Realizability Models

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MFPS Tutorial, New Orleans, 10 April 2007

Naïve Realizability Models

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Prerequisites

- partial functions $\mathbb{N} \rightharpoonup \mathbb{N}$ computed by Turing machines
- M(n) means that the function M computed by the Turing machine is defined on n and the output is $M(n)$.
- there are computable bijections

$$\begin{tikzcd}
(m,\ell) \arrow[r, mapsto] & \llparenthesis m,\ell \rrparenthesis \\
\mathbb{N}\times\mathbb{N} \arrow[r, bend left] & \mathbb{N} \arrow[l, bend left] \\
(n_0,n_1) & n \arrow[l, mapsto]
\end{tikzcd}$$

- "test on 0" is computable $(i,n,m) \mapsto \mathsf{case}(i,n,m) = \begin{cases} n & \text{if } i = 0 \\ m & \text{otherwise} \end{cases} : \mathbb{N}\times\mathbb{N}\times\mathbb{N} \to \mathbb{N}$
- since Turing machines are strings in a fixed alphabet, they can be encoded by numbers: the number which encodes the machine M is $\lfloor M \rfloor$, the machine encoded by the number e is $\lceil e \rceil$.
- U is a universal Turing machine so that

$$U(\llparenthesis e,n \rrparenthesis) = \lceil e \rceil(n) \qquad \text{any } n.$$

  and S satisfies

$$M(\llparenthesis m,\ell \rrparenthesis) = \lceil S(\lfloor M \rfloor, m) \rceil(\ell) = U(\llparenthesis S(\lfloor M \rfloor, m), \ell \rrparenthesis) \qquad \text{any } M, m, \ell.$$

- the fundamental structure is that of partial combinatory algebra given on the set $\mathbb{N}$ by the partial binary operation:

$$(t,n) \mapsto \lceil t \rceil(n) = \text{the output of the Turing machine encoded by } t \text{ on input } n$$

Assemblies

An *assembly* $X = (X_{\mathit{el}}, \Vdash_X)$ is

- a collection of elements X_{el} together with
- a relation $\Vdash_X$ between numbers and elements of X_{el}
  which is surjective, *i.e.* for each $x \in X_{\mathit{el}}$ there is a number such that $n \Vdash_X x$

Write $n \Vdash_X x$ as $n \Vdash x \in X$. Read it as "n *realizes* that x is in X".

Say that $X \subseteq Y$ when there is a computable function M such that
for every $x \in X_{\mathit{el}}$ and every $n \in \mathbb{N}$, if $n \Vdash x \in X$, then $M(n) \Vdash x \in Y$

$$\begin{tikzcd}
\textbf{if} & & x \in X & \textbf{then} & & x \in Y & & \textbf{any } x \\
& n \arrow[ur, phantom, sloped, "\Vdash"] \arrow[rrr, rightsquigarrow] & & & M(n) \arrow[ur, phantom, sloped, "\Vdash"] & & & \textbf{any } n
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if

$$\begin{tikzcd} & x \in X & & & x \in Y \\ n \arrow[ur, phantom, "\Vdash" sloped] \arrow[rrr, rightsquigarrow] & & & M(n) \arrow[ur, phantom, "\Vdash" sloped] & \end{tikzcd}$$

then

any x

any n

$X \subseteq X$

Assemblies

An *assembly* $X = (X_{e\ell}, \Vdash_X)$ is

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if **then**

$$\begin{tikzcd} & x \in X & & x \in Y \\ n \arrow[ur, phantom, "\Vdash"] \arrow[rr, squiggly] & & M(n) \arrow[ur, phantom, "\Vdash"] & \end{tikzcd}$$

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If $X \subseteq Y$ and $Y \subseteq Z$ then $X \subseteq Z$

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$$\begin{tikzcd}
& & & \textbf{if} & m \arrow[dr, phantom, "\Vdash" sloped] \arrow[rrr, rightsquigarrow] & & \textbf{then} & L(m) \arrow[dr, phantom, "\Vdash" sloped] & & \textbf{any } m \\
\textbf{if} & & x \in X & \textbf{then} & & x \in Y & & & x \in Z & \textbf{any } x \\
& n \arrow[ur, phantom, "\Vdash" sloped] \arrow[rrr, rightsquigarrow] & & & M(n) \arrow[ur, phantom, "\Vdash" sloped] & & & & & \textbf{any } n
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\textbf{if} & & x \in X & \textbf{then} & & x \in Y & & & x \in Z & \textbf{any } x \\
& n \arrow[ur, phantom, "\Vdash"] \arrow[rrr, squiggly] \arrow[rrrrrr, squiggly, bend right] & & & M(n) \arrow[ur, phantom, "\Vdash"] & & & L(M(n)) \arrow[ur, phantom, "\Vdash"] & & \textbf{any } n
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Note that $X = Y$ means that $X_{el} = Y_{el}$,

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for suitable M

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$$\begin{tikzcd}
& & & M'(m) \arrow[dl, phantom, sloped, "\Vdash"] & \textbf{when} & & m \arrow[lll, squiggly] \arrow[dl, phantom, sloped, "\Vdash"] & \textbf{any } m \\
\textbf{if} & & x \in X & \textbf{then} & & x \in Y & & \textbf{any } x \\
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for suitable M and M'

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for suitable M and M'

NO REQUEST that $M'(M(n)) = n$ or similar.

Examples of assemblies

- S: a set

  $\nabla S \overset{\text{def}}{=} (S, \Vdash_{\nabla S})$ where $\quad n \Vdash_{\nabla S} s \quad$ if $\quad s \in S$ NOTE: no condition on n

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  So $\nabla\emptyset \subseteq X$ for any assembly X

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Check $\mathbf{N} = \nabla\mathbb{N}$

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Check $\quad \mathbf{N} = \nabla\mathbb{N}$

if $\; n \Vdash x \in \mathbf{N} \;$ **then** $\; n \rightsquigarrow \; \Vdash x \in \nabla\mathbb{N}$

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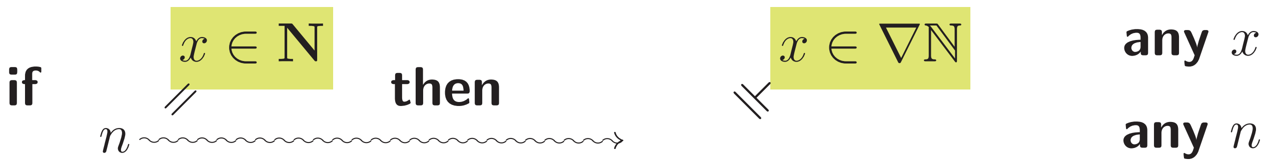

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if

$$\begin{tikzcd}
x \in \mathbf{N} & & x \in \nabla\mathbb{N} \\
n \arrow[u, phantom, sloped, "\Vdash"] \arrow[rr, squiggly] & & n \arrow[u, phantom, sloped, "\Vdash"]
\end{tikzcd}$$

then

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Examples of assemblies

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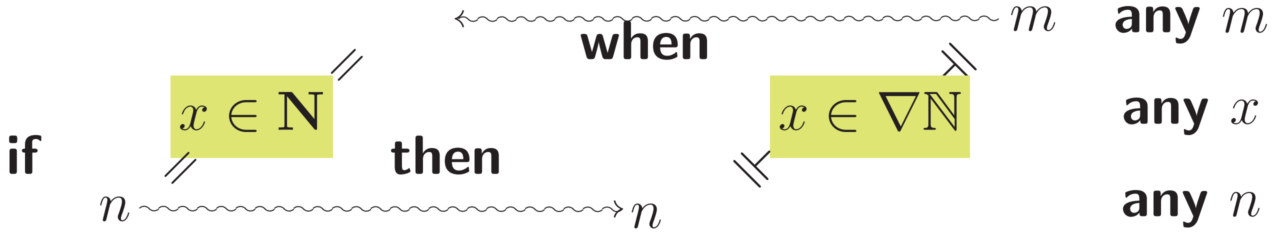

if $n \Vdash x \in \mathbf{N}$ **then** $n \Vdash x \in \nabla\mathbb{N}$

when $m \Vdash x \in \nabla\mathbb{N}$ then $m \Vdash x \in \mathbf{N}$

any m

any x

any n

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$$\begin{tikzcd}
 & & & {} & & & m \arrow[lll, squiggly, "??", "\textbf{when}"'] & \textbf{any } m \\
\textbf{if} & & x \in \mathbf{N} \arrow[ur, phantom, sloped, "\Vdash"] & \textbf{then} & & x \in \nabla\mathbb{N} \arrow[ur, phantom, sloped, "\Vdash"] & & \textbf{any } x \\
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  $\nabla S \overset{\text{def}}{=} (S, \Vdash_{\nabla S})$ where $n \Vdash_{\nabla S} s$ if $s \in S$ NOTE: no condition on n

  - $n \Vdash x \in \nabla\{a,b\}$ iff $x = a$ or $x = b$
  - $n \Vdash x \in \nabla\mathbb{N}$ iff $x \in \mathbb{N}$
  - $n \Vdash x \in \nabla\emptyset$ iff $\lightning$

  So $\nabla\emptyset \subseteq X$ for any assembly X

- $\mathbf{N} = (\mathbb{N}, \Vdash_{\mathbf{N}})$ where $n \Vdash_{\mathbf{N}} x$ if $n = x$.

  Then $n \Vdash x \in \mathbf{N}$ iff $n = x \in \mathbb{N}$

Check $\mathbf{N} \subsetneq \nabla\mathbb{N}$

if $n \Vdash x \in \mathbf{N}$ **then** $n \rightsquigarrow n \Vdash x \in \nabla\mathbb{N}$

any x

any n

Examples of assemblies

- $\Sigma = (\{0,1\}, \Vdash_\Sigma)$ where $n \Vdash_\Sigma x$ if either $\lceil n \rceil(n)\downarrow$ and $x = 1$ or $\lceil n \rceil(n)\uparrow$ and $x = 0$.

Then $n \Vdash x \in \Sigma$ iff
$$\begin{array}{rlll} \text{either} & \lceil n \rceil(n)\downarrow & \text{and} & x = 1 \\ \text{or} & \lceil n \rceil(n)\uparrow & \text{and} & x = 0 \end{array}$$

Examples of assemblies

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Check $\Sigma \subsetneq \nabla\{0,1\}$

Examples of assemblies

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Check $\Sigma \subsetneq \nabla\{0,1\}$

if

$$\begin{tikzcd} & x \in \nabla\{0,1\} & & x \in \Sigma \\ n \arrow[ur, phantom, "\Vdash" sloped] \arrow[rrr, squiggly] & & & {} \arrow[u, phantom, "\Vdash"] \end{tikzcd}$$

then

any x

any n

Examples of assemblies

- $\Sigma = (\{0,1\}, \Vdash_\Sigma)$ where $n \Vdash_\Sigma x$ if either $\lceil n \rceil(n)\downarrow$ and $x = 1$ or $\lceil n \rceil(n)\uparrow$ and $x = 0$.

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or $\lceil n \rceil(n)\uparrow$ and $x = 0$

Check $\Sigma \subsetneq \nabla\{0,1\}$

if **then** **any** x

$$\begin{tikzcd} & x \in \nabla\{0,1\} & & x \in \Sigma \\ n \arrow[ur, phantom, "\Vdash"] \arrow[rr, squiggly, "??"'] & & {} \arrow[ur, phantom, "\Vdash"] \end{tikzcd}$$

any n

Examples of assemblies

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  Check $\Sigma \subsetneq \nabla\{0,1\}$

  if $n \Vdash x \in \nabla\{0,1\}$ **then** $n \underset{??}{\rightsquigarrow} \Vdash x \in \Sigma$

  any x

  any n

- $\mathbf{1} = (\{0\}, \Vdash_{\mathbf{1}})$ where $n \Vdash_{\mathbf{1}} x$ if $n = 0 = x$.

Examples of assemblies

- $\Sigma = (\{0,1\}, \Vdash_\Sigma)$ where $n \Vdash_\Sigma x$ if either $\lceil n \rceil(n)\downarrow$ and $x = 1$ or $\lceil n \rceil(n)\uparrow$ and $x = 0$.

