Partial morphisms in categories of effective objects

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Abstract

This paper is divided in two parts. In the first one we analyse in great generality *data types* in relation to *partial morphisms*. We introduce partial function spaces, partial cartesian closed categories and complete objects, motivate their introduction and show some of their properties. In the second part we define the (partial) cartesian closed category $\mathring{G}EN$ of generalized numbered sets, prove that it is a *good extension* of the category of numbered sets and show how it is related to the recursive topos.

Introduction

By *data type* one usually means a set of objects of the same *kind*, suitable for manipulation by a computer program. Of course, computers actually manipulate formal representations of objects. The purpose of the mathematical semantics of programming languages, however, is to characterize data types (and functions on them) in a way which is independent of any specific representation mechanism. So the objects one deals with are mostly elements of *structures* borrowed from different areas of mathematics, whose meaning is well understood and does not depend on the practice of programming.

A *categorical* definition of data types, e.g. by *universal properties*, seems even more suitable to this goal, because it allows us to abstract not only from a specific computer representation, but also from the details of the mathematical structures. Unfortunately, sometimes it's not possible to give a categorical definition of a data type, because some *relevant* concept related to it does not have a counterpart at the abstract categorical level.

A concept, which is often ignored at a categorical level, is *partiality*. In *Section 1* we discuss the notion of partial map in very general terms. The usual categorical definition is modified slightly to allow us to consider only those maps whose domains are in a suitable sense *admissible subobjects*. A notion of *partial* *cartesian closed category* ($pCCC$) is introduced, which provides the axiomatic framework to discuss *higher types* containing partial maps. This, like the original notion of cartesian closed category, is given by a strictly categorical definition, and therefore is relevant to the consideration of partial maps between cpos and other mathematical structures, as well as sets.

Quite often, in the mathematical semantics of programming languages we lose the notion of effective computability, which has an intrinsic operational character. This notion may be recovered by a suitable definition of *computable element*. However, it is worth pursuing a general notion of effectiveness over abstract data types, since computable elements and maps provide the *regular* interpretation of programming constructs. Effectiveness for functions on natural numbers is well understood, a simple way of extending it to abstract data types is to establish a *correspondence* between natural numbers and the elements of an abstract data type, so that the recursive functions $\mathring{R}$ induce a natural definition of *effective morphism*:

Definition 1 *Numbered Sets and Effective Morphisms*

- $\underline{A} = (A, \mathring{e}_A)$ *is a* **numbered set** $\overset{\Delta}{\Longleftrightarrow}$ $\mathring{e}_A\colon \omega \to A$ *is surjective*
- *If* $\underline{A}$ *and* $\underline{B}$ *are numbered sets, then a total function* $g\colon A \to B$ *is an* **effective morphism** *from* $\underline{A}$ *to* $\underline{B}$ $\overset{\Delta}{\Longleftrightarrow}$ $\exists f \in \mathring{R}. \mathring{e}_B \circ f = g \circ \mathring{e}_A$

Since it's technically more convenient to have also a counterpart for the empty set, we will denote by $\mathring{E}N$ the category of numbered sets and effective morphisms extended with a *strict initial object* $\mathring{0}$ (see [Ers75]). In the above definition of numbered set we haven't assumed any structure on $\underline{A}$, so that $\mathring{E}N$ may be considered the *universe of all effective objects* (independently from the meaning of *effectively given data type*). For numbered sets there is a natural notion of partial morphism, suggested by the partial recursive functions $\mathring{P}R$:

Definition 2 *Partial Effective Morphisms*

If $\underline{A}$ *and* $\underline{B}$ *are numbered sets, then a partial function* $g\colon A \rightharpoonup B$ *is a* **partial effective morphism** *from* $\underline{A}$ *to* $\underline{B}$ $\overset{\Delta}{\Longleftrightarrow}$ $\exists f \in \mathring{P}R. \mathring{e}_B \circ f = g \circ \mathring{e}_A$

The categories one needs for interpreting high level programming languages must possess good *closure properties* so that the existence of objects, which are formally given by general definitional tools, is *a priori* assured. For instance, we require closure w.r.t. cartesian products, function spaces (and other data type constructions), since these constructions are commonly used in the design of programming languages. But unfortunately, $\mathring{E}N$ doesn't have function spaces. For instance, there is no *effective* and *uniform* way of numbering $\mathring{E}N(\underline{\omega}, \underline{\omega})$ (where $\underline{\omega}$ is the numbered set $(\omega, \mathring{i}d_\omega)$). As a matter of fact $\mathring{E}N$ is far from being cartesian closed. For this reason, in *Section 2* we extend (in a straightforward way) the definition of numbered set and (partial) effective morphism to obtain the *pCCC* (as well as *CCC*) $\mathring{G}EN$ of *generalized numbered sets*, and show that it is a *good extension* of $\mathring{E}N$. In that section we also abstract from the properties of partial recursive functions, and *relativize* the notion of numbered sets to an *acceptable* set of partial functions (see *Def* 24).

In [Mul81] there is another *good extension* of $\mathring{E}N$, the **recursive topos** $\mathcal{R}$, such that $\mathring{E}N$ is embedded in $\mathcal{R}$ and the embedding preserves limits (finite colimits) and function spaces. In *Section 4* we introduce a topos *similar* to $\mathcal{R}$ and give a *topos-theoretic characterization* of $\mathring{G}EN$, following quite closely Chapter 6 of [Ros86]. From this characterization and *general* facts in topos theory (see [Hyl82] Section 5 and [Ros86] Chapter 6), one can easily derive the properties of $\mathring{G}EN$, that in *Section 2* are proved by *elementary* means.

In *Section 3* we compare a type structure in $\mathring{G}EN$ with other ones, introduced in connection with Recursion Theory in higher types. From this comparison it turns out that morphisms in $\mathring{G}EN$ are similar to Banach-Mazur functionals (see [Rog67]), that is we lose the *uniformity* property of effective morphisms when we extend $\mathring{E}N$ to $\mathring{G}EN$.

A different proposal for a *universe of effective objects* is the *Effective Topos* $\mathcal{F}$, which is based on the realizability interpretation of logical connectives, and has been extensively investigated in [Hyl82] and [McC84]. In $\mathcal{F}$ *uniformity* holds but we have to give up *numberings*. $\mathring{E}N$ can be identified with a full sub-category of $\mathcal{F}$, but the embedding of $\mathring{E}N$ in $\mathcal{F}$ doesn't seem to preserve function spaces. A comparison between the Recursive and the Effective Topos can be found in [Ros86].

Throughout the paper, category theory is used in a fairly elementary way (see [BW85] Chapter 1), the only exception is *Section 4*, which relies on some knowledge of Grothendieck toposes (see [MR77], [Joh77]).

