

A MODEL OF INTUITIONISTIC
SET THEORY

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We shall present a realizability model for the intuitionistic theory IZF of sets and we will show that in it the class of trichotomous ordinals is a set. The present work, suggested by Prof. D.S.Scott, was worked out by D.C.MacCarty and the author at Oxford in the academic year 1980-81.

1. The model. It is essentially that studied by Scedrov in [3], but it is built in such a way that the natural membership relation be extensional. We availed ourselves of ideas presented in [2] and [1].

Let N be the set of natural numbers; if n is in N , n_0 and n_1 and..., will denote the canonical projections and $\{n\}$ the partial recursive function encoded by n . Before introducing the model, let us recall the clauses to extend realizability to non-atomic formulae. Let A and B be formulae and let \bar{a} be a list of elements in R . One will say that a natural number realizes a compound formula as follows:

$\underline{nr}A \wedge B[\bar{a}]$ iff $n_0 \underline{r}A[\bar{a}]$ & $n_1 \underline{r}B[\bar{a}]$;
 $\underline{nr}A \vee B[\bar{a}]$ iff $n_0 = 0 \Rightarrow n_1 \underline{r}A[\bar{a}]$ & $n_0 \neq 0 \Rightarrow n_1 \underline{r}B[\bar{a}]$;
 $\underline{nr}A \rightarrow B[\bar{a}]$ iff $\forall m \in N [\underline{mr}A[\bar{a}] \Rightarrow \{n\}(m) \downarrow \& \{n\}(m) \underline{r}B[\bar{a}]]$;
 $\underline{nr}\neg A[\bar{a}]$ iff $\forall m \in N. m \not\underline{r}A[\bar{a}]$;
 $\underline{nr}\forall x A[\bar{a}]$ iff $\forall b \in R. \underline{nr}A[\bar{a}][b/x]$;
 $\underline{nr}\exists x A[\bar{a}]$ iff $\exists b \in R. \underline{nr}A[\bar{a}][b/x]$.

Notation. For $X, Y \subseteq N$, $n \in N$, we shall write $X \stackrel{n}{\dashv} Y$ to mean that $X \subseteq \{n_0\}^{-1}(Y)$ and $Y \subseteq \{n_1\}^{-1}(X)$.

Definition. The model R is the class of all partial functions

$a: R \rightarrow P_*(N)$ into the non-empty parts of N such that
 (i) $\forall b, c \in R \forall n \in N [\underline{mr}b = c \Rightarrow [b \in \text{dom } a \Leftrightarrow c \in \text{dom } a]]$;
 (ii) $\exists t \in N \forall b, c, \in R \forall m \in N \underline{mr}b = c \Rightarrow \{t\}(m) \downarrow \& a(b) \stackrel{\{t\}(m)}{\dashv} a(c)$];
 where $\underline{mr}b = c$ is $\underline{mr}\forall x [x \in b \leftrightarrow x \in c]$ and $\underline{nr}d \in b$ is $d \in \text{dom } b$ &
 $n_0 \in b(d)$ & $\forall k \in N [\underline{kr}d = d' \Rightarrow \{n_1\}(k) \downarrow \& b(d) \stackrel{\{n_1\}(k)}{\dashv} b(d')]$.

Obviously the definition is circular, but we leave it to the interested reader to write out the details of the correct cumulative definition.

The two clauses for atomic formulae written above complete the definition of realizability of the first order language with symbols $\in, =$. One says that a sentence A is realized in the model R if there is a natural number that realizes it and we shall write $R \models A$.

Theorem. All the axioms of IZF are realized in R , including the logical axioms for an intuitionistic calculus and the axioms of extensionality, separation, pair, union, power, infinity, \in -induction, and collection.

For the proof we only warn the reader that one tricky point is in the realization of the axiom of \in -induction.

We remark that the natural numbers in R are defined as: $\underline{0}$ is the empty function; $\underline{n+1}(c) = \{m \in \underline{mr}c = 0 \vee \dots \vee c = n\}$. It is easy to show that, if $\underline{kr}n = \underline{m}$, then $n = m$. Moreover, the "set of all natural numbers" is ω defined by $\omega(c) = \{\langle n, m \rangle : \underline{mr}c = \underline{n}\}$.