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Check $\Sigma \subsetneq \nabla\{0,1\}$

if $n \Vdash x \in \nabla\{0,1\}$ **then** $\underset{??}{\rightsquigarrow} \Vdash x \in \Sigma$ **any** x

any n

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Examples of assemblies

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or $\lceil n \rceil(n)\uparrow$ and $x = 0$

Check $\Sigma \subsetneq \nabla\{0,1\}$

if **then** **any** x

$$\begin{tikzcd} & x \in \nabla\{0,1\} & & x \in \Sigma \\ n \arrow[ur, phantom, "\Vdash"] \arrow[rr, squiggly, "??"'] & & {} \arrow[ur, phantom, "\Vdash"] & \end{tikzcd}$$

any n

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Check $\mathbf{1} = \nabla\{0\}$

if **then** **any** x

$$\begin{tikzcd} & x \in \nabla\{0\} & & x \in \mathbf{1} \\ n \arrow[ur, phantom, "\Vdash"] \arrow[rr, squiggly] & & {} \arrow[ur, phantom, "\Vdash"] & \end{tikzcd}$$

any n

Examples of assemblies

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Check $\Sigma \subsetneq \nabla\{0,1\}$

if $n \Vdash x \in \nabla\{0,1\}$ **then** $\Vdash x \in \Sigma$ **any** x

$$\begin{tikzcd}[column sep=huge] n \arrow[r, rightsquigarrow, "??"'] & {} \end{tikzcd}$$

any n

- $\mathbf{1} = (\{0\}, \Vdash_{\mathbf{1}})$ where $n \Vdash_{\mathbf{1}} x$ if $n = 0 = x.$

Check $\mathbf{1} = \nabla\{0\}$

if $n \Vdash x \in \nabla\{0\}$ **then** $\Vdash x \in \mathbf{1}$ **any** x

$$\begin{tikzcd}[column sep=huge] n \arrow[r, rightsquigarrow] & 0 \end{tikzcd}$$

any n

Elementary constructions

- Intersection: $X \cap Y \subseteq X \qquad X \cap Y \subseteq Y$

if $Z \subseteq X$ and $Z \subseteq Y$, then $Z \subseteq X \cap Y$

Define: $X \cap Y \overset{\text{def}}{=} (X_{el} \cap Y_{el}, \Vdash_{X \cap Y})$

where $n \Vdash_{X \cap Y} z$ if $n_0 \Vdash_X z$ and $n_1 \Vdash_Y z$

i.e. $\langle\!|m, \ell|\!\rangle \Vdash z \in X \cap Y$ iff $m \Vdash z \in X$ and $\ell \Vdash z \in Y$

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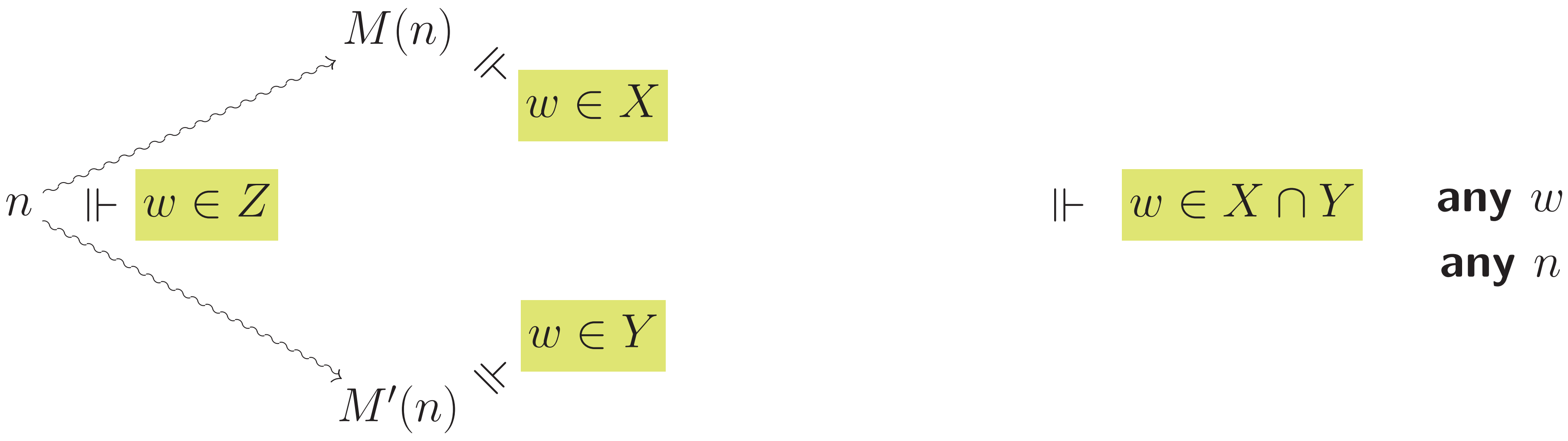

$$\begin{tikzcd}
 & M(n) \arrow[r, phantom, "\Vdash"] & w \in X & & \\
n \arrow[ru, rightsquigarrow] \arrow[rd, rightsquigarrow] \arrow[r, phantom, "\Vdash"] & w \in Z & & \Vdash & w \in X \cap Y \\
 & M'(n) \arrow[r, phantom, "\Vdash"] & w \in Y & &
\end{tikzcd}$$

any w

any n

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- Intersection: $X \cap Y \subseteq X$ $X \cap Y \subseteq Y$

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if $Z \subseteq X$ and $Z \subseteq Y$, then $Z \subseteq X \cap Y$

$$\begin{tikzcd}
 & M(n) \arrow[dr, phantom, "\Vdash"] & & \\
 & & w \in X & \\
n \Vdash w \in Z \arrow[uur, squiggly] \arrow[rrr, squiggly, bend left] \arrow[ddr, squiggly] & & & \langle M(n), M'(n) \rangle \Vdash w \in X \cap Y \\
 & & w \in Y & \\
 & M'(n) \arrow[ur, phantom, "\Vdash"] & &
\end{tikzcd}$$

any w

any n

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$X \cap Y \subseteq X$

$$\begin{tikzcd}
 & z \in X \cap Y & & z \in X \\
\langle\!| m, \ell |\!\rangle \arrow[ur, phantom, sloped, "\Vdash"] \arrow[rr, squiggly] & & m \arrow[ur, phantom, sloped, "\Vdash"] &
\end{tikzcd}$$

Elementary constructions

- Intersection: $X \cap Y \subseteq X$ $\quad$ $X \cap Y \subseteq Y$

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i.e. $\quad \langle m, \ell \rangle \Vdash z \in X \cap Y \quad$ iff $\quad m \Vdash z \in X \quad$ and $\quad \ell \Vdash z \in Y$

$X \cap Y \subseteq X$

$$\begin{tikzcd}
z \in X \cap Y & z \in X \\
\langle m, \ell \rangle \arrow[u, phantom, "\Vdash" sloped] \arrow[r, squiggly] & m \arrow[u, phantom, "\Vdash" sloped]
\end{tikzcd}$$

$X \cap Y \subseteq Y$

$$\begin{tikzcd}
z \in X \cap Y & z \in Y \\
\langle m, \ell \rangle \arrow[u, phantom, "\Vdash" sloped] \arrow[r, squiggly] & \ell \arrow[u, phantom, "\Vdash" sloped]
\end{tikzcd}$$

Elementary constructions

- Union: $X \subseteq X \cup Y \qquad Y \subseteq X \cup Y$

  if $X \subseteq Z$ and $Y \subseteq Z$ then $X \cup Y \subseteq Z$

  Define: $X \cup Y \overset{\text{def}}{=} (X_{el} \cup Y_{el}, \Vdash_{X \cup Y})$

  where $n \Vdash_{X \cup Y} z$ if either $n_1 = 0, n_0 \Vdash_X z$ or $n_1 = 1, n_0 \Vdash_Y z$

  i.e. $\langle\!\langle n, i \rangle\!\rangle \Vdash z \in X \cup Y$ if either $i = 0$ and $n \Vdash z \in X$

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either $i = 0$ and $n \Vdash z \in X$
or $i = 1$ and $n \Vdash z \in Y$

if $X \subseteq Z$ and $Y \subseteq Z$ then $X \cup Y \subseteq Z$

$$\begin{tikzcd}
 & n \arrow[d, phantom, "\Vdash" sloped] \arrow[rrr, squiggly] & & & M(n) \arrow[d, phantom, "\top"] & \\
 & w \in X & & & & \\
\langle n, i \rangle \Vdash w \in X \cup Y & & & & \Vdash w \in Z & \textbf{any } w \\
 & & & & \arrow[u, phantom, "\perp"] & \textbf{any } n \\
 & w \in Y & & & M'(n) & \\
 & n \arrow[u, phantom, "\Vdash" sloped] \arrow[rrruu, squiggly] & & & &
\end{tikzcd}$$

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or $i = 1$ and $n \Vdash z \in Y$

if $X \subseteq Z$ and $Y \subseteq Z$ then $X \cup Y \subseteq Z$

$$\begin{tikzcd}
 & n \Vdash w \in X \arrow[rr, squiggly] & & M(n) \arrow[d, phantom, "\Top"] \\
\langle\!\langle n, i \rangle\!\rangle \Vdash w \in X \cup Y \arrow[rr, squiggly, bend left] & & \mathsf{case}(i, M(n), M'(n)) \Vdash & w \in Z \\
 & n \Vdash w \in Y \arrow[rr, squiggly] & & M'(n) \arrow[u, phantom, "\Bot"]
\end{tikzcd}$$

any w

any n

Elementary constructions

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or $i = 1$ and $n \Vdash z \in Y$

$X \subseteq X \cup Y$

$$\begin{tikzcd} & z \in X & & z \in X \cup Y \\ n \arrow[ur, phantom, "\Vdash"] \arrow[rr, squiggly] & & {} \arrow[ur, phantom, "\Vdash"] & \end{tikzcd}$$

Elementary constructions

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Define: $X \cup Y \overset{\text{def}}{=} (X_{e\ell} \cup Y_{e\ell}, \Vdash_{X \cup Y})$

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$X \subseteq X \cup Y$

$$\begin{tikzcd}
 & z \in X & & z \in X \cup Y \\
n \arrow[ru, phantom, "\Vdash" sloped] \arrow[rr, squiggly] & & \langle\!\langle n, 0 \rangle\!\rangle \arrow[ru, phantom, "\Vdash" sloped] &
\end{tikzcd}$$

Elementary constructions

- Union: $X \subseteq X \cup Y$ $\quad$ $Y \subseteq X \cup Y$

  if $X \subseteq Z$ and $Y \subseteq Z$ then $X \cup Y \subseteq Z$

Define: $X \cup Y \overset{\text{def}}{=} (X_{el} \cup Y_{el}, \Vdash_{X \cup Y})$

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i.e. $\langle n, i \rangle \Vdash z \in X \cup Y$ if $\begin{cases} \text{either } i = 0 \text{ and } n \Vdash z \in X \\ \text{or } i = 1 \text{ and } n \Vdash z \in Y \end{cases}$

$X \subseteq X \cup Y$

$$\begin{tikzcd}
 & z \in X & & z \in X \cup Y \\
n \arrow[ur, phantom, sloped, "\Vdash"] \arrow[rr, squiggly] & & \langle n, 0 \rangle \arrow[ur, phantom, sloped, "\Vdash"] &
\end{tikzcd}$$

$Y \subseteq X \cup Y$

$$\begin{tikzcd}
 & z \in Y & & z \in X \cup Y \\
n \arrow[ur, phantom, sloped, "\Vdash"] \arrow[rr, squiggly] & & \langle n, 1 \rangle \arrow[ur, phantom, sloped, "\Vdash"] &
\end{tikzcd}$$

Elementary constructions

- (Pseudo)complement: $(X \Longrightarrow_A Y) \cap X \subseteq Y$ $(X \Longrightarrow_A Y) \subseteq A$

if $W \subseteq A$ and $W \cap X \subseteq Y$ then $W \subseteq (X \Longrightarrow_A Y)$

Define: $(X \Longrightarrow_A Y) \overset{\text{def}}{=} ((A_{\mathit{e\ell}} \smallsetminus X_{\mathit{e\ell}}) \cup Y_{\mathit{e\ell}}, \Vdash_{(X \Longrightarrow_A Y)})$

where $n \Vdash_{(X \Longrightarrow_A Y)} z$ if $n_0 \Vdash_A z$ and for all $k \Vdash_X z$ $\lceil n_1 \rceil (k) \Vdash_Y z$

i.e. $\langle\!\langle m, e \rangle\!\rangle \Vdash z \in (X \Longrightarrow_A Y)$ iff $m \Vdash z \in A$ and

for all $k \Vdash z \in X$ one has $\lceil e \rceil (k) \Vdash z \in Y$

Elementary constructions

- (Pseudo)complement: $(X \Longrightarrow_A Y) \cap X \subseteq Y$ $\quad (X \Longrightarrow_A Y) \subseteq A$