1 Partial Morphisms

When dealing with computable functions or with the semantics of programs partiality arises naturally. In this section we develop an abstract framework for partial functions, namely *categories with domains* (see *Def* 4). In defining partial morphisms we will simply mimic the *set-theoretic* definition of partial map (see also [Obt86], [Ros86] and [Mog86]). In the literature there exist other approaches to formalizing the concept of category of *partial morphisms* (see bibliography) and an overview of them can be found in [RR86].

After the basic definitions and some examples of categories of partial morphisms we consider the relations between data types and partial morphisms. More precisely, we investigate whether the operations associated to data types can be extended *naturally* to partial morphisms, and introduce new data types (lifting and partial function spaces) in connection with partial morphisms. These data types have already been introduced in denotational semantics, but their definitions were given by set-theoretic constructions, that work only in particular categories, and do not stress their *abstract properties*. We define *partial cartesian closed categories* (*pCCC*), that are the natural counterpart of cartesian closed categories, when *partial function spaces* are considered. There is also a variant of the lambda-calculus, the *partial lambda-calculus*, corresponding to the equational theory of partial cartesian closed categories (see [Mog86], [Ros86], [CO87] and [Mog87]).

Another notion related to partial morphisms is that of complete object (see *Def* 20), introduced in [Ers73]. Informally a complete object is like a set with an element $\perp$ to represent *undefined*.

An elementary approach to partial morphisms, complete objects and partial function spaces (in *concrete* categories of partial morphisms) can be found in [LM84a].

1.1 Categories with Domains

By analogy with the set-theoretic definition of partial map (from a to b) as a map from a subset of a to b, in a category C, we will identify a partial morphism from an object a to an object b with a morphism *from* a subobject of a to b.

Notation. We fix some notation, for basic categorical notions (see [BW85] Chapter 1)

- $i: a \hookrightarrow b$ means that i is a **mono** from a to b
- $[i: d \hookrightarrow a]$ is the **subobject** of a corresponding to i, i.e. the class of monos *isomorphic* to i. $\mathring{S}ubObj(a)$ is the class of subobjects of a
- If $f: a \to b$ and $[i: d \hookrightarrow b]$ is a subobject of b, then $f^{-1}([i])$ is the **inverse image** of $[i]$ along f, i.e. the subobject $[i']$ of a s.t. $a \overset{i'}{\hookleftarrow} d' \overset{f'}{\to} d$ is a pullback of $a \overset{f}{\to} b \overset{i}{\hookleftarrow} d$ for some f'

Definition 3 *Partial Morphisms in Categories*

A **witness** *for a partial morphism from a to b is a pair* (i, f) *s.t.* $i: d \hookrightarrow a$ *and* $f: d \to b$*, for some object d; and two witnesses* (i_1, f_1) *and* (i_2, f_2) *are* **isomorphic** $\overset{\Delta}{\iff}$ *there exists an isomorphism i s.t.* $i_1 = i_2 \circ i$ *and* $f_1 = f_2 \circ i$*.*

A **partial morphism** *from a to b is an equivalence class of isomorphic witnesses for a partial morphism from a to b*

It's easy to show that if two witnesses are isomorphic, then the isomorphism i is unique.

Notation. We fix some notation for partial morphisms

- $g: a \rightharpoonup b$ means that g is a partial morphism from a to b
- $\mathring{p}(i, f)$ is the partial morphism corresponding to the witness (i, f)

- $\mathring{p}(i_1, f_1) \leq \mathring{p}(i_2, f_2)$ iff there exists i s.t. $i_1 = i_2 \circ i$ and $f_1 = f_2 \circ i$

  If the i above exists, then it's a mono and is unique. Moreover, $\leq$ is a partial order

- the **domain** of a $\mathring{p}(i, f)\colon a \rightharpoonup b$ is the subobject $[i]$ of a

- the **composition** of $\mathring{p}(i_1, f_1)\colon a \rightharpoonup b$ and $\mathring{p}(i_2, f_2)\colon b \rightharpoonup c$ is the partial morphism $\mathring{p}(i_1 \circ i, f_2 \circ f)\colon a \rightharpoonup c$

  where $d_1 \overset{i}{\hookleftarrow} d \overset{f}{\to} d_2$ is a pullback of $d_1 \overset{f_1}{\to} b \overset{i_2}{\hookleftarrow} d_2$ (therefore, composition is well-defined iff such a pullback exists)

In general partial morphisms do not form a category, because composition may be undefined. Moreover, they are usually more than the *admissible* ones, for instance there are partial morphisms in $\mathring{E}N$ that are not effective. The criteria we use to describe the *admissible* partial morphisms is to impose some constraints on the class of *admissible* subobjects. For these reasons categories of partial morphisms will be defined by giving both a category and a *domain structure*, i.e. a collection of subobjects with certain properties:

Definition 4 *Domain Structure and Category of Partial Morphisms*
$\mathcal{M} \overset{\Delta}{=} \langle \mathcal{M}(a) | a \in C \rangle$ *is a* **domain structure** *on* $C \overset{\Delta}{\iff}$

1. $\mathcal{M}(a) \subseteq \mathring{S}ubObj(a)$
2. $[\mathring{i}d_a] \in \mathcal{M}(a)$
3. $[i'\colon c \hookrightarrow b] \in \mathcal{M}(b), [i\colon b \hookrightarrow a] \in \mathcal{M}(a) \Longrightarrow [i \circ i'\colon c \hookrightarrow a] \in \mathcal{M}(a)$
4. $f\colon a \to b, m \in \mathcal{M}(b) \Longrightarrow f^{-1}(m) \in \mathcal{M}(a)$

   and therefore $f^{-1}(m)$ *exists*

$(C, \mathcal{M})$ *is called a* **category with domains** *(dC).* $\mathring{P}(C, \mathcal{M})$ *is the* **category of partial morphisms** *(in C) with domain in* $\mathcal{M}$

When C and $\mathcal{M}$ are clear from the context, we write $a \lhd_{\mathring{p}} b$ for a is a retract of b in $\mathring{P}(C, \mathcal{M})$.

The properties of a domain structure are (necessary and) sufficient to make sure that $\mathring{P}(C, \mathcal{M})$ is a category. More precisely, properties (3) and (4) imply that composition of partial morphisms in $\mathring{P}(C, \mathcal{M})$ is well-defined, while property (2) implies that the identities $\mathring{p}(\mathring{i}d_a, \mathring{i}d_a)$ are in $\mathring{P}(C, \mathcal{M})$ and that there is a **canonical embedding** of C into $\mathring{P}(C, \mathcal{M})$, which maps $f\colon a \to b$ to $\mathring{p}(\mathring{i}d_a, f)\colon a \rightharpoonup b$. The partial order $\leq$ on partial morphisms enjoys the following properties:

Proposition 5 *If* $\mathring{P}(C, \mathcal{M})$ *is a category of partial morphisms, then*

1. *total morphisms (i.e. those of the form* $\mathring{p}(\mathring{i}d, f)$*) are maximal*

2. *composition* $\_\circ\_ : \mathring{P}(C,\mathcal{M})(a,b)\times\mathring{P}(C,\mathcal{M})(b,c)\to\mathring{P}(C,\mathcal{M})(a,c)$ *of partial morphisms is monotonic*

Both Obtulowicz and Rosolini, starting from the simple-minded definition of category of partial morphisms presented above, develop *equational axiomatizations* for categories of partial morphisms. In general the categories satisfying these axioms are not of the form $\mathring{P}(C,\mathcal{M})$. However, they can always be embedded fully and faithfully in a category of the form $\mathring{P}(C,\mathcal{M})$ (see [Ros86] and [RR86] for details).

