement $\mathcal{C}_X Y = X \setminus Y$ is open.*

Definition A.1.4 *Let X and X' topological spaces and let $f : X \to X'$ be a function of X in X'; f is said to be* continuous *if, for each A' open set of X', $f^{-1}(A')$ is an open set of X.*

Definition A.1.5 *Let X and X' topological spaces. A* homeomorphism *of X into X' is a bijective function $f : X \to X'$ such that f and f^{-1} are continuous.*

Definition A.1.6 *Let Y be a subset of a topological space X,*

- *the* interior *of Y, denoted $i(Y)$, is the union of all open sets contained in Y, i.e., the largest open set contained in Y;*
- *the* closure *of Y, denoted $\overline{Y}$, is the intersection of all closed sets that contain Y, i.e., the smallest closed set containing Y;*
- *the* border *of Y is the set of all points x in X such that, for each open set U containing x, $U \cap Y \neq \emptyset$ and $U \cap \mathcal{C}_X Y \neq \emptyset$ ($\mathcal{C}_X Y = X \setminus Y$);*
- *the* exterior *of Y is defined by $Y^- = X \setminus \overline{Y}$;*
- *the* regularization *of Y is defined by $r(Y) = \overline{i(Y)}$.*

The following relationship holds among interior, closure and border of Y: $b(Y) \cup i(Y) = \overline{Y}$.

When a topology is defined, it is useful to have a family of open sets that generate all other open sets of the topology. Thus we define the concept of *base* for a topology.

Definition A.1.7 *A* base *for a topological space X is a family $\mathcal{B}$ of open non-empty sets of X such that each open set of X is union of elements of $\mathcal{B}$.*

In the following, we define connectivity in a topological space and present a few results on mappings preserving connectivity (see [29, 41] as references for definitions and results which are not shown here).

Definition A.1.8 *A topological space X is connected if and only if it is not union of two open sets A and B such that $A \neq \emptyset$, $B \neq \emptyset$ and $A \cap B = \emptyset$.*

Obviously each set composed of a single element is connected.

Since closed sets are complements of open sets, we have the following proposition (see [29, 41] for a formal proof).

Proposition A.1.9 *A topological space X is not connected if and only if one of the following conditions holds:*

- *there exists $A \subseteq X$, with $A \neq \emptyset$ and $A \neq X$, such that A is both open and closed;*
- *there exist two closed sets $F \subseteq X$ and $G \subseteq X$ such that $F \neq \emptyset$, $G \neq \emptyset$, $F \cap G = \emptyset$ and $X = F \cup G$.*

The following proposition relates continuous functions and connected spaces (the proof can be found in [29, 41]).

Proposition A.1.10 *Let X and Y be topological spaces and let $f : X \rightarrow Y$ be a continuous surjective function. If X is connected, then Y is also connected.*

The following corollary states that continuous functions preserve connectivity.

Corollary A.1.11 *Let $f : X \rightarrow Y$ be a continuous function. For each connected subset X' of X, $f(X')$ is a connected subset of Y.*

Definition A.1.12 *Given a function $f : X \rightarrow Y$, we say that f* reflects connected subsets *(also,* preserves connected subsets by inverse image *of connected sets if, for each $Y' \subseteq Y$ such that Y' is connected, also $f^{-1}(Y')$ is connected.*

Appendix B

Preorders and orders

B.1 Definitions

Definition B.1.1 *A* preorder $\leq$ *on a set* X *is a relation that satisfies the following properties:*

- $\forall x \in X,\ x \leq x$ *(reflexivity);*
- $\forall x, y, z \in X,\ x \leq y$ *and* $y \leq z \ \Rightarrow\ x \leq z$ *(transitivity).*

Definition B.1.2 *A* partial order $\leq$ *on a set* X *is a relation that satisfies the following properties:*

- $\forall x \in X,\ x \leq x$ *(reflexivity);*
- $\forall x, y \in X,\ x \leq y$ *and* $y \leq x \ \Rightarrow\ x = y$ *(antisymmetry);*
- $\forall x, y, z \in X,\ x \leq y$ *and* $y \leq z \ \Rightarrow\ x \leq z$ *(transitivity).*

A set with an order is called a *poset* (partially ordered set).