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where $n \Vdash_{(X \Longrightarrow_A Y)} z$ if $n_0 \Vdash_A z$ and for all $k \Vdash_X z$ $\lceil n_1 \rceil(k) \Vdash_Y z$

i.e. $\langle m, e \rangle \Vdash z \in (X \Longrightarrow_A Y)$ iff $m \Vdash z \in A$ and

for all $k \Vdash z \in X$ one has $\lceil e \rceil(k) \Vdash z \in Y$

$(X \Longrightarrow_A Y) \cap X \subseteq Y$

$$\begin{tikzcd}
 & z \in (X \Longrightarrow_A Y) \cap X & & z \in Y \\
\langle \langle m, e \rangle, \ell \rangle \arrow[ur, phantom, sloped, "\Vdash"] \arrow[rr, rightsquigarrow] & & \lceil e \rceil(\ell) \arrow[ur, phantom, sloped, "\Vdash"] &
\end{tikzcd}$$

any z

any n, e, m

Elementary constructions

- (Pseudo)complement: $(X \Longrightarrow_A Y) \cap X \subseteq Y$ $(X \Longrightarrow_A Y) \subseteq A$

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if $W \subseteq A$ and $W \cap X \subseteq Y$ then $W \subseteq (X \Longrightarrow_A Y)$

Elementary constructions

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i.e. $\llparenthesis m, e \rrparenthesis \Vdash z \in (X \Longrightarrow_A Y)$ iff $m \Vdash z \in A$ and

for all $k \Vdash z \in X$ one has $\lceil e \rceil(k) \Vdash z \in Y$

if $W \subseteq A$ and $W \cap X \subseteq Y$ then $W \subseteq (X \Longrightarrow_A Y)$

Suppose

$$\begin{tikzcd}
z \in W & z \in A \\
n \arrow[u, phantom, sloped, "\Vdash"] \arrow[r, squiggly] & M(n) \arrow[u, phantom, sloped, "\Vdash"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
z \in W \cap X & z \in Y \\
\llparenthesis m, \ell \rrparenthesis \arrow[u, phantom, sloped, "\Vdash"] \arrow[r, squiggly] & L(\llparenthesis m, \ell \rrparenthesis) \arrow[u, phantom, sloped, "\Vdash"]
\end{tikzcd}$$

Elementary constructions

- (Pseudo)complement: $(X \Longrightarrow_A Y) \cap X \subseteq Y$ $\quad$ $(X \Longrightarrow_A Y) \subseteq A$

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where $n \Vdash_{(X \Longrightarrow_A Y)} z$ if $n_0 \Vdash_A z$ and for all $k \Vdash_X z$ $\lceil n_1 \rceil (k) \Vdash_Y z$

i.e. $\langle\!| m, e |\!\rangle \Vdash z \in (X \Longrightarrow_A Y)$ iff $m \Vdash z \in A$ and

for all $k \Vdash z \in X$ one has $\lceil e \rceil (k) \Vdash z \in Y$

if $W \subseteq A$ and $W \cap X \subseteq Y$ then $W \subseteq (X \Longrightarrow_A Y)$

Suppose

$$\begin{tikzcd}
z \in W & & z \in A \\
n \arrow[u, phantom, "\Vdash"] \arrow[rr, squiggly] & & M(n) \arrow[u, phantom, "\Vdash"]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
z \in W \cap X & & z \in Y \\
\langle\!| m, \ell |\!\rangle \arrow[u, phantom, "\Vdash"] \arrow[rr, squiggly] & & L(\langle\!| m, \ell |\!\rangle) \arrow[u, phantom, "\Vdash"]
\end{tikzcd}$$

$$\begin{tikzcd}
z \in W & & z \in (X \Longrightarrow_A Y) \\
n \arrow[u, phantom, "\Vdash"] \arrow[r, squiggly] & {} \arrow[ur, phantom, "\Vdash"] &
\end{tikzcd}$$

Elementary constructions

- (Pseudo)complement: $(X \Longrightarrow_A Y) \cap X \subseteq Y$ $(X \Longrightarrow_A Y) \subseteq A$

  if $W \subseteq A$ and $W \cap X \subseteq Y$ then $W \subseteq (X \Longrightarrow_A Y)$

Define: $(X \Longrightarrow_A Y) \overset{\text{def}}{=} ((A_{\mathit{el}} \setminus X_{\mathit{el}}) \cup Y_{\mathit{el}}, \Vdash_{(X \Longrightarrow_A Y)})$

where $n \Vdash_{(X \Longrightarrow_A Y)} z$ if $n_0 \Vdash_A z$ and for all $k \Vdash_X z$ $\lceil n_1 \rceil(k) \Vdash_Y z$

i.e. $\langle\!| m, e |\!\rangle \Vdash z \in (X \Longrightarrow_A Y)$ iff $m \Vdash z \in A$ and

for all $k \Vdash z \in X$ one has $\lceil e \rceil(k) \Vdash z \in Y$

if $W \subseteq A$ and $W \cap X \subseteq Y$ then $W \subseteq (X \Longrightarrow_A Y)$

Suppose
$$\begin{tikzcd} & z \in W & & z \in A \\ n \arrow[ur, phantom, "\Vdash"] \arrow[rr, rightsquigarrow] & & M(n) \arrow[ur, phantom, "\Vdash"] & \end{tikzcd}$$
and
$$\begin{tikzcd} & z \in W \cap X & & z \in Y \\ \langle\!| m, \ell |\!\rangle \arrow[ur, phantom, "\Vdash"] \arrow[rr, rightsquigarrow] & & L(\langle\!| m, \ell |\!\rangle) \arrow[ur, phantom, "\Vdash"] & \end{tikzcd}$$

$$\begin{tikzcd} & z \in W & & z \in (X \Longrightarrow_A Y) \\ n \arrow[ur, phantom, "\Vdash"] \arrow[rr, rightsquigarrow] & & \langle\!| M(n), S(\lfloor L \rfloor, n) |\!\rangle \arrow[ur, phantom, "\Vdash"] & \end{tikzcd}$$

where $\lceil S(\lfloor L \rfloor, m) \rceil(\ell) = L(\langle m, \ell \rangle)$.

Elementary constructions

- (Pseudo)complement: $(X \Longrightarrow_A Y) \cap X \subseteq Y$ $(X \Longrightarrow_A Y) \subseteq A$

  if $W \subseteq A$ and $W \cap X \subseteq Y$ then $W \subseteq (X \Longrightarrow_A Y)$

Elementary constructions

- complement: $(X \Longrightarrow_A \nabla\emptyset) \cap X \subseteq \nabla\emptyset$ $(X \Longrightarrow_A \nabla\emptyset) \subseteq A$

  if $W \subseteq A$ and $W \cap X \subseteq \nabla\emptyset$ then $W \subseteq (X \Longrightarrow_A \nabla\emptyset)$

Elementary constructions

- complement: $(X \Longrightarrow_A \nabla\emptyset) \cap X \subseteq \nabla\emptyset$ $\quad (X \Longrightarrow_A \nabla\emptyset) \subseteq A$

  if $W \subseteq A$ and $W \cap X \subseteq \nabla\emptyset$ then $W \subseteq (X \Longrightarrow_A \nabla\emptyset)$

So $\langle\!\langle m, e\rangle\!\rangle \Vdash z \in (X \Longrightarrow_A Y \;\;)$ iff $m \Vdash z \in A$ and
for all $k \Vdash z \in X$ one has $\lceil e \rceil(k) \Vdash z \in Y$

Elementary constructions

- complement: $(X \Longrightarrow_A \nabla\emptyset) \cap X \subseteq \nabla\emptyset$ $(X \Longrightarrow_A \nabla\emptyset) \subseteq A$

  if $W \subseteq A$ and $W \cap X \subseteq \nabla\emptyset$ then $W \subseteq (X \Longrightarrow_A \nabla\emptyset)$

So $\langle\!\langle m, e \rangle\!\rangle \Vdash z \in (X \Longrightarrow_A \nabla\emptyset)$ iff $m \Vdash z \in A$ and
for all $k \Vdash z \in X$ one has $\lceil e \rceil(k) \Vdash z \in \nabla\emptyset$

Elementary constructions

- complement: $(X \Longrightarrow_A \nabla\emptyset) \cap X \subseteq \nabla\emptyset$ $(X \Longrightarrow_A \nabla\emptyset) \subseteq A$

  if $W \subseteq A$ and $W \cap X \subseteq \nabla\emptyset$ then $W \subseteq (X \Longrightarrow_A \nabla\emptyset)$

So $\langle\!\langle m, e\rangle\!\rangle \Vdash z \in (X \Longrightarrow_A \nabla\emptyset)$ iff $m \Vdash z \in A$ and
for all $k \Vdash z \in X$ one has $\lceil e\rceil(k) \Vdash z \in \nabla\emptyset$

iff $m \Vdash z \in A$ and no $k \Vdash z \in X$

Elementary constructions

- complement: $(X \Longrightarrow_A \nabla\emptyset) \cap X \subseteq \nabla\emptyset$ $(X \Longrightarrow_A \nabla\emptyset) \subseteq A$

  if $W \subseteq A$ and $W \cap X \subseteq \nabla\emptyset$ then $W \subseteq (X \Longrightarrow_A \nabla\emptyset)$

So $\langle\!| m, e |\!\rangle \Vdash z \in (X \Longrightarrow_A \nabla\emptyset)$ iff $m \Vdash z \in A$ and
for all $k \Vdash z \in X$ one has $\lceil e \rceil (k) \Vdash z \in \nabla\emptyset$

iff $m \Vdash z \in A$ and no $k \Vdash z \in X$

Define: $A \smallsetminus X \overset{\text{def}}{=} (A_{\mathit{el}} \smallsetminus X_{\mathit{el}}, \Vdash_{A \smallsetminus X})$
where $n \Vdash_{A \smallsetminus X} z$ if $n \Vdash_A z$

Elementary constructions

- complement: $(X \Longrightarrow_A \nabla\emptyset) \cap X \subseteq \nabla\emptyset$ $\quad$ $(X \Longrightarrow_A \nabla\emptyset) \subseteq A$

  if $W \subseteq A$ and $W \cap X \subseteq \nabla\emptyset$ then $W \subseteq (X \Longrightarrow_A \nabla\emptyset)$

So $\langle\!\langle m, e \rangle\!\rangle \Vdash z \in (X \Longrightarrow_A \nabla\emptyset)$ iff $m \Vdash z \in A$ and
for all $k \Vdash z \in X$ one has $\lceil e \rceil(k) \Vdash z \in \nabla\emptyset$

iff $m \Vdash z \in A$ and no $k \Vdash z \in X$

Define: $A \smallsetminus X \overset{\text{def}}{=} (A_{el} \smallsetminus X_{el}, \Vdash_{A \smallsetminus X})$
where $n \Vdash_{A \smallsetminus X} z$ if $n \Vdash_A z$

Exercise: Check if $(X \Longrightarrow_A Y) \;=\; (A \smallsetminus X) \cup Y$

Elementary constructions

- Cartesian product: $A \times B \overset{\text{def}}{=} (A_{el} \times B_{el}, \Vdash_{A\times B})$

  where $n \Vdash_{A\times B} (z, w)$ if $n_0 \Vdash_A z$ and $n_1 \Vdash_B w$

  i.e. $\langle\!|m, \ell|\!\rangle \Vdash (z, w) \in A \times B$ iff $m \Vdash z \in A$ and $\ell \Vdash w \in B$

Elementary constructions

- Cartesian product: $A \times B \overset{\text{def}}{=} (A_{el} \times B_{el}, \Vdash_{A \times B})$

  where $n \Vdash_{A \times B} (z, w)$ if $n_0 \Vdash_A z$ and $n_1 \Vdash_B w$

  i.e. $\langle\!|m, \ell|\!\rangle \Vdash (z, w) \in A \times B$ iff $m \Vdash z \in A$ and $\ell \Vdash w \in B$