Any category C has a **trivial domain structure**, namely $\mathcal{M}(a)=\{[\mathring{id}_a]\}$, such that $\mathring{P}(C,\mathcal{M})$ is isomorphic to C.

Example. The simplest example of a non trivial category of partial morphisms is the category of sets and partial functions, which corresponds to the domain structure $\langle \mathring{S}ubObj(a)|a\in \mathring{S}ET\rangle$ over $\mathring{S}ET$. In fact, if C has all finite limits, then $\langle \mathring{S}ubObj(a)|a\in C\rangle$ is a domain structure over C, actually it's the biggest.

Most of the categories C we consider are concrete, i.e. the behaviour of a morphism is uniquely determined by its behaviour on *global elements*; so that a morphism $f: a\to b$ in C can be identified with a function from $C(\mathring{1},a)$ to $C(\mathring{1},b)$. For categories with domains, we *strengthen* the definition of concreteness by requiring partial morphisms with domain in $\mathcal{M}$ (in particular total morphisms) to be uniquely determined by their behaviour on *global elements*:

Definition 6 *Concrete Category with Domains*

$(C,\mathcal{M})$ *is a* **concrete** $dC \overset{\Delta}{\iff} C$ *has a terminal object* $\mathring{1}$ *and*

$$\forall a,b\in C.\forall f,g: a\rightharpoonup b.(\forall h:\mathring{1}\to a.f\circ h=g\circ h)\longrightarrow f=g$$

If $(C,\mathcal{M})$ is concrete, then $\mathring{P}(C,\mathcal{M})$ can be treated as a category of sets and partial functions.

Example. In the category of numbered sets $\mathring{E}N$ there is a natural counterpart to the recursively enumerable (r.e.) sets, namely:

Definition 7 $X\subseteq A$ *is an* **r.e.-subset** *of* $\underline{A} \overset{\Delta}{\iff} \mathring{e}_A^{-1}(X)(\subseteq\omega)$ *is an r.e.-set*

Every nonempty r.e.-subset X of $\underline{A}$ can be numbered by $\mathring{e}_A\circ f$, where $f\in\mathring{R}$ is an enumeration of $\mathring{e}_A^{-1}(X)$, so $\underline{X}\triangleq(X,\mathring{e}_A\circ f)$ becomes a numbered set and $[incl_X:\underline{X}\hookrightarrow\underline{A}]$ becomes a subobject of $\underline{A}$. On the other hand, if X is empty we can take the subobject $[\mathring{0}_A:\mathring{0}\hookrightarrow\underline{A}]$, where $\mathring{0}_A$ is the unique morphism from the strict initial object to $\underline{A}$. Therefore, for every r.e.-subset X of $\underline{A}$ there is a subobject of $\underline{A}$, we call it the **r.e.-subobject** corresponding to X. It's easy to show that the collection $\mathcal{M}_{r.e.}$ of r.e.-subobjects is a domain structure and that $\mathring{P}(\mathring{E}N,\mathcal{M}_{r.e.})$ is isomorphic to the category of numbered sets and partial effective morphisms.

Example. Another category with a natural domain structure on it is the category of topological spaces $\mathring{T}op$:

Definition 8 *If* $\mathbf{X}$ *is a topological space, then* m *is an* **open subobject** *of* $\mathbf{X}$ $\stackrel{\Delta}{\Longleftrightarrow}$ *m is the subobject corresponding to an open subset of* $\mathbf{X}$ *with the induced topology*

It's easy to show that the collection $\mathcal{M}_{open}$ of open subobjects is a domain structure. A simple way to represent a partial morphism $g: X \rightharpoonup Y$ in $\mathring{P}(\mathring{T}op, \mathcal{M}_{open})$ is by a partial map from X to Y s.t. the inverse image $g^{-1}(A)$ of an open subset A of $\mathbf{Y}$ is an open subset of $\mathbf{X}$. A full sub-category of $\mathring{T}op$ is the category $\mathring{C}PO$ of partial orders complete w.r.t. lubs of ω-chains and monotonic maps preserving lubs of ω-chains (see [Plo85]). We don't assume the existence of a least element in a cpo, because we want an open subset of a cpo (with the induced partial order) to be a cpo, so that $\mathcal{M}_{open}$ *restricted* to $\mathring{C}PO$ is a domain structure over $\mathring{C}PO$.

1.2 Data types and partial morphisms

Having introduced partial morphisms, we have to check whether they fit into the categorical constructions corresponding to the usual data types, or whether some modifications are required. We will investigate the most *relevant* data types, namely: products, function spaces, partial function spaces and lifting. Every data type will be specified by a *universal property.* The categorical definitions of product and function space are already familiar; however, the operations associated with them need to be extended to partial morphisms in order to give an interpretation of terms (programs) as partial morphisms in a *denotational* style. This extension is not always possible for the abstraction operation $\mathring{\Lambda}(\_)$ associated with function spaces, therefore, in the context of categories of partial morphisms, function spaces will be replaced by partial function spaces. Throughout this section $(C, \mathcal{M})$ denotes a category with domains.

The usual data types are defined without any reference to partial morphisms, and we recall their definition in order to fix the notation:

Definition 9 *Standard data types*

- $\mathring{1}$ *is a terminal object* $\stackrel{\Delta}{\Longleftrightarrow}$

  for any $a \in C$ *there exists unique* $!: a \to \mathring{1}$

- $a \stackrel{\pi_1}{\leftarrow} a \times b \stackrel{\pi_2}{\to} b$ *is a* **product** *of* a *and* b $\stackrel{\Delta}{\Longleftrightarrow}$

  for any $f: c \to a$ *and* $f': c \to b$ *there exists unique* $\langle f, f' \rangle: c \to a \times b$ *s.t.* $f = \pi_1 \circ \langle f, f' \rangle$ *and* $f' = \pi_2 \circ \langle f, f' \rangle$.