2. Realizability transferred inside the model. Suppose a is a given element of R . One can define a relation in R from a to ω which is defined on all of a and which links an element of a and a number realizing its membership to a . More precisely, let $F: R^2 \rightarrow P_*(N)$ be defined by

$F(b, c) = \{\langle n, k, m \rangle : \exists d \in R [\underline{nr}d \in a \& \underline{kr}d = b \& \underline{mr}c = \underline{n}]\}$.
 One shows that F is a set and it determines an element in R such

that $\text{sr}F \subseteq a \times \omega \wedge \forall x \in a \exists y. \langle x, y \rangle \in F$, for some s in N which does not depend on a (we apologize to the reader if, for lack of space, we skip the proof of the previous statement which is nonetheless essential for what follows; we hope it will be found an interesting exercise). From this one gets that there is f in R such that $\text{tr}f: a \rightarrow P_*(\omega)$ for some t which does not depend on a , where $P_*(N)$ is the set in R of the inhabited parts of ω .

3. The trichotomous ordinals in R . First of all some notation: $\text{Trans}(x)$ is $\forall y \in x \forall z \in y. z \in x$; $\text{Ord}(x)$ is $\text{Trans}(x) \wedge \forall y \in x. \text{Trans}(x)$; $\text{Tri}(x)$ is $\text{Ord}(x) \wedge \forall y, z \in x [y \in z \vee y = z \vee z \in y]$. We intend to show that

$$R \models \forall x [\text{Tri}(x) \rightarrow \exists f [f: x \rightarrow P_*\omega \wedge \forall y, z \in x [\exists w. w \in f(y) \cap f(z) \rightarrow y = z]]].$$

So let a be in R and take f as defined in §2. We must find a number p in N which does not depend on a such that

$$\forall m \in N [\text{mr}Tri(a) \Rightarrow \{p\}(m) \downarrow \&\{p\}(m) \text{r}\forall y, z [y \in a \wedge z \in a \wedge \exists w. w \in f(y) \cap f(z) \rightarrow y = z]].$$

Suppose that $\text{mr}Tri(a)$. Then

$$\begin{aligned} \forall b, c \in R \forall i \in N [\text{ir}b \in a \wedge c \in a \Rightarrow \{m\}(i) \downarrow \&\{m\}(i)_0 = 0 \Rightarrow \{m\}(i)_1 \text{r}b \in c \\ \{m\}(i)_0 = 1 \Rightarrow \{m\}(i)_1 \text{r}b = c \\ \{m\}(i)_0 \neq 0, 1 \Rightarrow \{m\}(i)_1 \text{r}c \in b]. \end{aligned}$$

Suppose moreover that $\text{nr}b \in a$ and $\text{kr}c \in a$ and that there exists $d \in R$ such that $\langle \ell, j \rangle \text{rd} \in f(b) \cap f(c)$. By definition of f , one may assume that $\langle \ell, j \rangle \text{rd} \in F(b, d) \cap F(c, d)$. Hence there are b' and c' in R such that $\ell_0 \text{r}b' \in a \& \ell_1 \text{r}b' = b \& \ell_2 \text{rd} = \ell_0 \& j_0 \text{r}c' \in a \& j_1 \text{r}c' = c \& j_2 \text{rd} = j_0$. Thus $\ell_0 = j_0$. On the other hand, it must be $\{m\}(\langle \ell_0, \ell_0 \rangle)_0 = 1$, hence $\{m\}(\langle \ell_0, j_0 \rangle)_0 = 1$. Therefore, $\{q\}(\langle \ell_1, \{m\}(\langle \ell_0, j_0 \rangle)_1, j_1 \rangle) \text{r}b = c$, where q realizes $\forall x, y, z, w [y = x \wedge y = z \wedge z = w \Rightarrow x = w]$. Take p such that $\{\{p\}(m)\}(\langle n, k, \langle \ell, j \rangle \rangle) = \{q\}(\langle \ell_1, \{m\}(\langle \ell_0, j_0 \rangle)_1, j_1 \rangle)$.

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