If $A \subseteq C$ and $B \subseteq D$ then $A \times B \subseteq C \times D$

Elementary constructions

- Cartesian product: $A \times B \overset{\text{def}}{=} (A_{el} \times B_{el}, \Vdash_{A \times B})$

  where $\quad n \Vdash_{A \times B} (z, w) \quad$ if $\quad n_0 \Vdash_A z \quad$ and $\quad n_1 \Vdash_B w$

  i.e. $\quad \langle\!|m, \ell|\!\rangle \Vdash (z, w) \in A \times B \quad$ iff $\quad m \Vdash z \in A \quad$ and $\quad \ell \Vdash w \in B$

  If $A \subseteq C$ and $B \subseteq D$ then $A \times B \subseteq C \times D$

- Diagonal: $\Delta_A \overset{\text{def}}{=} (\{(z, z) \mid z \in A_{el}\}, \Vdash_{\Delta_A})$

  where $\quad n \Vdash_{\Delta_A} (z, w) \quad$ if $\quad z = w \quad$ and $\quad n \Vdash z \in A$

Elementary constructions

- Cartesian product: $A \times B \overset{\text{def}}{=} (A_{el} \times B_{el}, \Vdash_{A\times B})$

  where $n \Vdash_{A\times B} (z, w)$ if $n_0 \Vdash_A z$ and $n_1 \Vdash_B w$

i.e. $\langle m, \ell \rangle \Vdash (z, w) \in A \times B$ iff $m \Vdash z \in A$ and $\ell \Vdash w \in B$

If $A \subseteq C$ and $B \subseteq D$ then $A \times B \subseteq C \times D$

- Diagonal: $\Delta_A \overset{\text{def}}{=} (\{(z, z) \mid z \in A_{el}\}, \Vdash_{\Delta_A})$

  where $n \Vdash_{\Delta_A} (z, w)$ if $z = w$ and $n \Vdash z \in A$

Check $\Delta_A \subseteq A \times A$

$$\begin{tikzcd}
 & (z,w) \in \Delta_A & & (z,w) \in A \times A \\
n \arrow[ur, phantom, "\Vdash"] \arrow[rr, squiggly] & & {} \arrow[ur, phantom, "\Vdash"] &
\end{tikzcd}$$

any z, w

Elementary constructions

- Cartesian product: $A \times B \overset{\text{def}}{=} (A_{\mathit{el}} \times B_{\mathit{el}}, \Vdash_{A\times B})$

  where $n \Vdash_{A\times B} (z, w)$ if $n_0 \Vdash_A z$ and $n_1 \Vdash_B w$

i.e. $\langle m, \ell \rangle \Vdash (z, w) \in A \times B$ iff $m \Vdash z \in A$ and $\ell \Vdash w \in B$

If $A \subseteq C$ and $B \subseteq D$ then $A \times B \subseteq C \times D$

- Diagonal: $\Delta_A \overset{\text{def}}{=} (\{(z, z) \mid z \in A_{\mathit{el}}\}, \Vdash_{\Delta_A})$

  where $n \Vdash_{\Delta_A} (z, w)$ if $z = w$ and $n \Vdash z \in A$

Check $\Delta_A \subseteq A \times A$

$$\begin{tikzcd} & (z,w) \in \Delta_A & & (z,w) \in A \times A \\ n \arrow[ur, phantom, sloped, "\Vdash"] \arrow[rr, squiggly] & & {} \arrow[ur, phantom, sloped, "\Vdash"] & \end{tikzcd}$$

any z, w

any n

Elementary constructions

- Cartesian product: $A \times B \overset{\text{def}}{=} (A_{e\ell} \times B_{e\ell}, \Vdash_{A\times B})$

  where $\quad n \Vdash_{A\times B} (z,w) \quad$ if $\quad n_0 \Vdash_A z \quad$ and $\quad n_1 \Vdash_B w$

i.e. $\quad \langle\!| m, \ell |\!\rangle \Vdash (z,w) \in A \times B \quad$ iff $\quad m \Vdash z \in A \quad$ and $\quad \ell \Vdash w \in B$

If $A \subseteq C$ and $B \subseteq D$ then $A \times B \subseteq C \times D$

- Diagonal: $\Delta_A \overset{\text{def}}{=} (\{(z,z) \mid z \in A_{e\ell}\}, \Vdash_{\Delta_A})$

  where $\quad n \Vdash_{\Delta_A} (z,w) \quad$ if $\quad z = w \quad$ and $\quad n \Vdash z \in A$

Check $\quad \Delta_A \subseteq A \times A$

$$\begin{tikzcd}
 & (z,w) \in \Delta_A & & (z,w) \in A \times A & \textbf{any } z,w \\
n \arrow[ur, phantom, "\Vdash"] \arrow[rr, rightsquigarrow] & & \langle\!| n, n |\!\rangle \arrow[ur, phantom, "\Vdash"] & & \textbf{any } n
\end{tikzcd}$$

Elementary constructions

- Projection: for $X \subseteq A \times B$ $\quad$ $X \subseteq \pi_1^{\mathsf{p}}(X) \times B$

  if $X \subseteq Y \times B$ then $\pi_1^{\mathsf{p}}(X) \subseteq Y$

Define: $\pi_1^{\mathsf{p}}(X) \overset{\text{def}}{=} (\{z \in A_{el} \mid \exists w \in B_{el} \quad (z, w) \in X_{el}\}, \Vdash_{\pi_1^{\mathsf{p}}(X)})$

where $\quad n \Vdash_{\pi_1^{\mathsf{p}}(X)} z \quad$ if $\quad$ for some $w \in B_{el} \qquad n \Vdash_X (z, w)$

Elementary constructions

- Projection: for $X \subseteq A \times B$ $X \subseteq \pi_1^{\mathsf{p}}(X) \times B$

  if $X \subseteq Y \times B$ then $\pi_1^{\mathsf{p}}(X) \subseteq Y$

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where $n \Vdash_{\pi_1^{\mathsf{p}}(X)} z$ if for some $w \in B_{el}$ $n \Vdash_X (z,w)$

Suppose $X \subseteq A \times B$

Elementary constructions

- Projection: for $X \subseteq A \times B$ $X \subseteq \pi_1^{\mathsf{p}}(X) \times B$

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where $n \Vdash_{\pi_1^{\mathsf{p}}(X)} z$ if for some $w \in B_{\mathit{el}}$ $n \Vdash_X (z, w)$

Suppose $X \subseteq A \times B$

$$\begin{tikzcd}
 & x \in X & & x \in A \times B \\
n \arrow[ur, phantom, "\Vdash"] \arrow[rr, rightsquigarrow] & & M(n) \arrow[ur, phantom, "\Vdash"] &
\end{tikzcd}$$

Elementary constructions

- Projection: for $X \subseteq A \times B$ $\quad X \subseteq \pi_1^{\mathsf{p}}(X) \times B$

  if $X \subseteq Y \times B$ then $\pi_1^{\mathsf{p}}(X) \subseteq Y$

Define: $\pi_1^{\mathsf{p}}(X) \overset{\text{def}}{=} (\{z \in A_{\mathit{el}} \mid \exists w \in B_{\mathit{el}} \quad (z,w) \in X_{\mathit{el}}\}, \Vdash_{\pi_1^{\mathsf{p}}(X)})$

where $\quad n \Vdash_{\pi_1^{\mathsf{p}}(X)} z \quad$ if $\quad$ for some $w \in B_{\mathit{el}} \qquad n \Vdash_X (z,w)$

Suppose $X \subseteq A \times B$

$$\begin{tikzcd}
& (z,w) \in X & & (z,w) \in A \times B \\
n \arrow[ur, phantom, sloped, "\Vdash"] \arrow[rrr, squiggly] & & & \langle\!| M'(n), M''(n) |\!\rangle \arrow[u, phantom, sloped, "\Vdash"]
\end{tikzcd}$$

Elementary constructions

- Projection: for $X \subseteq A \times B$ $\quad$ $X \subseteq \pi_1^{\mathsf{p}}(X) \times B$

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where $\quad n \Vdash_{\pi_1^{\mathsf{p}}(X)} z \quad$ if $\quad$ for some $w \in B_{el} \qquad n \Vdash_X (z, w)$

Suppose $X \subseteq A \times B$

$$\begin{tikzcd}
 & (z,w) \in X & & (z,w) \in A \times B \\
n \arrow[ur, phantom, "\Vdash" sloped] \arrow[rrr, squiggly] & & & \langle\!| M'(n), M''(n) |\!\rangle \arrow[u, phantom, "\Vdash" sloped]
\end{tikzcd}$$

$X \subseteq \pi_1^{\mathsf{p}}(X) \times B$

Elementary constructions

- Projection: for $X \subseteq A \times B$ $\quad X \subseteq \pi_1^{\mathsf{p}}(X) \times B$

  if $X \subseteq Y \times B$ then $\pi_1^{\mathsf{p}}(X) \subseteq Y$

Define: $\pi_1^{\mathsf{p}}(X) \overset{\text{def}}{=} (\{z \in A_{\mathit{el}} \mid \exists w \in B_{\mathit{el}} \quad (z,w) \in X_{\mathit{el}}\}, \Vdash_{\pi_1^{\mathsf{p}}(X)})$

where $\quad n \Vdash_{\pi_1^{\mathsf{p}}(X)} z \quad$ if $\quad$ for some $w \in B_{\mathit{el}} \qquad n \Vdash_X (z,w)$

Suppose $X \subseteq A \times B$

$$\begin{tikzcd}
 & (z,w) \in X & & (z,w) \in A \times B \\
n \arrow[ur, phantom, "\Vdash" sloped] \arrow[rr, rightsquigarrow] & & \langle\!| M'(n), M''(n) |\!\rangle \arrow[ur, phantom, "\Vdash" sloped] &
\end{tikzcd}$$

$X \subseteq \pi_1^{\mathsf{p}}(X) \times B$

$$\begin{tikzcd}
 & (z,w) \in X & & (z,w) \in \pi_1^{\mathsf{p}}(X) \times B \\
n \arrow[ur, phantom, "\Vdash" sloped] \arrow[rr, rightsquigarrow] & & \langle\!| n, M''(n_1) |\!\rangle \arrow[ur, phantom, "\Vdash" sloped] &
\end{tikzcd}$$

Elementary constructions

- Projection: for $X \subseteq A \times B$ $X \subseteq \pi_1^{\mathsf{p}}(X) \times B$

  if $X \subseteq Y \times B$ then $\pi_1^{\mathsf{p}}(X) \subseteq Y$

Define: $\pi_1^{\mathsf{p}}(X) \overset{\text{def}}{=} (\{z \in A_{e\ell} \mid \exists w \in B_{e\ell} \quad (z,w) \in X_{e\ell}\}, \Vdash_{\pi_1^{\mathsf{p}}(X)})$

where $n \Vdash_{\pi_1^{\mathsf{p}}(X)} z$ if for some $w \in B_{e\ell}$ $n \Vdash_X (z,w)$

Suppose $X \subseteq A \times B$

$$\begin{tikzcd}
 & (z,w) \in X & & (z,w) \in A \times B \\
n \arrow[ur, phantom, "\Vdash" sloped] \arrow[rrr, squiggly] & & & \langle\!| M'(n), M''(n) |\!\rangle \arrow[u, phantom, "\Vdash"]
\end{tikzcd}$$

if $X \subseteq Y \times B$ then $\pi_1^{\mathsf{p}}(X) \subseteq Y$

Elementary constructions

- Projection: for $X \subseteq A \times B$ $\quad X \subseteq \pi_1^{\mathsf{p}}(X) \times B$

  if $X \subseteq Y \times B$ then $\pi_1^{\mathsf{p}}(X) \subseteq Y$

Define: $\pi_1^{\mathsf{p}}(X) \overset{\text{def}}{=} (\{z \in A_{e\ell} \mid \exists w \in B_{e\ell} \quad (z,w) \in X_{e\ell}\}, \Vdash_{\pi_1^{\mathsf{p}}(X)})$

where $\quad n \Vdash_{\pi_1^{\mathsf{p}}(X)} z \quad$ if $\quad$ for some $w \in B_{e\ell} \qquad n \Vdash_X (z,w)$

Suppose $X \subseteq A \times B$

$$\begin{tikzcd} & (z,w) \in X & & (z,w) \in A \times B \\ n \arrow[ur, phantom, "\Vdash" sloped] \arrow[rr, squiggly] & & \llparenthesis M'(n), M''(n) \rrparenthesis \arrow[ur, phantom, "\Vdash" sloped] & \end{tikzcd}$$

if $X \subseteq Y \times B$ $\qquad$ then $\pi_1^{\mathsf{p}}(X) \subseteq Y$

$$\begin{tikzcd} & (z,w) \in X & & (z,w) \in Y \times B \\ m \arrow[ur, phantom, "\Vdash" sloped] \arrow[rr, squiggly] & & \llparenthesis L(m), L'(m) \rrparenthesis \arrow[ur, phantom, "\Vdash" sloped] & \end{tikzcd}$$