- $(a \to b) \times a \stackrel{\mathring{e}val}{\to} b$ *is a* **function space** *from* a *to* b $\stackrel{\Delta}{\Longleftrightarrow}$

  for any $f: c \times a \to b$ *there exists unique* $\mathring{\Lambda}(f): c \to (a \to b)$ *s.t.* $f = \mathring{e}val \circ (\mathring{\Lambda}(f) \times \mathring{i}d_a)$

In order to extend the operations on data types to partial morphisms we examine what happens in the category of sets and partial functions. If $g: c \rightharpoonup a$ and $g': c \rightharpoonup b$, then we take $\langle g, g' \rangle$ to be the *most defined* partial function h s.t. $\pi_1 \circ h \leq g$ and $\pi_2 \circ h \leq g'$, i.e.

$$\langle g, g' \rangle (x) \triangleq \begin{cases} \langle g(x), g'(x) \rangle & \text{if both } g(x) \text{ and } g'(x) \text{ are defined} \\ undefined & \text{otherwise} \end{cases}$$

In general

Definition 10 *If $\mathring{p}(i_1, f_1): c \rightharpoonup a$ and $\mathring{p}(i_2, f_2): c \rightharpoonup b$, then*

$$\langle \mathring{p}(i_1, f_1), \mathring{p}(i_2, f_2) \rangle \triangleq \mathring{p}(i, \langle f_1 \circ i'_1, f_2 \circ i'_2 \rangle)$$

where $d_1 \overset{i'_1}{\hookleftarrow} d \overset{i'_2}{\hookrightarrow} d_2$ is a pullback of $d_1 \overset{i_1}{\hookrightarrow} c \overset{i_2}{\hookleftarrow} d_2$ and $i \triangleq i_1 \circ i'_1 = i_2 \circ i'_2$ (the subobject $[i]$ is called the **intersection** *of $[i_1]$ and $[i_2]$)*

The existence of these subobjects follows from properties 3 and 4 for a domain structure (see *Def* 4). Products in $\mathring{P}(C, \mathcal{M})$ are quite different from those in C, and they do not seem to be a *natural* data type. Unlike products, equalizers do not present any problem, in fact equalizers in C are also equalizers in $\mathring{P}(C, \mathcal{M})$. In $\mathring{SET}$ one can extend $\mathring{\Lambda}(\_)$ to partial morphisms, namely if $g: c \times a \rightharpoonup b$, then we take $\mathring{\Lambda}(g)$ to be the *most defined* partial function h s.t. $\mathring{eval}(h \times \mathring{id}_a) \leq g$, i.e.

$$\mathring{\Lambda}(g)(x) \triangleq \begin{cases} \lambda y: a.g(x, y) & \text{if } g(x, y) \text{ is defined for all } y \in a \\ undefined & \text{otherwise} \end{cases}$$

However, in general it is not possible to extend $\mathring{\Lambda}(\_)$ to partial morphisms, so that $\mathring{\Lambda}(g: c \times a \rightharpoonup b)$ is the *most defined* $h: c \rightharpoonup (a \to b)$ s.t. $\mathring{eval}(h \times \mathring{id}_a) \leq g$, and at the same time $\mathring{\Lambda}(\_)$ is natural in c, i.e. is a natural transformation from $\mathring{P}(C, \mathcal{M})(\_ \times a, b)$ to $\mathring{P}(C, \mathcal{M})(\_, a \to b)$. For instance, in $\mathring{CPO}$ the first condition implies that the domain of $\mathring{\Lambda}(g: \mathbf{X} \times \mathbf{Y} \rightharpoonup \mathbf{Z})$ is the interior of $\{x \in X | \forall y \in Y.g(x, y) \text{ is undefined}\}$, but then $\mathring{\Lambda}$ cannot be natural, because the inverse image of the interior of a subset S (along a continuous map f) is not necessarily the interior of the inverse image of S. This problem does not arise in the case of partial function spaces, because $\mathring{p}\Lambda(\_)$ is already defined on partial morphisms.

Definition 11 *Partial function space and lifting*

- $(a \rightharpoonup b) \times a \overset{\mathring{p}eval}{\rightharpoonup} b$ *is a* **partial function space** *from a to b* $\overset{\Delta}{\iff}$

  for any $g: c \times a \rightharpoonup b$ there exists unique $\mathring{p}\Lambda(g): c \to (a \rightharpoonup b)$ s.t. $g = \mathring{p}eval \circ (\mathring{p}\Lambda(g) \times \mathring{id}_a)$

- $b_\perp \overset{\mathring{o}pen}{\rightharpoonup} b$ *is a* **lifting** *of b* $\overset{\Delta}{\iff}$

  for any $g: c \rightharpoonup b$ there exists unique $\overline{g}: c \to b_\perp$ s.t. $g = \mathring{o}pen \circ \overline{g}$

The previous considerations concerning function spaces suggest that we replace them with partial function spaces and this leads to the following definition:

Definition 12 *pCCC*
$(C, \mathcal{M})$ *is a* **partial cartesian closed category** *(pCCC)* $\stackrel{\Delta}{\Longleftrightarrow}$ *C has a terminal object, products and partial function spaces*

The familiar notion of CCC is just a *degenerate* instance of $pCCC$, corresponding to a trivial domain structure.

In the category of sets and partial functions, partial function spaces are what one expects and lifting corresponds to adding an extra element (representing *undefined*) to a set. Lifting and partial function spaces can be defined one in terms of the other, namely:

Proposition 13

- $\mathring{open} \triangleq (\mathring{1} \rightharpoonup b) \stackrel{\langle \mathring{id}, ! \rangle}{\rightarrow} (\mathring{1} \rightharpoonup b) \times \mathring{1} \stackrel{\mathring{peval}}{\rightharpoonup} b$ *is a lifting of b*
- $\mathring{peval} \triangleq (a \rightarrow b_\perp) \times a \stackrel{\mathring{eval}}{\rightarrow} b_\perp \stackrel{\mathring{open}}{\rightharpoonup} b$ *is a partial function space from a to b*

We haven't considered coproducts (more generally colimits) in relation to partial morphisms, but they do not present any problem, and in fact

Proposition 14 *Colimits and Partial Morphisms*
If each object of C has a lifting, then the canonical embedding of C in $\mathring{P}(C, \mathcal{M})$ has a right adjoint and therefore it preserves colimits

Proof

The right adjoint of the canonical embedding is the **lifting** functor $\_{}_\perp \colon \mathring{P}(C, \mathcal{M}) \rightarrow C$, s.t. $\_{}_\perp \colon a \mapsto a_\perp$ and $\_{}_\perp \colon f \mapsto \overline{f \circ \mathring{open}}$ ∎

Proposition 15 *In a pCCC with coproducts the following (natural) isomorphisms hold:*

- $a \times (b + c) \cong (a \times b) + (a \times c)$
- $a \rightarrow (b \rightharpoonup c) \cong (a \times b) \rightharpoonup c$
- $(a + b) \rightharpoonup c \cong (a \rightharpoonup c) \times (b \rightharpoonup c)$

If in the proposition above one replaces partial function spaces with function spaces, one gets the usual isomorphisms that hold in cartesian closed categories (see [LS86]), except $a \rightarrow (b \times c) \cong (a \rightarrow b) \times (a \rightarrow c)$, which doesn't have a counterpart in $pCCC$.