Definition B.1.3 *Let* $(X, \preceq_X)$ *and* $(Y, \preceq_Y)$ *be two posets. A function* $f : X \to Y$ *is* monotonic *if it preserves the ordering, i.e.,* $x \preceq_X x'$ *(and* $x \neq x'$*) implies* $f(x) \preceq_Y f(x')$*.*

B.2 Topology Induced by a Preorder

In this Section, we show that a preorder (i.e., a reflexive and transitive binary relation) always induces a topology. Furthermore, we show that a mapping between two topological spaces with such topology is continuous if and only if it preserves the preorder. Connected sets in this topology are also characterized.

Proposition B.2.1 *A preorder induces the following topology on X: $A \subseteq X$ is open in X if and only if $\forall x \in A$ and $\forall y \in X$ such that $x \leq y$, $y \in A$.*

Proof.

- $\emptyset$ and X satisfy the condition.
- Every union of sets satisfying the above condition results in a set satisfying the condition: given a family $\{A_i\}_{i \in I}$ of sets such that $\forall i \in I$, $\forall x \in A_i$ and $\forall y \in X$ such that $x \leq y$, $y \in A_i$, we have that $A_I = \cup_{i \in I} A_i$ is such that $\forall x \in A_I$ and $\forall y \in X$ such that $x \leq y$, let $j \in I$ and $x \in A_j$, then $y \in A_j$; thus, $y \in A_I$.
- Every finite intersection of sets satisfying such condition results in a set satisfying the condition: given A and A' such that $\forall x \in A[A']$ and $\forall y \in X$ such that $x \leq y$, $y \in A[A']$, we have that $A \cap A' = \{x \mid x \in A, \textit{ and } x \in A'\}$ is such that $\forall x \in A \cap A'$ and $\forall y \in X$ such that $x \leq y$, $y \in A$ since $x \in A$ and $y \in A'$ since $x \in A'$, thus, $y \in A \cap A'$. □

In the following, we present some results related to the topology induced by a preorder. In particular, Proposition B.2.2 shows that the *star*[1] of an entity is a basis of open sets for this topology, i.e., it establishes an equivalence between continuity and monotonicity (in the case of posets).

Proposition B.2.2 *Let B_X be the family $B_X = \{U_x\}_{x \in X}$ with $U_x = \{y \in X \mid x \leq y\}$ (called the* star *of x). B_X is a basis of open sets for the topology induced by the preorder $\leq$ on X.*

Proof. For each $x \in X$, U_x is open in X because, $\forall y \in U_x$ and $\forall z \in X$ with $y \leq z$, by transitivity we have that $x \leq z$ and thus $z \in U_x$. U_x is indeed the smallest open set containing x.

For each A open in X, we must show that A can be obtained as union of elements of B_X. Let $A \subseteq X$ be an open set in X. We show that $A = \cup_{a \in A} U_a$.

[1] Here the overloading of term *star*, already defined in the context of an ACC as the set of entities which are bounded by a given entity, is intentional.

(a) We show that $A \subseteq \cup_{a\in A} U_a$, i.e., $\forall x \in A$, $x \in \cup_{a\in A} U_a$.

Let $x \in A$. Since $\forall x \in X$, $x \in U_x$, we have that $x \in \cup_{a\in A} U_a$.

(b) We show that $\cup_{a\in A} U_a \subseteq A$, i.e., $\forall x \in \cup_{a\in A} U_a$, $x \in A$.

Let $x \in \cup_{a\in A} U_a$. Let $a \in A$ such that $x \in U_a$. Since $U_a = \{y \in X \mid a \leq y\}$, then $a \leq x$. But A is open and thus it must contain also x. $\square$

The following Proposition shows that a function is continuous in this topology if and only if it preserves the preorder.

Proposition B.2.3 *Let $f : X \to Y$ be a function between preordered sets. f is continuous for the topologies induced by the preorders if and only if $\forall x, y \in X,\ x \leq y \Rightarrow f(x) \leq f(y)$ (i.e., f preserves the preorder).*

Proof.

1. By hypothesis, we have that $\forall x, y \in X,\ x \leq y \Rightarrow f(x) \leq f(y)$.

   We have to show that if $U \subseteq X$ is open in X then $f^{-1}(U)$ is open in X. If U is open, then, $\forall p \in U$ and $\forall q \in X$ such that $p \leq q$, $q \in U$. We must show that $\forall a \in f^{-1}(U)$ and $\forall b \in X$ such that $a \leq b$, then $b \in f^{-1}(U)$. If $a \in f^{-1}(U)$, then $f(a) \in U$. Let $b \in X$ be such that $a \leq b$. By hypothesis this implies that $f(a) \leq f(b)$. Since $f(a) \in U$ and U is open, then it must be $f(b) \in U$ too. But this means that $b \in f^{-1}(U)$.