Elementary constructions

- Projection: for $X \subseteq A \times B$ $X \subseteq \pi_1^{\mathsf{p}}(X) \times B$

  if $X \subseteq Y \times B$ then $\pi_1^{\mathsf{p}}(X) \subseteq Y$

Define: $\pi_1^{\mathsf{p}}(X) \overset{\text{def}}{=} (\{z \in A_{\mathit{el}} \mid \exists w \in B_{\mathit{el}} \quad (z,w) \in X_{\mathit{el}}\}, \Vdash_{\pi_1^{\mathsf{p}}(X)})$

where $n \Vdash_{\pi_1^{\mathsf{p}}(X)} z$ if for some $w \in B_{\mathit{el}}$ $n \Vdash_X (z,w)$

Suppose $X \subseteq A \times B$

$$\begin{tikzcd}
& (z,w) \in X & & (z,w) \in A \times B \\
n \arrow[ur, phantom, "\Vdash" sloped] \arrow[rrr, rightsquigarrow] & & & \langle\!\langle M'(n), M''(n) \rangle\!\rangle \arrow[u, phantom, "\Vdash" sloped]
\end{tikzcd}$$

if $X \subseteq Y \times B$ then $\pi_1^{\mathsf{p}}(X) \subseteq Y$

$$\begin{tikzcd}
& (z,w) \in X & & (z,w) \in Y \times B \\
m \arrow[ur, phantom, "\Vdash" sloped] \arrow[rr, rightsquigarrow] & & \langle\!\langle L(m), L'(m) \rangle\!\rangle \arrow[ur, phantom, "\Vdash" sloped] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& z \in \pi_1^{\mathsf{p}}(X) & & z \in Y \\
m \arrow[ur, phantom, "\Vdash" sloped] \arrow[rr, rightsquigarrow] & & L(m) \arrow[ur, phantom, "\Vdash" sloped] &
\end{tikzcd}$$

Elementary constructions

- Direct Image: $\pi_1^{\mathsf{d}}(X) \subseteq A$ $\quad$ $\pi_1^{\mathsf{d}}(X) \times B \subseteq X$

  for $Y \subseteq A$ $\quad$ if $Y \times B \subseteq X$ then $Y \subseteq \pi_1^{\mathsf{d}}(X)$

Define: $\pi_1^{\mathsf{d}}(X) \overset{\mathsf{def}}{=} (\{z \in A_{el} \mid \forall w \in B_{el} \quad (z, w) \in X_{el}\}, \Vdash_{\pi_1^{\mathsf{d}}(X)})$

where $n \Vdash_{\pi_1^{\mathsf{d}}(X)} z$ if $n_0 \Vdash_A z$ and for all $k \Vdash_B w$ $\quad \lceil n_1 \rceil(k) \Vdash_X (z, w)$

Elementary constructions

- Direct Image: $\pi_1^{\mathsf{d}}(X) \subseteq A$ $\qquad$ $\pi_1^{\mathsf{d}}(X) \times B \subseteq X$

  for $Y \subseteq A$ $\qquad$ if $Y \times B \subseteq X$ then $Y \subseteq \pi_1^{\mathsf{d}}(X)$

Define: $\pi_1^{\mathsf{d}}(X) \overset{\mathsf{def}}{=} (\{z \in A_{\mathit{el}} \mid \forall w \in B_{\mathit{el}} \quad (z, w) \in X_{\mathit{el}}\}, \Vdash_{\pi_1^{\mathsf{d}}(X)})$

where $\quad n \Vdash_{\pi_1^{\mathsf{d}}(X)} z \quad$ if $\quad n_0 \Vdash_A z \quad$ and $\quad$ for all $k \Vdash_B w \quad \lceil n_1 \rceil (k) \Vdash_X (z, w)$

$\pi_1^{\mathsf{d}}(X) \times B \subseteq X$

Elementary constructions

- Direct Image: $\pi_1^{\mathsf{d}}(X) \subseteq A$ $\quad \pi_1^{\mathsf{d}}(X) \times B \subseteq X$

  for $Y \subseteq A$ $\quad$ if $Y \times B \subseteq X$ then $Y \subseteq \pi_1^{\mathsf{d}}(X)$

Define: $\pi_1^{\mathsf{d}}(X) \overset{\mathsf{def}}{=} (\{z \in A_{\mathit{el}} \mid \forall w \in B_{\mathit{el}} \quad (z,w) \in X_{\mathit{el}}\}, \Vdash_{\pi_1^{\mathsf{d}}(X)})$

where $\quad n \Vdash_{\pi_1^{\mathsf{d}}(X)} z \quad$ if $\quad n_0 \Vdash_A z \quad$ and $\quad$ for all $k \Vdash_B w \quad \lceil n_1 \rceil(k) \Vdash_X (z,w)$

$\pi_1^{\mathsf{d}}(X) \times B \subseteq X$

$$\begin{tikzcd}
 & (z,w) \in \pi_1^{\mathsf{d}}(X) \times B & & (z,w) \in X \\
\{\!|\{\!|m, e|\!\}, \ell|\!\} \arrow[ur, phantom, "\Vdash"] \arrow[rr, squiggly] & & \lceil e \rceil(\ell) \arrow[ur, phantom, "\Vdash"] &
\end{tikzcd}$$

Elementary constructions

- Direct Image: $\pi_1^{\mathsf{d}}(X) \subseteq A$ $\pi_1^{\mathsf{d}}(X) \times B \subseteq X$

  for $Y \subseteq A$ if $Y \times B \subseteq X$ then $Y \subseteq \pi_1^{\mathsf{d}}(X)$

Define: $\pi_1^{\mathsf{d}}(X) \overset{\mathsf{def}}{=} (\{z \in A_{\mathit{el}} \mid \forall w \in B_{\mathit{el}} \quad (z,w) \in X_{\mathit{el}}\}, \Vdash_{\pi_1^{\mathsf{d}}(X)})$

where $n \Vdash_{\pi_1^{\mathsf{d}}(X)} z$ if $n_0 \Vdash_A z$ and for all $k \Vdash_B w$ $\lceil n_1 \rceil(k) \Vdash_X (z,w)$

if $Y \times B \subseteq X$ then $Y \subseteq \pi_1^{\mathsf{d}}(X)$

$$\begin{tikzcd} & (z,w) \in Y \times B & & (z,w) \in X \\ \langle\!\langle m, \ell \rangle\!\rangle \arrow[ur, phantom, "\Vdash"] \arrow[rr, squiggly] & & M(\langle\!\langle m, \ell \rangle\!\rangle) \arrow[ur, phantom, "\Vdash"] & \end{tikzcd}$$

Elementary constructions

- Direct Image: $\pi_1^{\mathsf{d}}(X) \subseteq A$ $\pi_1^{\mathsf{d}}(X) \times B \subseteq X$

  for $Y \subseteq A$ if $Y \times B \subseteq X$ then $Y \subseteq \pi_1^{\mathsf{d}}(X)$

Define: $\pi_1^{\mathsf{d}}(X) \overset{\text{def}}{=} (\{z \in A_{\mathit{el}} \mid \forall w \in B_{\mathit{el}} \quad (z,w) \in X_{\mathit{el}}\}, \Vdash_{\pi_1^{\mathsf{d}}(X)})$

where $n \Vdash_{\pi_1^{\mathsf{d}}(X)} z$ if $n_0 \Vdash_A z$ and for all $k \Vdash_B w$ $\lceil n_1 \rceil(k) \Vdash_X (z,w)$

if $Y \times B \subseteq X$ then $Y \subseteq \pi_1^{\mathsf{d}}(X)$

$$\begin{tikzcd}
& (z,w) \in Y \times B & & (z,w) \in X & & z \in Y & & z \in \pi_1^{\mathsf{d}}(X) \\
\langle m, \ell \rangle \arrow[ur, phantom, sloped, "\Vdash"] \arrow[rr, rightsquigarrow] & & M(\langle m, \ell \rangle) \arrow[ur, phantom, sloped, "\Vdash"] & & m \arrow[ur, phantom, sloped, "\Vdash"] \arrow[rr, rightsquigarrow] & & S(\lfloor M \rfloor, m) \arrow[ur, phantom, sloped, "\Vdash"] &
\end{tikzcd}$$

Examples

- $\mathbf{N}\times\mathbf{N}\subseteq\Delta_{\mathbf{N}}\cup((\mathbf{N}\times\mathbf{N})\setminus\Delta_{\mathbf{N}})$

$$\begin{tikzcd}
 & & & \text{any } z,w \\
 & (z,w)\in\mathbf{N}\times\mathbf{N} & & (z,w)\in\Delta_{\mathbf{N}}\cup((\mathbf{N}\times\mathbf{N})\setminus\Delta_{\mathbf{N}}) \\
\Lbag m,\ell\Rbag \arrow[ur, phantom, "\Vdash" sloped] \arrow[rr, rightsquigarrow] & & \Lbag m,\mathsf{sign}\,|m-\ell|\Rbag \arrow[ur, phantom, "\Vdash" sloped] & \\
m=z,\ell=w & & & \text{any } m,\ell
\end{tikzcd}$$

Examples

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$$\begin{tikzcd} & (z,w)\in\mathbf{N}\times\mathbf{N} & & (z,w)\in\Delta_{\mathbf{N}}\cup((\mathbf{N}\times\mathbf{N})\smallsetminus\Delta_{\mathbf{N}}) \\ \langle\!| m,\ell |\!\rangle \arrow[ur, phantom, "\Vdash"] \arrow[rr, rightsquigarrow] & & \langle\!| m, \mathsf{sign}\,|m-\ell| |\!\rangle \arrow[ur, phantom, "\Vdash"] & \end{tikzcd}$$

$m = z, \ell = w$

any m, ℓ

- $(\nabla\mathbb{N}\smallsetminus(\nabla\mathbb{N}\smallsetminus\mathbf{N})) = \nabla\mathbb{N}$

$$\begin{tikzcd} & z\in\nabla\mathbb{N} & & z\in(\nabla\mathbb{N}\smallsetminus(\nabla\mathbb{N}\smallsetminus\mathbf{N})) \\ n \arrow[ur, phantom, "\Vdash"] \arrow[rr, rightsquigarrow] & & n \arrow[ur, phantom, "\Vdash"] & \end{tikzcd}$$

any z

any n

Recap on elementary constructions

$n \Vdash x \in \mathbf{N}$	$n = x \in \mathbb{N}$
$n \Vdash x \in \nabla S$	$z \in S$
$n \Vdash x \in \nabla \emptyset$	$\lightning$
$n \Vdash x \in \mathbf{1}$	$n = 0 = x.$
$n \Vdash x \in \Sigma$	either $\lceil n \rceil (n) \downarrow$ and $x = 1$ or $\lceil n \rceil (n) \uparrow$ and $x = 0$
$\langle m, \ell \rangle \Vdash (z, w) \in A \times B$	$m \Vdash z \in A$ and $\ell \Vdash w \in B$
$n \Vdash (z, w) \in \Delta_X$	$z = w$ and $n \Vdash z \in X$
$\langle m, \ell \rangle \Vdash z \in X \cap Y$	$m \Vdash z \in X$ and $\ell \Vdash z \in Y$
$\langle n, i \rangle \Vdash z \in X \cup Y$	either $i = 0$ and $n \Vdash z \in X$ or $i = 1$ and $n \Vdash z \in Y$
$\langle m, e \rangle \Vdash z \in (X \Longrightarrow_A Y)$	$m \Vdash z \in A$ and $\lceil e \rceil (k) \Vdash z \in Y$ for all $k \Vdash z \in X$
$\langle m, e \rangle \Vdash z \in A \setminus X$	$m \Vdash z \in A$ and no $k \Vdash z \in X$
$n \Vdash z \in \pi_1^{\mathsf{p}}(X)$	$n \Vdash (z, w) \in X$ for some $k \Vdash w \in B$
$\langle m, e \rangle \Vdash z \in \pi_1^{\mathsf{d}}(X)$	$m \Vdash z \in A$ and $\lceil n \rceil (k) \Vdash (z, w) \in X$ for all $k \Vdash w \in B$