Since the domain structure $\mathcal{M}$ plays a key role in defining partial morphisms, it's important to have a data type that *represents* $\mathcal{M}$:

Definition 16 *Dominance*

A subobject true of Σ *is a* **dominance** $\stackrel{\Delta}{\Longleftrightarrow}$ $true \in \mathcal{M}(\Sigma)$ *and for any* $m \in \mathcal{M}(a)$ *there exists unique* $\phi_m : a \to \Sigma$ *s.t.* $m = {\phi_m}^{-1}(true)$

In the category of sets and partial functions a dominance is the same as a subobject classifier. The characteristic feature of a dominance is the natural isomorphism $\mathcal{M}(\_) \dot{\cong} C(\_, \Sigma)$, where $\mathcal{M}$ is the functor from $C^{\mathring{op}}$ to $\mathring{S}ET$ s.t. $\mathcal{M}(f : a \to b) : m \in \mathcal{M}(b) \mapsto f^{-1}(m) \in \mathcal{M}(a)$. A dominance is definable by more familiar data types:

Proposition 17 $[\overline{\mathring{id}_{\mathring{1}}} : \mathring{1} \hookrightarrow \mathring{1}_\perp] \in \mathcal{M}(\mathring{1}_\perp)$ *is a dominance*

Therefore a $pCCC$ always has a dominance.

In [Sco80] the category $\mathring{S}h(C)$ of presheaves over C is proposed as *conservative cartesian closed extension* of C, because $\mathring{S}h(C)$ has all (small) limits and function spaces and the Yoneda embedding of C in $\mathring{S}h(C)$ preserves limits and function spaces that already exist in C. Therefore, a *construction* in C involving only limits and function spaces gives the same result in $\mathring{S}h(C)$. Moreover, if some intermediate steps of a construction require limits or function spaces that do not exist in C, then we can always *imagine* that it is carried out in $\mathring{S}h(C)$. A similar *conservative extension* is possible for categories with domains (see [Ros86]):

Proposition 18 *Domain Structure over Presheaves*

If $(C, \mathcal{M})$ *is a category with domains, then there exists a domain structure* $\mathring{S}h(\mathcal{M})$ *over the category* $\mathring{S}h(C)$ *of presheaves on* C *s.t.* $(\mathring{S}h(C), \mathring{S}h(\mathcal{M}))$ *has a dominance and the Yoneda embedding* $\mathring{Y}$ *embeds* $\mathcal{M}$ *fully and faithfully in* $\mathring{S}h(\mathcal{M})$, *i.e.* $\mathring{S}h(\mathcal{M})(\mathring{Y}a) = \{[\mathring{Y}i] | [i] \in \mathcal{M}(a)\}$

In particular, the Yoneda embedding $\mathring{Y} : C \to \mathring{S}h(C)$ can be extended to partial morphisms $(\mathring{Y} : \mathring{p}(i, f) \mapsto \mathring{p}(\mathring{Y}i, \mathring{Y}f))$, so that it becomes a full and faithful embedding of $\mathring{P}(C, \mathcal{M})$ in $\mathring{P}(\mathring{S}h(C), \mathring{S}h(\mathcal{M}))$. In a topos with a dominance one can define lifting and partial function spaces (as shown in [Ros86]), therefore, $(\mathring{S}h(C), \mathring{S}h(\mathcal{M}))$ is a $pCCC$.

Proposition 19 *The Yoneda embedding preserves lifting and partial function spaces*

1.3 Complete objects

A notion related to partial morphisms is that of complete object (see [Ers73]). The characteristic feature of a complete object a is that every partial morphism with codomain a can be *extended* to a total morphism. In set-theoretic terms this means that a has an element *representing* undefined.

Definition 20 *Complete Object*

An object a *of a* dC $(C, \mathcal{M})$ *is* **complete** $\stackrel{\Delta}{\Longleftrightarrow}$
for all $g : c \rightharpoonup a$ *there exists* $f : c \to a$ *s.t.* $g \leq f$ *(i.e.* f *extends* g*)*

If we restrict attention to complete objects, partial morphisms become *redundant*, because we can always extend them to total morphisms. The sub-category of complete objects is closed w.r.t. most of the data type constructions:

Proposition 21 *Complete Objects and Data Types*

1. *if $a \lhd b$ in C and b is complete, then so is a*
2. *if a and b are complete, then $a \times b$ is complete*
3. *if b is complete, then $a \to b$ is complete*
4. *$a \rightharpoonup b$ is complete*

Proof

- Suppose that $a \lhd b$ via (in, out). Given $f : c \rightharpoonup a$, let g be an extension of $in \circ f$, then $out \circ g$ extends f
- Given $f : c \rightharpoonup a \times b$, let g be an extension of $\pi_1 \circ f$ and h an extension of $\pi_2 \circ f$, then $\langle g, h \rangle$ extends f
- Given $f : c \rightharpoonup (a \to b)$, let g be an extension of $\mathring{e}val \circ (f \times \mathring{id}_a)$, then $\mathring{\Lambda}(g)$ extends f
- If $f : c \rightharpoonup (a \rightharpoonup b)$, then $\mathring{p}\Lambda(\mathring{p}eval \circ (f \times \mathring{id}_a))$ extends f

∎

There is a simpler way of checking whether an object is complete, provided the lifting exists:

Proposition 22 *a is complete $\iff a \lhd a_\perp$ in C*

Proof

If a is complete, then there exists $f : a_\perp \to a$ extending $\mathring{open} : a_\perp \rightharpoonup a$. But $\mathring{id}_a = \mathring{open} \circ \overline{\mathring{id}_a} \le f \circ \overline{\mathring{id}_a}$, because composition is monotonic; since $\mathring{id}_a$ is maximal, the equality $f \circ \overline{\mathring{id}_a} = \mathring{id}_a$ must hold. In other words $a \lhd a_\perp$ in C.