2. By hypothesis, we have that if f is continuous.

   Let us consider $x \leq x'$ and the open set $U = \{y \in Y \mid f(x) \leq y\}$ of Y. Then $f^{-1}(U)$ is open in X and $x \in f^{-1}(U)$ implies $x' \in f^{-1}(U)$. Thus $f(x') \in U$, i.e., $f(x) \leq f(x')$. $\square$

Fact B.2.4 shows that the star of an entity is connected.

Fact B.2.4 *Let X be a topological space with the topology induced by a preorder $\leq$. $\forall x \in X$, U_x is connected.*

Proof. Let us assume by contradiction that U_x is not connected. Then there exist two open non-empty disjoint proper subsets of U_x that cover U_x. But U_x is the smallest open set containing x. Thus U_x must be connected. $\square$

Obviously not all open sets are necessarily connected.

The following fact characterizes connectivity in the topology induced by a preorder. Note that a topological space with this topology can be interpreted as a direct graph. A direct graph is said to be connected when, forgetting the orientation of arcs, there exists a walk between each pair of nodes.

Fact B.2.5 *Let X be a topological space with the topology induced by a preorder $\leq$. X is connected if and only if, by closing $\leq$ first for symmetry and then for transitivity, it becomes the total (equivalence) relation on X.*

Proof.

1. By hypothesis if we close $\leq$ first by symmetry and then by transitivity, then it becomes a total relation on X. We want to show that X is connected.

   By contradiction: we want to show that if X is not connected, then by closing $\leq$ for symmetry, it does not result in a total relation on X. If X is not connected, then by definition there exist two open subsets A and B of X such that $A \neq \emptyset$, $B \neq \emptyset$, $A \cap B = \emptyset$ and $X = A \cup B$. By definition of open set, $\forall a \in A$ and $\forall a' \in X$ such that $a \leq a'$, $a' \in A$. Let $b \in B$. Then if $a \leq b$, it should be $b \in A$ but $A \cap B = \emptyset$. If $b \leq a$ then it should be $a \in B$, again this is impossible. Thus, even if we close $\leq$ by symmetry and transitivity elements of A and B are not related, thus $\leq$ is not total.

2. By hypothesis X is connected. We want to show that if we close $\leq$ by symmetry and transitivity, then we obtain a relation $\sim$ which is total on X.

   For each $x \in X$, let $[x]$ be the equivalence class of x with respect to $\sim$; $[x]$ is open in X. Thus the equivalence classes obtained through $\sim$ form a covering of X by means of disjoint open sets. Being X connected, there can only be one such equivalence class, i.e., $\sim$ is total. $\square$

As a consequence of Fact B.2.5 a topological space X is not connected when there exists two non-empty subsets X' and X'' of X covering X and such that no element of X' is related through $\leq$ with elements of X''.

Note that continuity guarantees the preservation of connected sets but not the preservation of connected sets by inverse image. An example is a function that maps two "disjoint regions" (corresponding to elements which are not in relation) to the same region (in the simplified map). For example, in Figure B.1 a function is depicted that maps region r' and r'' onto r_1 and region r onto r_0. Thus the inverse image of r_1 is not connected.

If we consider the characterization of connected sets, the preservation of the connected sets by inverse image can be expressed in the following way: for each $x, y \in Y$ such that $x \leq y$, then $\exists x' \in f^{-1}(x)$, $\exists y' \in f^{-1}(y)$, such that either $x' \leq y'$ or $y' \leq x'$. In particular,

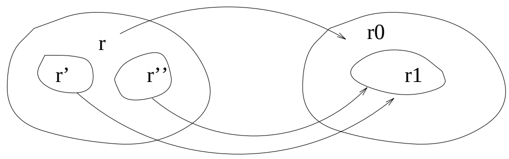

Figure B.1: An example of function that does not preserve connected sets by inverse image.

if f preserves the preorder, then for each $x, y \in Y$ such that $x \leq y$ and $x \neq y$, there exist $x' \in f^{-1}(x)$ and $y' \in f^{-1}(y)$ such that $x' \preceq y'$.

Appendix C

Graph Theory Concepts

Definition C.1.1 *A graph consists of a non-empty set V, a (possibly empty) set E disjoint from V, and a mapping ϕ of E into $V \times V$ (where each pair (v, v') is considered as unordered). The elements of V and E are called the* vertices *(or* nodes*) and* edges *(or* arcs*) of the graph, respectively, and ϕ is called the* incidence mapping *associated with the graph.*

If $\phi(e) = (v, w)$, e is said to be *incident* at v and w, and v and w are called the *endpoints* of e. A graph is usually denoted by (V, E, ϕ) or by (V, E) when the incident mapping is implicit. A graph is said to be *connected* if for every pair of distinct vertices there exists a chain of edges joining them.