Functions

$f\colon A \to B$ must be $f \subseteq A \times B$ such that

single-valued $\forall a \in A \forall b \in B \forall b' \in B \quad [(a,b) \in f \wedge (a,b') \in f] \Rightarrow b = b'$

total $\forall a \in A \exists b \in B \quad (a,b) \in f$

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$[\pi^*_{12}(f) \cap \pi^*_{13}(f)] \subseteq \pi^*_{23}\Delta_B$

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$$A \subseteq \pi_1^{\mathsf{p}}(f)$$

$f \subseteq A \times B$:

$$\begin{tikzcd}
 & (a,b) \in f & & (a,b) \in A \times B \\
n \arrow[ur, phantom, "\Vdash"] \arrow[rr, rightsquigarrow] & & \langle\!| M'(n), M''(n) |\!\rangle \arrow[ur, phantom, "\Vdash"] &
\end{tikzcd}$$

any a, b

any n

Functions

$f\colon A\longrightarrow B$ must be $f\subseteq A\times B$ such that

single-valued $\forall a\in A\forall b\in B\forall b'\in B\quad [(a,b)\in f\wedge(a,b')\in f]\Rightarrow b=b'$

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\end{tikzcd}$$

any a, b

any n

$A\subseteq\pi^{\mathsf{p}}_1(f)$:

$$\begin{tikzcd}
 & a\in A & (a,b)\in f \\
m \arrow[ur, phantom, sloped, "\Vdash"] \arrow[r, rightsquigarrow] & M(m) \arrow[ur, phantom, sloped, "\Vdash"] &
\end{tikzcd}$$

any a, **some** b

any m

Functions

$f\colon A \to B$ must be $f \subseteq A \times B$ such that

single-valued $\forall a \in A \forall b \in B \forall b' \in B \quad [(a,b) \in f \wedge (a,b') \in f] \Rightarrow b = b'$

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$$\begin{tikzcd}
 & (a,b) \in f & & (a,b) \in A \times B \\
n \arrow[ur, phantom, "\Vdash"] \arrow[rr, squiggly] & & \{\!|M'(n), M''(n)|\!\} \arrow[ur, phantom, "\Vdash"] &
\end{tikzcd}$$

any a, b

any n

$A \subseteq \pi^{\mathsf{p}}_1(f)$:

$$\begin{tikzcd}
 & a \in A & & (a,b) \in f \\
m \arrow[ur, phantom, "\Vdash"] \arrow[rr, squiggly] & & M(m) \arrow[ur, phantom, "\Vdash"] &
\end{tikzcd}$$

any a, **some** b

any m

$[\pi^*_{12}(f) \cap \pi^*_{13}(f)] \subseteq \pi^*_{13}(\Delta_B)$:

$$\begin{tikzcd}
 & (a,b) \in f, (a,b') \in f & & (b,b') \in \Delta_B \\
\{\!|n', n''|\!\} \arrow[ur, phantom, "\Vdash"] \arrow[rr, squiggly] & & L(\{\!|n', n''|\!\}) \arrow[ur, phantom, "\Vdash"] &
\end{tikzcd}$$

any a, b, b'

any n', n''

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$f\colon A \longrightarrow B$ must be $f \subseteq A \times B$ such that

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 & (a,b) \in f & & (a,b) \in A \times B & \textbf{any } a,b \\
n \arrow[ur, phantom, "\Vdash"] \arrow[rr, rightsquigarrow] & & \llparenthesis M'(n), M''(n) \rrparenthesis \arrow[ur, phantom, "\Vdash"] & & \textbf{any } n
\end{tikzcd}$$

$A \subseteq \pi^{\mathsf{p}}_1(f)$:

$$\begin{tikzcd}
 & a \in A & & (a,b) \in f & \textbf{any } a, \textbf{ some } b \\
m \arrow[ur, phantom, "\Vdash"] \arrow[rr, rightsquigarrow] & & M(m) \arrow[ur, phantom, "\Vdash"] & & \textbf{any } m
\end{tikzcd}$$

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$$\begin{tikzcd}
 & (a,b) \in f, (a,b') \in f & & (b,b') \in \Delta_B & \textbf{any } a,b,b' \\
\llparenthesis n', n'' \rrparenthesis \arrow[ur, phantom, "\Vdash"] \arrow[rr, rightsquigarrow] & & L(\llparenthesis n', n'' \rrparenthesis) \arrow[ur, phantom, "\Vdash"] & & \textbf{any } n', n'' \\
 & & & b = b' &
\end{tikzcd}$$

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any a, b

any n

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\end{tikzcd}$$

any a, **some** b unique

any m

$[\pi^*_{12}(f) \cap \pi^*_{13}(f)] \subseteq \pi^*_{13}(\Delta_B)$:

$$\begin{tikzcd}
 & (a,b) \in f, (a,b') \in f & & (b,b') \in \Delta_B \\
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\end{tikzcd}$$

any a, b, b'

any n', n''

$b = b'$

Functions

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\end{tikzcd}$$

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\end{tikzcd}$$

any a, **some** b unique
say $b = f_1(a)$
any m

$[\pi^*_{12}(f) \cap \pi^*_{13}(f)] \subseteq \pi^*_{13}(\Delta_B)$:

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 & (a,b) \in f, (a,b') \in f & & (b,b') \in \Delta_B \\
\lbag n', n'' \rbag \arrow[ur, phantom, "\Vdash"] \arrow[rr, squiggly] & & L(\lbag n', n'' \rbag) \arrow[ur, phantom, "\Vdash"] &
\end{tikzcd}$$

any a, b, b'

any n', n''

$b = b'$

Functions

$f\colon A \longrightarrow B$ must be $f \subseteq A \times B$ such that $A \subseteq \pi_1^{\mathsf{p}}(f)$ and $[\pi_{12}^*(f) \cap \pi_{13}^*(f)] \subseteq \pi_{13}^*(\Delta_B)$

determines $f_1\colon A_{e\ell} \longrightarrow B_{e\ell}$ uniquely

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determines $f_1\colon A_{\mathit{el}} \longrightarrow B_{\mathit{el}}$ uniquely

There is also a computable function H which
for every $a \in A_{\mathit{el}}$ and $n \in \mathbb{N}$, if $n \Vdash a \in A$ then $H(n) \Vdash f_1(a) \in B$

$$\begin{tikzcd} \textbf{if} & & a \in A & \textbf{then} & & f_1(a) \in B & \textbf{any } a \\ & n \arrow[ur, phantom, "\Vdash"] \arrow[rrr, squiggly] & & & H(n) \arrow[ur, phantom, "\Vdash"] & & \textbf{any } n \end{tikzcd}$$

Functions

$f\colon A \longrightarrow B$ must be $f \subseteq A \times B$ such that $A \subseteq \pi_1^{\mathsf{p}}(f)$ and $[\pi_{12}^*(f) \cap \pi_{13}^*(f)] \subseteq \pi_{13}^*(\Delta_B)$

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There is also a computable function H which
for every $a \in A_{e\ell}$ and $n \in \mathbb{N}$, if $n \Vdash a \in A$ then $H(n) \Vdash f_1(a) \in B$

if **then**

$$\begin{tikzcd} & a \in A & & & f_1(a) \in B \\ n \arrow[ur, phantom, "\Vdash" sloped] \arrow[rrr, squiggly] & & & H(n) \arrow[ur, phantom, "\Vdash" sloped] & \end{tikzcd}$$

any a

any n

Moreover, define: $[f_1] \overset{\text{def}}{=} (\{(a, f_1(a)) \mid a \in A_{e\ell}, \Vdash_{[f_1]})$
where $n \Vdash_{[f_1]} (a, f_1(a))$ if $n \Vdash_A a$.

Then $f = [f_1]$

Examples

- $[f_1]\colon A \to \nabla S$ **if** $$\begin{tikzcd} & a \in A & & f_1(a) \in \nabla S \\ n \arrow[ur, phantom, "\Vdash"] \arrow[rrr, squiggly] & & {} & {} \arrow[u, phantom, "\Vdash"] \end{tikzcd}$$ **then**

  any a

  any n

To give a function from an assembly A to ∇S is to give a set-function from $A_{e\ell}$ to S.

Examples

- $[f_1]\colon A \to \nabla S$

$$\begin{tikzcd} & & a \in A & & f_1(a) \in \nabla S & \textbf{any } a \\ \textbf{if} & n \arrow[ur, phantom, "\Vdash"] \arrow[rrr, rightsquigarrow] & & \textbf{then} & 0 \arrow[u, phantom, "\Vdash"] & \textbf{any } n \end{tikzcd}$$

To give a function from an assembly A to ∇S is to give a set-function from A_{el} to S.

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- $[f_1]\colon A \to \nabla S$

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To give a function from an assembly A to ∇S is to give a set-function from A_{el} to S.

- $[f_1]\colon \mathbf{N} \to \mathbf{N}$

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To give a function from an assembly A to ∇S is to give a set-function from A_{el} to S.

- $[f_1]\colon \mathbf{N} \to \mathbf{N}$ for some computable function H

$$\begin{tikzcd} \textbf{if} & a \in \mathbf{N} & \textbf{then} & f_1(a) \in \mathbf{N} & \textbf{any } a \\ n \arrow[ur, phantom, "\Vdash"] \arrow[rr, rightsquigarrow] & & H(n) \arrow[ur, phantom, "\Vdash"] & & \textbf{any } n \end{tikzcd}$$

Examples

- $[f_1]\colon A \to \nabla S$

$$\begin{tikzcd}
 & & a \in A & & & f_1(a) \in \nabla S & \textbf{any } a \\
\textbf{if} & & & \textbf{then} & & & \\
 & n \arrow[ru, phantom, "\Vdash"] \arrow[rrr, squiggly] & & & 0 \arrow[ru, phantom, "\Vdash"] & & \textbf{any } n
\end{tikzcd}$$

To give a function from an assembly A to ∇S is to give a set-function from A_{el} to S.

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 & & a \in \mathbf{N} & & & f_1(a) \in \mathbf{N} & \textbf{any } a \\
\textbf{if} & & & \textbf{then} & & & \\
 & n \arrow[ru, phantom, "\Vdash"] \arrow[rrr, squiggly] & & & H(n) \arrow[ru, phantom, "\Vdash"] & & \textbf{any } n
\end{tikzcd}$$

Examples

- $[f_1]\colon A \to \nabla S$

$$\begin{tikzcd}[column sep=small]
\text{\textbf{if}} & & a \in A & \text{\textbf{then}} & & f_1(a) \in \nabla S & \text{\textbf{any} } a \\
& n \arrow[ur, phantom, "\Vdash" sloped] \arrow[rrr, squiggly] & & & 0 \arrow[ur, phantom, "\Vdash" sloped] & & \text{\textbf{any} } n
\end{tikzcd}$$

To give a function from an assembly A to ∇S is to give a set-function from $A_{e\ell}$ to S.

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\text{\textbf{if}} & & a \in \mathbf{N} & \text{\textbf{then}} & & f_1(a) \in \mathbf{N} & \text{\textbf{any} } a \\
& n \arrow[ur, phantom, "\Vdash" sloped] \arrow[rrr, squiggly] & & & H(n) \arrow[ur, phantom, "\Vdash" sloped] & & \text{\textbf{any} } n
\end{tikzcd}$$

Examples

- $[f_1]\colon A \to \nabla S$

$$\begin{tikzcd}
\textbf{if} & a \in A \arrow[dl, phantom, "\Vdash" sloped] & \textbf{then} & & f_1(a) \in \nabla S \arrow[dl, phantom, "\Vdash" sloped] & \textbf{any } a \\
n \arrow[rrr, rightsquigarrow] & & & 0 & & \textbf{any } n
\end{tikzcd}$$

To give a function from an assembly A to ∇S is to give a set-function from A_{el} to S.

- $[f_1]\colon \mathbf{N} \to \mathbf{N}$ for some computable function H

$$\begin{tikzcd}
\textbf{if} & a \in \mathbf{N} \arrow[dl, phantom, "\Vdash" sloped] & \textbf{then} & & f_1(a) \in \mathbf{N} \arrow[dl, phantom, "\Vdash" sloped] & \textbf{any } a \\
n \arrow[rrr, rightsquigarrow] & & & H(n) & & \textbf{any } n
\end{tikzcd}$$

To give a endofunction on $\mathbf{N}$ is to give a computable set-function on $\mathbb{N}$.

