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Studying repleteness in the category of cpos

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Abstract

We consider the notion of replete object in the category of directed complete partial orders and Scott-continuous functions, and show that, contrary to previous expectations, there are non-replete objects. The same happens in the case of ω-complete posets.

Synthetic Domain Theory developed from an idea of Dana Scott: it is consistent with intuitionistic set theory that all functions between domains are continuous. He never wrote about this point of view explicitly, though he presented his ideas in many lectures also suggesting that the model offered by Kleene's realizability was appropriate, and influenced various thesis works, *e.g.* [10,13,11,8,12], see also [14].

SDT can now be recognized as defining the "good properties" required on a category $\mathbf{C}$ (usually, a topos with a dominance $t: 1 \longrightarrow \Sigma$) in order to develop domain theory within a theory of sets.

One of the problems addressed early in the theory was the identification of the sets to be considered as the Scott domains. As one would expect in a synthetic approach, the collection of these should be determined by the "good properties" of the universe, in an intrinsic way. The best suggestion so far for such a collection comes from [6,15,5] and is that of repleteness.

It is an orthogonality condition, see [2], and determines the replete objects of $\mathbf{C}$ as those which are completely recoverable from their properties detected by Σ. Say that A is *replete* (wrt. Σ) if it is orthogonal to all $f: X \longrightarrow Y$ in

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C such that $\Sigma^f\colon \Sigma^Y \longrightarrow \Sigma^X$ is iso, *i.e.* for such f and all α there is a unique β such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ & {}_{\alpha}\searrow & \Big\downarrow{\scriptstyle \beta} \\ & & A. \end{array}$$

The following results are known, see [6,3,1].

- The full subcategory of the replete objects is a reflective exponential ideal of **C**. It is the smallest such containing Σ.
- Scott domains and continuous functions are replete in the topos of presheaves on the monoid **L** of continuous endofunctions on the complete chain $\overline{\omega}$, with $t\colon 1 \longrightarrow \Sigma$ the top element of the Sierpinsky space.
- dI-domains and stable continuous functions are replete in the topos of presheaves on the category of finite products of $\overline{\omega}$ and stable continuous maps—this category is the completion wrt. finite products $\mathrm{Prod}_{\mathrm{fin}}(\mathbf{L})$ of **L**. The dominance $t\colon 1 \longrightarrow \Sigma$ is as before.
- Effectively given domains are replete in the effective topos, see [4], where Σ is the equivalence relation on numbers with two classes: the one which defines $t\colon 1 \longrightarrow \Sigma$ consists of the (codes of) Turing machines converging on input 0.

1 Replete cpos

The results raise an obvious question: do the replete objects form a known collection of domains in some of the examples above. For instance, in case of presheaves on **L**, are all the ω-complete posets replete.

We shall answer this question in the negative, approaching the problem from the slightly different perspective of the category **CPO** of directed-complete posets.

First of all, note that there are many replete complete posets. For instance, every sober cpo A is replete because

$$A \xrightarrow{\ \sim\ } \mathbf{Frm}(\mathrm{Open}(A), \Sigma) \rightarrowtail \Sigma^{\mathbf{CPO}(A,\Sigma)} \rightrightarrows \Sigma^{\mathbf{CPO}(\Sigma^{\mathbf{CPO}(A,\Sigma)},\Sigma)}$$

presents A as an equalizer of replete objects.

Lemma 1.1 *Suppose A is replete, and $f\colon A \longrightarrow B$ is such that $\Sigma^f\colon \Sigma^B \xrightarrow{\ \sim\ } \Sigma^A$. Then f has a retraction.*

In order to show that there are non-replete cpos we shall prove

Theorem 1.2 *There is a continuous embedding $f\colon A \rightarrowtail B$ of countable cpos such that*