$a_\perp$ is complete (by *Prop 13* and *Prop 21* (4)), therefore, by *Prop 21* (1) and $a \lhd a_\perp$, a is complete ∎

In categories like $\mathring{E}N$ and $\mathring{C}PO$, complete objects have particularly interesting properties. For instance the complete objects in $\mathring{C}PO$ are exactly the cpos with a least element, and therefore they have a least fixpoint operator. In $\mathring{E}N$ the complete objects satisfy a property similar to the II Recursion Theorem for partial functions:

- for all $f : \underline{A} \to \underline{A}$ in $\mathring{E}N$ there exists an $a \in A$ s.t. $f(a) = a$

This property is actually characteristic of the so called **precomplete** objects (see [Ers73]):

Definition 23 *An object* $\underline{A}$ *of* $\mathring{E}N$ *is* **precomplete** $\overset{\Delta}{\Longleftrightarrow}$
for all $g\colon \underline{\omega} \rightharpoonup \underline{A}$ *there exists* $f\colon \underline{\omega} \to \underline{A}$ *s.t.* f *extends* g

Example. The numbered set $(\mathring{P}R, \phi)$ of partial recursive functions (ϕ is the *standard* gödel-numbering, which maps an index for a partial recursive function to the corresponding partial recursive function) is a complete object in $\mathring{E}N$.

Example. ([Vis80]) Take an *elementary* bijective coding $\lceil \_ \rceil$ of Λ (the set of λ-terms) in ω. For any λ-theory T (considered as an equivalence relation over Λ) the term model $M_T \triangleq \Lambda/T$ for T can be numbered by the function λ_T, which maps the coding $\lceil M \rceil$ for M to the equivalence class $[M]_T$. The resulting numbered set (M_T, λ_T) is always precomplete, but in general is not complete.

2 Generalized Numbered Sets

In the category of numbered sets $\mathring{E}N$, the lack of (partial) function spaces (and also equalizers) is due to the fact that sometimes there is no *onto numbering* of $\mathring{E}N(\underline{A}, \underline{B})$. For instance, in the case of total recursive functions we don't have an *effective* numbering, as for the partial ones. Nevertheless we have *effective* functions from ω to $\mathring{R}$, like $\lambda x.\lambda y.f(\langle x, y\rangle)$ where $f \in \mathring{R}$ and $\langle \_, \_ \rangle$ is some *effective coding* for pairs of natural numbers, but none of these functions can be *onto*. This observation leads to the definition of *Generalized Numbered Sets* (see [LM84a] and *Def* 29). In this section we prove that the category $\mathring{G}EN$ of Generalized Numbered Sets is a *pCCC* with all limits, colimits and function spaces. Moreover, we show that $\mathring{E}N$ is embedded fully and faithfully in $\mathring{G}EN$ and that the embedding preserves limits, finite colimits and (partial) function spaces already existing in $\mathring{E}N$. Therefore, $\mathring{G}EN$ is a *conservative extension* of $\mathring{E}N$ with *good closure properties*.

The definitions of $\mathring{E}N$ and $\mathring{G}EN$ rely on few simple properties of *partial* recursive functions, so we can *relativize* both of them w.r.t. an *acceptable* set L of partial endofunctions on a set A. This leaves us with some degree of freedom in choosing the set of *computable* functions. For instance the set of partial functions representable in a Uniform Reflexive Structure[1] is acceptable and can replace the partial recursive functions.

Notation. In the sequel we use the following notational conventions

- A denotes a non-empty set

[1] i.e. a partial combinatory algebra with a combinator IF for testing equality ($IFxyuv \triangleq$ if $x = y$ then u else v). URS were proposed to capture some properties of partial recursive functions, like the s-m-n Theorem and the II Recursion Theorem

- L denotes an *acceptable* set (see *Def* 24), unless stated otherwise
- If *op* is an n-ary operation, and $X_1, \ldots, X_n$ are sets, then $op(X_1, \ldots, X_n)$ denotes the set $\{op(x_1, \ldots, x_n) | x_1 \in X_1, \ldots, x_n \in X_n\}$. Sometimes we will write x instead of singleton $\{x\}$
- $\mathring{K}_x$ is the constant function $(\lambda y \in Y.x)$, where the set Y will be apparent from the context

Definition 24 *Acceptable Set*
$L \subseteq A \rightharpoonup A$ *is* **acceptable** $\overset{\Delta}{\Longleftrightarrow}$

1. *L is a monoid, i.e. $L \circ L \subseteq L$ and $\mathring{id}_A \in L$*
2. *L has all constants functions, i.e. $\mathring{K}_a \in L$ for any $a \in A$*
3. *there is an effective coding of pairs w.r.t. L, i.e. $(A \times A) \lhd_{\mathring{p}} A$ via $(in_\times, out_\times)$ so that $p_i \triangleq \pi_i \circ out_\times \in L$ and $in_\times \circ \langle L, L\rangle \subseteq L$*

   when there is a coding of pairs, we will write $\langle x, y\rangle$ instead of $in_\times(\langle x, y\rangle)$, and it will be apparent from the context whether $\langle x, y\rangle$ denotes a real pair or its coding

4. *there is an effective coding of sum w.r.t. L, i.e. $A + A \lhd_{\mathring{p}} A$ via (in_+, out_+) so that $in_i \triangleq in_+ \circ \mathring{inj}_i \in L$ and $(L \sqcup L) \circ out_+ \subseteq L$*

   if f and g are functions whose domain is included in A, then we will write $f \vee g$ for $(f \sqcup g) \circ out_+$

5. *L is closed w.r.t. definition by cases, i.e.*

$$switch \triangleq in_+ \circ (in_\times + in_\times) \circ \alpha \circ (out_+ \times \mathring{id}_A) \circ out_\times \in L$$

   where $\alpha: (A + A) \times A \to (A \times A) + (A \times A)$ is the canonical isomorphism that maps $\langle \mathring{inj}_i(a), b\rangle$ to $\mathring{inj}_i(\langle a, b\rangle)$

It is easy to show that two effective codings, (in, out) and (in', out'), are *effectively equivalent*, i.e. there exist f and g in L s.t. $out = out' \circ f$ and $out' = out \circ g$. So we don't care about which specific effective coding we are using. The name "definition by cases", given to the last property for an acceptable set, requires some justification:

Proposition 25 *If g, f_1 and f_2 are in L, then the partial function*

$$h(a) \triangleq \begin{cases} f_1(a) & \text{if } out_+(g(a)) \in \mathring{img}(\mathring{inj}_1) \\ f_2(a) & \text{if } out_+(g(a)) \in \mathring{img}(\mathring{inj}_2) \\ undefined & otherwise \end{cases}$$

is also in L

Proof

$$h = ((f_1 \circ p_2) \vee (f_2 \circ p_2)) \circ switch \circ in_{\times} \circ \langle g, \mathring{id}_A \rangle \qquad \blacksquare$$

This property of L is essential in proving that $\mathring{G}EN_L$ has function spaces.

In the definition of L-numbered set (see *Def* 28), we cannot simply take a set X together with a surjective total function $\mathring{e}_X: A \to X$, because on the resulting category one cannot define the appropriate domain structure[2]. Therefore, we relax the *totality* requirement for $\mathring{e}_X$:

Definition 26 *$F(L)$ is the class of set-theoretic functions whose domain coincides with the domain of some function in L, i.e.*
$F(L) \stackrel{\Delta}{=} \{f | \exists g \in L. \mathring{dom}(f) = \mathring{dom}(g)\}$.