Examples

- $[f_1]\colon A \to \nabla S$

if $\quad$ **then**

$$\begin{tikzcd} & a \in A & & f_1(a) \in \nabla S \\ n \arrow[ur, phantom, "\Vdash" sloped] \arrow[rr, squiggly] & & 0 \arrow[ur, phantom, "\Vdash" sloped] & \end{tikzcd}$$

any a

any n

To give a function from an assembly A to ∇S is to give a set-function from A_{el} to S.

- $[f_1]\colon \mathbf{N} \to \mathbf{N}$ $\quad$ for some computable function H

if $\quad$ **then**

$$\begin{tikzcd} & a \in \mathbf{N} & & f_1(a) \in \mathbf{N} \\ n \arrow[ur, phantom, "\Vdash" sloped] \arrow[rr, squiggly] & & H(n) \arrow[ur, phantom, "\Vdash" sloped] & \end{tikzcd}$$

any a

any n

To give a endofunction on $\mathbf{N}$ is to give a computable set-function on $\mathbb{N}$.

- $[f_1]\colon \nabla S \to \mathbf{N}$

Examples

- $[f_1]\colon A \to \nabla S$

$$\begin{tikzcd} \textbf{if} & a \in A & \textbf{then} & f_1(a) \in \nabla S & \textbf{any } a \\ & n \arrow[u, phantom, "\Vdash"] \arrow[rr, squiggly] & & 0 \arrow[u, phantom, "\Vdash"] & \textbf{any } n \end{tikzcd}$$

To give a function from an assembly A to ∇S is to give a set-function from A_{el} to S.

- $[f_1]\colon \mathbf{N} \to \mathbf{N}$ for some computable function H

$$\begin{tikzcd} \textbf{if} & a \in \mathbf{N} & \textbf{then} & f_1(a) \in \mathbf{N} & \textbf{any } a \\ & n \arrow[u, phantom, "\Vdash"] \arrow[rr, squiggly] & & H(n) \arrow[u, phantom, "\Vdash"] & \textbf{any } n \end{tikzcd}$$

To give a endofunction on $\mathbf{N}$ is to give a computable set-function on $\mathbb{N}$.

- $[f_1]\colon \nabla S \to \mathbf{N}$ for some computable function H s

$$\begin{tikzcd} \textbf{if} & a \in \nabla S & \textbf{then} & f_1(a) \in \mathbf{N} & \textbf{any } a \\ & {} \arrow[u, phantom, "\Vdash"] \arrow[rr, squiggly] & & {} \arrow[u, phantom, "\Vdash"] & \textbf{any } n \end{tikzcd}$$

Examples

- $[f_1]\colon A \to \nabla S$

$$\begin{tikzcd}[column sep=small]
 & & a \in A & & & f_1(a) \in \nabla S & \textbf{any } a \\
\textbf{if} & & & \textbf{then} & & & \\
 & n \arrow[uur, phantom, "\Vdash"] \arrow[rrr, squiggly] & & & 0 \arrow[uur, phantom, "\Vdash"] & & \textbf{any } n
\end{tikzcd}$$

To give a function from an assembly A to ∇S is to give a set-function from A_{el} to S.

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$$\begin{tikzcd}[column sep=small]
 & & a \in \mathbf{N} & & & f_1(a) \in \mathbf{N} & \textbf{any } a \\
\textbf{if} & & & \textbf{then} & & & \\
 & n \arrow[uur, phantom, "\Vdash"] \arrow[rrr, squiggly] & & & H(n) \arrow[uur, phantom, "\Vdash"] & & \textbf{any } n
\end{tikzcd}$$

To give a endofunction on $\mathbf{N}$ is to give a computable set-function on $\mathbb{N}$.

- $[f_1]\colon \nabla S \to \mathbf{N}$ for some computable function H s

$$\begin{tikzcd}[column sep=small]
 & & a \in \nabla S & & & f_1(a) \in \mathbf{N} & \textbf{any } a \\
\textbf{if} & & & \textbf{then} & & & \\
 & 0 \arrow[uur, phantom, "\Vdash"] \arrow[rrr, squiggly] & & & {} \arrow[uur, phantom, "\Vdash"] & &
\end{tikzcd}$$

Examples

- $[f_1]\colon A \to \nabla S$

$$\begin{tikzcd}
\textbf{if} & & a \in A & \textbf{then} & f_1(a) \in \nabla S & \textbf{any } a \\
& n \arrow[ur, phantom, "\Vdash" sloped] \arrow[rr, squiggly] & & 0 \arrow[ur, phantom, "\Vdash" sloped] & & \textbf{any } n
\end{tikzcd}$$

To give a function from an assembly A to ∇S is to give a set-function from $A_{e\ell}$ to S.

- $[f_1]\colon \mathbf{N} \to \mathbf{N}$ for some computable function H

$$\begin{tikzcd}
\textbf{if} & & a \in \mathbf{N} & \textbf{then} & f_1(a) \in \mathbf{N} & \textbf{any } a \\
& n \arrow[ur, phantom, "\Vdash" sloped] \arrow[rr, squiggly] & & H(n) \arrow[ur, phantom, "\Vdash" sloped] & & \textbf{any } n
\end{tikzcd}$$

To give a endofunction on $\mathbf{N}$ is to give a computable set-function on $\mathbb{N}$.

- $[f_1]\colon \nabla S \to \mathbf{N}$ for some computable function H s

$$\begin{tikzcd}
\textbf{if} & & a \in \nabla S & \textbf{then} & f_1(a) \in \mathbf{N} & \textbf{any } a \\
& 0 \arrow[ur, phantom, "\Vdash" sloped] \arrow[rr, squiggly] & & H(0) \arrow[ur, phantom, "\Vdash" sloped] & &
\end{tikzcd}$$

Examples

- $[f_1]\colon A \to \nabla S$

$$\begin{tikzcd} & a \in A & & f_1(a) \in \nabla S & \textbf{any } a \\ \textbf{if} & & \textbf{then} & & \\ & n \arrow[uu, phantom, sloped, "\Vdash"] \arrow[rr, squiggly] & & 0 \arrow[uu, phantom, sloped, "\Vdash"] & \textbf{any } n \end{tikzcd}$$

To give a function from an assembly A to ∇S is to give a set-function from A_{el} to S.

- $[f_1]\colon \mathbf{N} \to \mathbf{N}$ for some computable function H

$$\begin{tikzcd} & a \in \mathbf{N} & & f_1(a) \in \mathbf{N} & \textbf{any } a \\ \textbf{if} & & \textbf{then} & & \\ & n \arrow[uu, phantom, sloped, "="] \arrow[rr, squiggly] & & H(n) \arrow[uu, phantom, sloped, "="] & \textbf{any } n \end{tikzcd}$$

To give a endofunction on $\mathbf{N}$ is to give a computable set-function on $\mathbb{N}$.

- $[f_1]\colon \nabla S \to \mathbf{N}$ for some computable function H s

$$\begin{tikzcd} & a \in \nabla S & & f_1(a) \in \mathbf{N} & \textbf{any } a \\ \textbf{if} & & \textbf{then} & & \\ & 0 \arrow[uu, phantom, sloped, "\Vdash"] \arrow[rr, squiggly] & & H(0) \arrow[uu, phantom, sloped, "="] & \end{tikzcd}$$

Examples

- $[f_1]\colon A \to \nabla S$

$$\begin{tikzcd}
\textbf{if} & a \in A \arrow[d, dash, "\Vdash"] & \textbf{then} & f_1(a) \in \nabla S & \textbf{any } a \\
& n \arrow[rr, rightsquigarrow] & & 0 \arrow[u, dash, "\Vdash"] & \textbf{any } n
\end{tikzcd}$$

To give a function from an assembly A to ∇S is to give a set-function from $A_{e\ell}$ to S.

- $[f_1]\colon \mathbf{N} \to \mathbf{N}$ for some computable function H

$$\begin{tikzcd}
\textbf{if} & a \in \mathbf{N} \arrow[d, dash, "\Vdash"] & \textbf{then} & f_1(a) \in \mathbf{N} & \textbf{any } a \\
& n \arrow[rr, rightsquigarrow] & & H(n) \arrow[u, dash, "\Vdash"] & \textbf{any } n
\end{tikzcd}$$

To give a endofunction on $\mathbf{N}$ is to give a computable set-function on $\mathbb{N}$.

- $[f_1]\colon \nabla S \to \mathbf{N}$ for some computable function H s

$$\begin{tikzcd}
\textbf{if} & a \in \nabla S \arrow[d, dash, "\Vdash"] & \textbf{then} & f_1(a) \in \mathbf{N} & \textbf{any } a \\
& 0 \arrow[rr, rightsquigarrow] & & H(0) \arrow[u, dash, "\Vdash"] &
\end{tikzcd}$$

To give a function from an assembly X to ∇S is to give a number in $\mathbf{N}$.

Some category structure on assemblies

- natural number object: $\mathbf{N}$
- finite products: $A \times B$ $\quad$ $\mathbf{1} \overset{\text{def}}{=} \nabla\{0\}$
- coproducts: $A + B \overset{\text{def}}{=} (A \times \nabla\{0\}) \cup (B \times \nabla\{1\})$
- equalizers: for $[f_1], [g_1] \colon A \longrightarrow B$ an equalizer is $A \cap \nabla\{a \in A_{el} \mid f_1(a) = g_1(a)\} \subseteq A$
- exponentials: $B^A \overset{\text{def}}{=} (\{f_1 \mid [f_1] \colon A \longrightarrow B\}, \Vdash_{B^A})$

  where $e \Vdash_{B^A} f_1$ means that

$$\begin{tikzcd} \textbf{if} & a \in A & \textbf{then} & f_1(a) \in B & \textbf{any } a \\ & n \arrow[u, phantom, sloped, "\Vdash"] \arrow[rr, squiggly] & & \lceil e \rceil(n) \arrow[u, phantom, sloped, "\Vdash"] & \textbf{any } n \end{tikzcd}$$

- factorization:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, two heads] & & B \\ & \pi_2^{\mathsf{p}}(f) \arrow[ur, tail] & \end{tikzcd}$$

Some assemblies of truth values

$$\begin{tikzcd}
\Sigma \arrow[r, phantom, "\subsetneq"] \arrow[dr, bend left] & \nabla\{0,1\} \arrow[loop right, "\mathsf{neg}"] \\
\mathbf{1}+\mathbf{1} \arrow[u, phantom, sloped, "\subsetneq"] \arrow[r, phantom, "\subsetneq"] \arrow[loop left] & \Pi \arrow[u, phantom, sloped, "\subsetneq"] \arrow[ul, bend left]
\end{tikzcd}$$

$n \Vdash x \in \Sigma$ if either $\lceil n \rceil(n)\downarrow$ and $x = 1$
or $\lceil n \rceil(n)\uparrow$ and $x = 0$

$n \Vdash x \in \Pi$ if either $\lceil n \rceil(n)\uparrow$ and $x = 1$
or $\lceil n \rceil(n)\downarrow$ and $x = 0$

neg swaps 0 and 1.

- $f\colon A \to \nabla\{0,1\}$ is completely determined by the subset ${f_1}^{-1}\{1\} \subseteq A_{el}$

- for $f\colon A \to \Sigma$ the subset
  $\{n \mid n \Vdash_A x \text{ for some } x \in {f_1}^{-1}\{1\}\} \subseteq \{n \mid n \Vdash_A x \text{ for some } x \in A_{el}\}$ is relatively r.e.
  i.e. it is of the form $\{n \mid n \Vdash_A x \text{ for some } x \in A_{el}\} \cap W$ for some $W \subseteq_{\text{r.e.}} \mathbb{N}$

- for $f\colon A \to \mathbf{1}+\mathbf{1}$ the subset
  $\{n \mid n \Vdash_A x \text{ for some } x \in {f_1}^{-1}\{1\}\} \subseteq \{n \mid n \Vdash_A x \text{ for some } x \in A_{el}\}$ is relatively recursive
  i.e. both it and its complement are relatively r.e.

$$\Sigma = \{p \mid \exists f \in \mathbf{N}^{\mathbf{N}} \quad p \Leftrightarrow \exists n \in \mathbf{N} \; f(n) = 0\}$$

Take the relatively r.e. subset

$$\{e \mid \lceil e \rceil(n) = 0 \text{ for some } n \in \mathbb{N}\} \cap (\mathbf{N}^{\mathbf{N}})_{\mathit{el}} \subseteq (\mathbf{N}^{\mathbf{N}})_{\mathit{el}}$$

and let $s \colon \mathbf{N}^{\mathbf{N}} \to \Sigma$ be the induced function.