(i) $\Sigma^f\colon \Sigma^B \xrightarrow{\ \sim\ } \Sigma^A$,

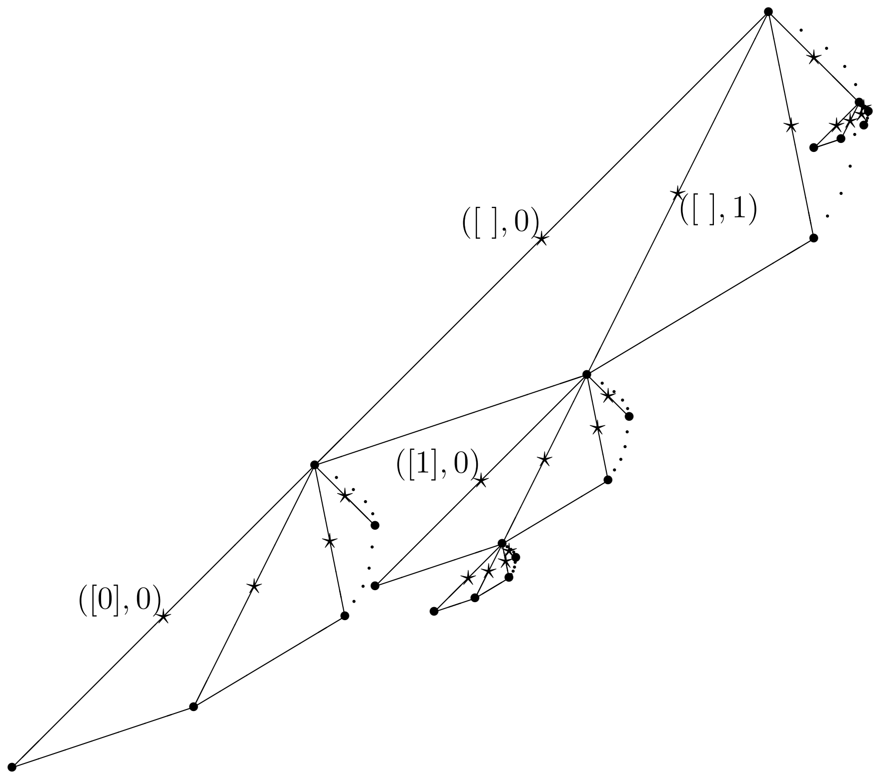

Fig. 1. The cpo B_0 with elements in $A_0 \subset B_0$ drawn with $\star$

(ii) *B has a top element,*

(iii) *A does not have a top element.*

Hence A cannot be a retract of B, neither can it be replete.

Note that Σ^A can be seen as the collection of Scott-open subsets of A as well as that of Scott-closed subsets:

$$\Sigma^A \xrightarrow{\sim} \mathrm{Open}(A) \xrightarrow{\sim} \mathrm{Closed}(A).$$

In what follows, it will be useful to think in terms of Scott-closed subsets.

The embedding f will be obtained after grafting on the poset C of lists $\ell = [i_1, \ldots, i_n]$ of numbers, ordered according to the clauses

(a) $\ell' < \ell$, if $\ell' = \ell@\ell_0$ for some list ℓ_0,

(b) $\ell@[i] < \ell@[j]$, if $i < j$. Hence $\bigvee^{\uparrow}_i \ell@[i] = \ell$.

The cpo A will be B without C; the cpo B is obtained by grafting in *appropriate* ways copies of the poset of lazy natural numbers, to points newly added to C.

So the first step is to add new points to C: for each pair $\ell@[i] < \ell$, add (ℓ, i) with the conditions that

(c) $\ell@[i] < (\ell, i) < \ell$.

The poset thus obtained is exactly domain E in [9]; but, since it appears as the first step for our construction, we shall like to call it B_0. The top part of B_0 looks something like in Fig. 1.

Proposition 1.3 *Let A_0 be the cpo on $B_0 - C$ with the order induced from B_0, and consider the inclusion $f_0: A_0 \rightarrowtail B_0$. Then*

$$f_0^{-1}: \text{Closed}(B_0) \longrightarrow \text{Closed}(A_0)$$

as its adjoint $f_0^-: X \longmapsto \overline{f_0[X]}$ as a retraction.

To force surjectivity of $f_0^{-1}: \text{Closed}(B_0) \rightarrowtail \text{Closed}(A_0)$, one notices at first how the equality in $X \subseteq f_0^{-1}(f_0^-(X))$ may fail: for instance, when X is the Scott-closed subset of A_0 generated by the set $\{([\,], n) \mid n \geq 1\}$.

In fact, the fixed points of $f_0^{-1} f_0^-$ are characterized as follows

Proposition 1.4 *For X is a Scott-closed subset of A_0, one has that*

$$f_0^{-1}(f_0^-(X)) = X$$

if and only if X satisfies the following conditions:

(i) *if there are infinitely many n such that $(\ell, n) \in X$, then $(\ell, n) \in X$ for all n,*

(ii) *if $(\ell @ [k+1], n) \in X$ for infinitely many n, then $(\ell @ [k], n) \in X$ for all n,*

(iii) *if there are infinitely many j such that $(\ell @ [j], n) \in X$ for infinitely many n, then $(\ell, n) \in X$ for all n.*

Clearly, conditions (ii) and (iii) may be rewritten, under (i), with "all n" instead of "infinitely many n". Note that, in order to get (i), we shall use new points and conditions very similar to that in [7], see (d) below.

Hence, to obtain an isomorphism between the closed subsets, add new elements and clauses:

(d) for each ℓ, i, elements $(\ell, i)_n$ with
- $(\ell, i)_n < (\ell, n)$
- $\bigvee^{\uparrow}_n (\ell, i)_n = (\ell, i)$

(e) for each $\ell @ [k]$, and i, elements $(\ell @ [k], i, \downarrow)_n$ with
- $(\ell @ [k], i, \downarrow)_n < (\ell @ [k+1], n)$
- $\bigvee^{\uparrow}_n (\ell @ [k], i, \downarrow)_n = (\ell @ [k], i)$

(f) for each ℓ, i, elements $(\ell, i, \infty)^j$ and $(\ell, i, \infty)^j_n$ with
- $(\ell, i, \infty)^j_n < (\ell @ [j], n)$
- $\bigvee^{\uparrow}_n (\ell, i, \infty)^j_n = (\ell, i, \infty)^j$
- $\bigvee^{\uparrow}_j (\ell, i, \infty)^j = (\ell, i)$

The conditions above define the required directed-complete poset B; the subposet A is $B - C$.

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