*L-**reducibility** is the preorder over $F(L)$ s.t. $f \leq g \stackrel{\Delta}{\iff}$ there exists $h \in L$ s.t. $f = g \circ h$*

The following property of $F(L)$ is crucial in establishing most of the results about $\mathring{G}EN_L$:

Proposition 27 *If f_1 and f_2 are in $F(L)$, then $f_1 \vee f_2$ is in $F(L)$ and its a lub of f_1 and f_2 w.r.t. the L-reducibility preorder*

Proof

$f_1 \vee f_2$ is clearly in $F(L)$. $f_i \leq f_1 \vee f_2$, because $f_i = (f_1 \vee f_2) \circ in_i$. If $f_i = f \circ g_i$, with g_i in L, then $f_1 \vee f_2 = f \circ (g_1 \vee g_2)$ $\blacksquare$

Since L-reducibility is a preorder lubs are not unique.

Before introducing generalized numbered sets, we describe how numbered sets are relativized to L:

Definition 28 *The category $\mathring{E}N_L$ of L-**numbered sets** and L-**effective morphisms** is defined as follows:*

- *the objects are pairs $\underline{X} = (X, \mathring{e}_X)$ s.t. $\mathring{e}_X \in F(L)$ and X is the image of $\mathring{e}_X$*
- *f is a morphism from $\underline{X}$ to $\underline{Y}$ $\stackrel{\Delta}{\iff}$ f is a function from X to Y and there exists an $f' \in L$ s.t. $f \circ \mathring{e}_X = \mathring{e}_Y \circ f'$*

  We can assume that f' and $\mathring{e}_X$ have the same domain

For instance, the category of numbered set (as defined in *Def* 1) without the strict initial object is just $\mathring{E}N_{\mathring{R}}$, while the category of numbered sets with the strict initial object is equivalent to $\mathring{E}N_{\mathring{P}R}$. In fact if $\underline{X} \in \mathring{E}N_{\mathring{P}R}$ is not initial, then there is a total recursive function f that enumerates $\mathring{dom}(\mathring{e}_X)$, so that $(X, \mathring{e}_X \circ f)$ is isomorphic to $\underline{X}$ in $\mathring{E}N_{\mathring{P}R}$ and $\mathring{e}_X \circ f: \omega \to X$ is onto.

$\mathring{E}N_L$ is a concrete category with finite products and finite colimits, but it's neither a *CCC* nor a *pCCC* (see *Def* 29 for the definition of L-effective partial morphisms).

[2]this kind of problem can be avoided by using the notion of p-category (see [Ros86])

Definition 29 *The category* $\mathring{G}EN_L$ *of* **generalized** L**-numbered sets** *and* L**-effective morphisms** *is defined as follows:*

- *the objects are pairs* $\underline{X} = (X, E_X)$ *s.t.*
  1. E_X *is a subset of* $F(L)$
  2. $X = \bigcup\{\mathring{img}(e) | e \in E_X\}$
  3. $\forall f_1, f_2 \in E_X . \exists f \in E_X . f_1, f_2 \leq f$*, i.e.* E_X *is directed*
- f *is a (partial) morphism from* $\underline{X}$ *to* $\underline{Y}$ $\stackrel{\Delta}{\iff}$ f *is a (partial) function from* X *to* Y *and* $f \circ E_X \subseteq E_Y \circ L$

The intuition behind generalized numbered sets is that one cannot gödelize $\mathring{R}$, but one can *effectively* enumerate it *piecewise*. It's immediate from the definition that the objects of $\mathring{G}EN_L$ are in one-one correspondence with the directed subsets of $F(L)$ and that $\mathring{E}N_L$ is a full subcategory of $\mathring{G}EN_L$, by identifying the L-numbered set $(X, \mathring{e}_X)$ with the generalized L-numbered set $(X, \{\mathring{e}_X\})$. L-effective partial morphisms are induced by the domain structure of L-subobjects:

Definition 30 $\mathcal{M}_L$

$X \subseteq Y$ *is an* L**-subset** *of* $\underline{Y} \in \mathring{G}EN_L$ $\stackrel{\Delta}{\iff}$ $e^{-1}(X)$ *is the domain of a function in* L*, for any* $e \in E_Y$*. The* L**-subobjects** *of* $\underline{Y}$ *are the equivalence classes of monos* $incl_X : \underline{X} \hookrightarrow \underline{Y}$*, where* X *is an* L*-subset of* $\underline{Y}$ *and* $E_X \stackrel{\Delta}{=} \{e \in E_Y \circ L | \mathring{img}(e) \subseteq X\}$

$\mathcal{M}_L$ *is the* **domain structure of** L**-subobjects**

The set A has a *canonical* L-numbered set structure, $\underline{A} \stackrel{\Delta}{=} (A, \mathring{id}_A)$, so that L is the set of L-effective partial endomorphisms on $\underline{A}$.

Proposition 31 *Properties of* $\mathcal{M}_L$

- *the family* $\mathcal{M}_L$ *of* L*-subobjects is a domain structure on* $\mathring{G}EN_L$*, and* $(\mathring{G}EN_L, \mathcal{M}_L)$ *is a concrete category with domains*
- f *is an* L*-effective partial morphism from* $\underline{X}$ *to* $\underline{Y}$ $\stackrel{\Delta}{\iff}$ $Z \stackrel{\Delta}{=} \mathring{dom}(f)$ *is an* L*-subset of* $\underline{X}$ *and* f *is an* L*-effective morphism from* $\underline{Z}$ *to* $\underline{Y}$
- *If* $[i : \underline{X} \hookrightarrow \underline{Y}]$ *is an* L*-subobject and* $\underline{Y}$ *is an* L*-numbered set, then* $\underline{X}$ *is (isomorphic to) an* L*-numbered set*

The last property means that $\mathcal{M}_L$ can be *restricted* to $\mathring{E}N_L$.

There are three functors that will be used in analysing the structure of $\mathring{G}EN_L$: the forgetful functor and two full and faithful embeddings of $\mathring{S}ET$ in $\mathring{G}EN_L$

Definition 32 Γ *,* Δ *and* $\_^A$

- $\Gamma: \mathring{G}EN_L \to \mathring{S}ET$ *is the forgetful functor, i.e.* $\Gamma: \underline{X} \mapsto X$
- $\Delta: \mathring{S}ET \to \mathring{G}EN_L$ *is the functor which maps* X *in* (X, E_X^{min}), *where* E_X^{min} *is the smallest ideal of* $F(L)$ *s.t.* (X, E_X^{min}) *is a generalized* L*-numbered set, more explicitly*

$$E_X^{min} = \{f \in F(L) | \exists X_0 \subseteq_{fin} X. f \leq \bigvee_{x \in X_0} \mathring{K}_x\}$$

- $\_^A: \mathring{S}ET \to \mathring{G}EN_L$ *is the functor which maps* X *in* (X, E_X^{max}), *where* E_X^{max} *is the biggest ideal of* $F(L)$ *s.t.* (X, E_X^{max}) *is a generalized* L*-numbered set, more explicitly*

$$E_X^{max} = \{f \in F(L) | \mathring{i}mg(f) \subseteq X\}$$

Δ and $\_^A$ are well defined, because lubs in $F(L)$ exist and are *well-behaved* (see *Prop 27*).