- $\forall p \in \Sigma \exists f \in \mathbf{N}^{\mathbf{N}} \quad (f, p) \in s$

  $\Sigma \subseteq \pi_2^{\mathsf{p}}(s)$:

$$\begin{tikzcd} & p \in \Sigma & & (f,p) \in s \\ n \arrow[ur, phantom, "\Vdash"] \arrow[rr, squiggly] & & C(n) \arrow[ur, phantom, "\Vdash"] & \end{tikzcd}$$

any p, **some** f

take $f \colon k \mapsto \lceil C(n) \rceil(k)$

any n

$\lceil C(n) \rceil(k)$ computes k steps of the computation $\lceil n \rceil(n)$ and returns 0 if it stops, 1 otherwise.

- $\forall f, g \in \mathbf{N}^{\mathbf{N}} \quad s(f) = s(g) \Leftrightarrow [\exists n \; f(n) = 0 \Leftrightarrow \exists m \; g(m) = 0]$

  $\{(f, g) \in \mathbf{N}^{\mathbf{N}} \times \mathbf{N}^{\mathbf{N}} \mid s(f) = s(g)\} = \{(f, g) \in \mathbf{N}^{\mathbf{N}} \times \mathbf{N}^{\mathbf{N}} \mid \exists n \; f(n) = 0 \Leftrightarrow \exists m \; g(m) = 0\}$

There is an enumeration $\mathcal{W}\colon \mathbf{N} \twoheadrightarrow \Sigma^{\mathbf{N}}$

- $\Sigma^{\mathbf{N}}$ is the assembly $(\Sigma^{\mathbf{N}})_{el}$ consists of the r.e. subsets of $\mathbb{N}$

  $e \Vdash V \in \Sigma^{\mathbf{N}}$ iff $V = \{n \in \mathbb{N} \mid \lceil e \rceil (n)\downarrow\}$

$\mathcal{W}_1\colon \mathbb{N} \to \mathrm{RE}\colon e \mapsto \{n \in \mathbb{N} \mid \lceil e \rceil (n)\downarrow\}$

$$\begin{tikzcd}
 & e \in \mathbf{N} & & \mathcal{W}_1(e) \in \Sigma^{\mathbf{N}} \\
n \arrow[ur, phantom, "\Vdash"] \arrow[rr, rightsquigarrow] & & n \arrow[ur, phantom, "\Vdash"] &
\end{tikzcd}$$

any e

any n

$$\begin{tikzcd}
 & W \in \Sigma^{\mathbf{N}} & & n \in \mathbf{N} \\
e \arrow[ur, phantom, "\Vdash"] \arrow[rr, rightsquigarrow] & & e \arrow[ur, phantom, "\Vdash"] &
\end{tikzcd}$$

any W, **some** n with $\mathcal{W}_1(n) = W$

take $n = e$

any e

Markov's Rule

Suppose $X \subseteq \mathbf{N} \times A$ is such that $\forall a \in A \forall n \in \mathbf{N}\ [(n, a) \in X \vee (n, a) \notin X]$.

$$\mathbf{N} \times A = X \cup [(\mathbf{N} \times A) \smallsetminus X]$$

If $A \smallsetminus \pi_1^{\mathsf{p}}(X) = \nabla\emptyset$ then $A \subseteq \pi_1^{\mathsf{p}}(X)$. If $A \smallsetminus \pi_1^{\mathsf{p}}(X) = \nabla\emptyset$ then $A \subseteq \pi_1^{\mathsf{p}}(X)$.

Since $\mathbf{N} \times A = X \cup [(\mathbf{N} \times A) \smallsetminus X]$, the relation $(X \times \nabla\{0\}) \cup [((\mathbf{N} \times A) \smallsetminus X) \times \nabla\{1\}] \subseteq (\mathbf{N} \times A) \times (1 + 1)$ is total and single-valued.

Hence there are two r.e.

Markov's Rule

Let A and $P \subseteq \mathbf{N} \times A$ be assemblies such that $\forall z \in \mathbf{N} \forall x \in A \, ((z,x) \in P \vee (z,x) \notin P)$

i.e. $\mathbf{N} \times A = P \cup [(\mathbf{N} \times A) \smallsetminus P]$

Then $\forall x \in A \, (\neg\neg\exists z \in \mathbf{N} \, (z,x) \in P \Rightarrow \exists z \, (z,x) \in P)$

i.e. $(A \smallsetminus \pi_1^{\mathsf{p}}(P)) \smallsetminus \pi_1^{\mathsf{p}}(P) = \pi_1^{\mathsf{p}}(P)$

Suppose

$$\begin{tikzcd}
 & (n,a) \in \mathbf{N} \times A & & (n,a) \in P \cup [(\mathbf{N} \times A) \smallsetminus P] \\
\llangle n, m \rrangle \arrow[ur, phantom, "\Vdash"] \arrow[rr, squiggly] & & \llangle M'(\llangle n, m \rrangle), M''(\llangle n, m \rrangle) \rrangle \arrow[ur, phantom, "\Vdash"] &
\end{tikzcd}$$

Note that, for any n, m, the value $M''(\llangle n, m \rrangle)$ is either 0 or 1.

Consider

$$\begin{tikzcd}
 & a \in (A \smallsetminus \pi_1^{\mathsf{p}}(P)) \smallsetminus \pi_1^{\mathsf{p}}(P) & & a \in \pi_1^{\mathsf{p}}(P) \\
m \arrow[ur, phantom, "\Vdash"] \arrow[rr, squiggly] & & \llangle M'(\llangle \min n. \, [M''(\llangle n, m \rrangle) = 0], m \rrangle) \rrangle \arrow[ur, phantom, "\Vdash"] &
\end{tikzcd}$$

Caucuses

An assembly A is a *caucus* if the relation $a \;{}_A\!\dashv\!\Vert\; n$ is single-valued.

Recall that, by definition, $\Vdash_A$ is surjective. Hence, on a caucus, ${}_A\!\dashv\!\Vert$ is a function $r\colon A_{el} \longrightarrow \mathbb{N}$.

The rule of choice holds for every caucus. [Otherwise said: every caucus is projective.]

Take any assembly B and any $R \subseteq A \times B$ such that $\forall a \in A \exists b \in B\ (a, b) \in R$ *i.e.* $A \subseteq \pi_1^{\mathsf{p}}(R)$

$$\begin{tikzcd} & a \in A & & (a,b) \in R \\ m \arrow[ur, phantom, "\Vdash"] \arrow[rr, squiggly] & & M(m) \arrow[ur, phantom, "\Vdash"] & \end{tikzcd}$$

any $a, z, w,$
some b
any m, n

$$\begin{tikzcd} & (z,w) \in R & & (z,w) \in A \times B \\ n \arrow[ur, phantom, "\Vdash"] \arrow[rr, squiggly] & & \llparenthesis M'(n), M''(n) \rrparenthesis \arrow[ur, phantom, "\Vdash"] & \end{tikzcd}$$

By the standard Axiom of Choice, get $g\colon A_{el} \times \mathbb{N} \longrightarrow B_{el}$ such that

$$\begin{tikzcd} & a \in A & & (a, g(a,m)) \in R \\ m \arrow[ur, phantom, "\Vdash"] \arrow[rr, squiggly] & & M(m) \arrow[ur, phantom, "\Vdash"] & \end{tikzcd}$$

any a

any m

Define $f_1\colon A_{el} \longrightarrow B_{el}\colon a \mapsto g(a, r(a))$ and check that $\forall a \in A\ (a, [f_1](a)) \in R$ *i.e.* $[f_1] \subseteq R$

$$\begin{tikzcd} & a \in A & & (a, g(a, r(a))) \in R \\ m \arrow[ur, phantom, "\Vdash"] \arrow[rr, squiggly] & & M''(M(m)) \arrow[ur, phantom, "\Vdash"] & \end{tikzcd}$$

$$m = r(a)$$

any a

any m

Projective Assemblies/Caucuses

- $\mathbf{N}$ is projective
- Every sub-assembly of $\mathbf{N}$ of the form $\mathbf{N} \cap \nabla Q$ for $Q \subseteq \mathbb{N}$ is projective
- $\mathbf{1}$ is projective
- Every assembly A where a single number k may realize, *i.e.* if $n \Vdash a \in A$ then $n = k$.
- Every assembly of the form ∇S is projective.

Projective Covers

Given any assembly A, consider the assembly

$$P_A \overset{\text{def}}{=} (A_{el} \times \mathbb{N}, \Vdash_{P_x})$$

where $\quad n \Vdash_{P_A} (a, m) \quad$ if $\quad n = m \Vdash_A a$

P_A is projective.

Moreover, the first projection

$$\begin{tikzcd} p_1 \colon P_{A\,el} \arrow[rr] \arrow[dr, phantom, "\parallel" sloped] & & A_{el} \\ & A_{el} \times \mathbb{N} & \end{tikzcd}$$

defines a surjection $[p_1] \colon P_A \twoheadrightarrow A$

since

$$\begin{tikzcd} & (a, m) \in P_A & & a \in A \\ n \arrow[rrr, rightsquigarrow] \arrow[ur, phantom, "\parallel" sloped] & & & n \arrow[u, phantom, "\Vdash" sloped] \end{tikzcd}$$

$n = m \Vdash_A a$

any a, m

any n

$$\begin{tikzcd} & a \in A & & (a, m) \in P_A \\ n \arrow[rr, rightsquigarrow] \arrow[ur, phantom, "\Vdash" sloped] & & n \arrow[ur, phantom, "\parallel" sloped] & \end{tikzcd}$$

any a, **some** m with $p_1((a, m)) = a$
take $m = n$
any n

Discrete/modest assemblies

Call an assembly D *discrete* (or *modest*) if every function $\nabla\{0,1\} \longrightarrow D$ is constant.

- $\mathbf{N}$ is discrete

- a sub-assembly of $\mathbf{N}$ is discrete

- a surjective image of a discrete assembly is discrete

- a discrete assembly D is a surjective image of a sub-assembly of $\mathbf{N}$,
  i.e. is totally determined by a partial equivalence relation on $\mathbb{N}$.

Exercise: D is discrete if and only if the map const: $D \longrightarrow D^{\nabla\{0,1\}}$ is a bijection.

Characterization of discrete/modest assemblies

General properties of the projective cover P_A:

- the second projection $$\begin{tikzcd} p_2\colon P_{A_{\mathit{el}}} \arrow[rr] \arrow[dr, equal] & & \mathbb{N} \\ & A_{\mathit{el}} \times \mathbb{N} & \end{tikzcd}$$ defines a function $[p_2]\colon P_A \to \mathbf{N}$

- the pairing $([p_1], [p_2])\colon P_A \longrightarrow A \times \mathbf{N}$ is one-one

- the parallel maps

$$\begin{tikzcd}
 & {P_A}^{\mathbf{1+1}} \arrow[dr, "\pi_1"] & \\
 {P_A}^{\nabla\{0,1\}} \arrow[ur] \arrow[dr] \arrow[rr, bend left] \arrow[rr, bend right] & & P_A \\
 & {P_A}^{\mathbf{1+1}} \arrow[ur, "\pi_2"'] &
\end{tikzcd}$$

  form the kernel pair of the map $[p_2]\colon P_A \to \mathbf{N}$.

For D discrete, prove that $[p_2]\colon P_D \to \mathbf{N}$ is one-one.

The effective topos

Assemblies are not enough.

The reason is “lack of quotient of equivalence relations”.

Define composition of relations as

$$R; S \overset{\mathsf{def}}{=} \pi^{\mathsf{p}}_{13}(\pi^{*}_{12}(R) \cap \pi^{*}_{23}(S))$$

An equivalence relation is $R \subseteq A \times A$ $\quad \Delta_A \subseteq R \quad R^o \subseteq R \quad R; R \subseteq R$

Formally add quaotients by taking the equivalence relations themselves as objects and define a *map* $F \colon R \to S$ to be $F \subseteq A \times B$ such that

$$R; F; S = F \qquad R \subseteq F; F^o \qquad F^o; F \subseteq R$$

Every object is a surjective image of a projective assembly.

There is a surjection $[p_1] \colon \nabla(\mathrm{P}\,(\mathbb{N})) \twoheadrightarrow \Omega$.

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