Definition C.1.2 *A geometric graph in $\mathbb{E}^n$ is a set $V = \{v_i\}$ of points in $\mathbb{E}^n$ and a set $E = \{e_j\}$ of simple curves satisfying the following conditions:*

- *every closed curve in E contains precisely one point of V;*
- *every open curve in E contains precisely two points of V, and these are its endpoints;*
- *the curves in E have no common points, except for points of V.*

A geometric graph is clearly a particular instance of a graph, in which the vertices and edges are respectively points and simple curves in $\mathbb{E}^n$ and in which $\phi(e) = (v, w)$ and v and w are the endpoints of curve e (if $v = w$, e is closed).

Every graph is abstractly identical (isomorphic) to a geometric graph. Isomorphisms between graphs are defined as follows: graph $G = (V, E)$ and $G' = (V', E')$ are said to be *isomorphic* if there exist 1-1 correspondences between V and V' and between E and E'

which preserve incidences. In other words, edge e is incident at vertex v in G if and only if the corresponding elements e' and v' are incident at G'. If G is isomorphic to a geometric graph G', then G' is called a *geometric realization* of G.

Definition C.1.3 *A graph is said to be* planar *if and only if it has a geometric realization in* $\mathbb{E}^2$.

A *plane graph* is any geometric realization of a planar graph in $\mathbb{E}^2$. A plane graph induces a subdivision of the plane into *faces*, which are, by definition, the connected components of the complement in $\mathbb{E}^2$ of the union of the edges of the graph. This definition also includes as a face of the graph also an unbounded portion of plane surrounding the union of all other vertices, edges and faces of the graph. Such face is called the *infinite face* (denoted f_∞).

The following formula, called *Euler formula* relates the numbers v, e, f, c, of vertices, edges, faces (with the exception of f_∞), connected components, respectively, of a plane graph.

$v - e + f = c.$

A *straight line plane graph (SLPG)* is a plane graph whose edges are straight line segments. A *plane subdivision* is induced by a connected SLPG such that every vertex is endpoint of at least two edges (i.e., it does not contain dangling edges).

Appendix D

Category Theory Concepts

Definition D.1.1 *A* category $\mathcal{C}$ *consists of*

- *a set $\mathcal{O}$ of objects;*
- *a set $\mathcal{A}$ of morphisms (arrows);*
- *operations assigning to each morphism $f \in \mathcal{A}$ objects $dom(f)$ and $cod(f)$ called* domain *and* codomain *of f, respectively;*
- *an operation assigning to each object $a \in \mathcal{O}$ morphism id_a called the* identity *morphism on a;*
- *an operation assigning to each pair of morphisms f, g such that*

$$cod(f) = dom(g)$$

a morphism $g \circ f$ called their composite *(f and g are said to be* composable*),*

satisfying the following properties:

1. $dom(id_a) = a = cod(id_a)$, $dom(g \circ f) = dom(f)$, $cod(g \circ f) = cod(g)$;
2. *for each $f \in \mathcal{A}$ with $a = dom(f)$ and $b = cod(f)$,*

$$f \circ id_a = f = id_b \circ f;$$

3. *for each f, g and g, h composable pairs (from 1. it follows that also the pairs $g \circ f, h$ and $f, h \circ g$ are composable), then*

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Definition D.1.2 *An arrow $f : a \to b$ in a category is an* isomorphism *if and only if there exists an arrow $h : b \to a$ such that $h \circ f = id_a$ and $f \circ h = id_b$. Two objects a and b are* isomorphic *if there exists an isomorphism $f : a \to b$.*

For example, in the category of posets (with monotonic functions[1]), a morphism f is an isomorphism if and only if f is bijective and reflects the order relation.

Definition D.1.3 *A subcategory $\mathcal{D} \subset \mathcal{C}$ is a category whose objects and morphisms are subsets of the objects and morphisms of $\mathcal{C}$, respectively, and whose operations are induced by the ones of $\mathcal{C}$. $\mathcal{D}$ is said to be* full *if for every pair of x, y of objects in $\mathcal{D}$, all morphisms of x into y in $\mathcal{C}$, are also morphisms in $\mathcal{D}$.*

[1] It is easy to show that it is a category.