Lemma 33 Δ *and* $\_^A$ *are respectively left and right adjoints of* Γ

The immediate consequence of this lemma is that Γ preserves both limits and colimits. Now we show that $\mathring{G}EN_L$ is closed w.r.t. limits, colimits and (partial) function spaces.

Theorem 34 $\mathring{G}EN_L$ *has limits and colimits*

Proof

If $F: I \to \mathring{G}EN_L$ is a small diagram in $\mathring{G}EN_L$ and $\pi: \underline{X} \dot{\to} F$ is a limit cone, then $\pi: X \dot{\to} \Gamma \circ F$ must be a limit cone in $\mathring{S}ET$, because Γ preserves limits (and is the identity on L-effective morphisms). So we are left to find E_X that makes π a limit cone in $\mathring{G}EN_L$, let

$$E_X = \{f \in F(L) | \mathring{i}mg(f) \subseteq X \wedge \forall i \in I. \pi_i \circ f \in E_{Fi} \circ L\}$$

we have to check that:

- $\underline{X}$ is a generalized L-numbered set (see *Def* 29).
  The first condition is obvious. The second is true, because all constant functions $\mathring{K}_x$ (for $x \in X$) are in E_X. The third follows, because if f and g are in E_X, then $f \vee g$ is in E_X; in fact
  $\pi_i \circ (f \vee g) = \pi_i \circ (f \sqcup g) \circ out_+ = (\pi_i \circ f \sqcup \pi_i \circ g) \circ out_+ = (\pi_i \circ f) \vee (\pi_i \circ g) \in E_{Fi} \circ L$, because $E_{Fi} \circ L$ is an ideal
- π is an L-effective morphism from $\underline{X}$ to Fi.
  This follows immediately from the definition of E_X

- If $\eta: \underline{Y} \dot{\to} F$ is a cone in $\mathring{G}EN_L$, then the mediating morphism $f: Y \to X$ in $\mathring{S}ET$ (s.t. $\eta = \pi \circ f$) is an L-effective morphism from $\underline{Y}$ to $\underline{X}$ (see *Def* 29)
  If $e \in E_Y$, then $f \circ e \in E_X$, because
  $\pi \circ (f \circ e) = (\pi \circ f) \circ e = \eta_i \circ e \in E_{Fi}$

The uniqueness of the mediating morphism in $\mathring{G}EN_L$ follows from the faithfulness of Γ.

Similarly for colimits one shows that $\mathring{inj}: F \dot{\to} \underline{X}$ is a colimit cone in $\mathring{G}EN_L$ iff $\mathring{inj}: \Gamma \circ F \dot{\to} X$ is a colimit cone in $\mathring{S}ET$ and E_X is the smallest ideal containing $\mathring{inj}_i \circ E_{Fi}$ for all $i \in I$. In fact

- $\underline{X}$ is a generalized L-numbered set (see *Def* 29)
  The first and third conditions are obvious. The second condition follows from $X = \bigcup\{\mathring{img}(\mathring{inj}_i) | i \in I\}$ and the second condition for E_{Fi}
- $\mathring{inj}_i$ is an L-effective morphism from Fi to $\underline{X}$
  This follows immediately from the definition of E_X
- If $\eta: F \dot{\to} \underline{Y}$ is a cone in $\mathring{G}EN_L$, then the mediating morphism $f: X \to Y$ in $\mathring{S}ET$ (s.t. $\eta = f \circ \mathring{inj}$) is an L-effective morphism from $\underline{X}$ to $\underline{Y}$ (see *Def* 29)
  In fact, $f \circ E_X$ is the smallest ideal containing $f \circ \mathring{inj}_i \circ E_{Fi}$ for all $i \in I$, but each of them is contained in the ideal $E_Y \circ L$, therefore $f \circ E_X \subseteq E_Y \circ L$

∎

Now we turn our attention to function spaces. There is a small full subcategory of $\mathring{E}N_L$, which plays an important role in the study of $\mathring{G}EN_L$:

Definition 35 *$\mathring{D}om_L$ is the full sub-category of $\mathring{E}N_L$, whose objects are $\underline{D} \triangleq (D, \mathring{id}_D)$, where D is an L-subset of $\underline{A}$.*

$\_'$ is the functor from $\mathring{G}EN_L$ to the category of presheaves over $\mathring{D}om_L$ given by currying the functor $(\underline{X}, \underline{D}) \mapsto \mathring{G}EN_L(\underline{D}, \underline{X})$ from $\mathring{G}EN_L \times \mathring{D}om_L{}^{op}$ to $\mathring{S}ET$

The main properties of the functor $\_'$ are:

Lemma 36 *The functor $\_'$ is full and faithful, and preserves limits and function spaces*

Proof

faithfulness is obvious. Fix an element $*$ of A. If $\eta: \underline{X}' \dot{\to} \underline{Y}'$, let $g(x) \triangleq \eta_A(\mathring{K}_x)(*)$ for all $x \in X$, then for any $\underline{D} \in \mathring{D}om_L$, L-effective morphism $f: \underline{D} \to \underline{X}$ and $a \in D$, the following equalities hold:

$$\eta_D(f)(a) = (\eta_D(f) \circ \mathring{K}_a)(*) = \eta_D(f \circ \mathring{K}_a)(*) = \eta_A(\mathring{K}_{fa})(*) = g(fa)$$

therefore $\eta_D(f) = g \circ f$ and to complete the proof of fullness we have to show that g is an L-effective morphism, i.e. $g \circ E_X \subseteq E_Y \circ L$. Take $e \in E_X$, then $e: \underline{D} \to \underline{X}$, where D is the domain of e, therefore

$$g \circ e = \eta_D(e) \in \mathring{G}EN_L(\underline{D}, \underline{Y}) \subseteq E_Y \circ L$$

Preservation of limits and function spaces is trivial ∎

By this lemma, two generalized L-numbered sets are isomorphic iff as presheaves they are isomorphic.

Theorem 37 *$\mathring{G}EN_L$ has function spaces and partial function spaces*

Proof

By *Lemma 36*, the only possible candidate for the function space from $\underline{X}$ to $\underline{Y}$ is a generalized L-numbered set $\underline{Z}$ s.t. $\mathring{G}EN_L(\underline{D}, \underline{Z}) \dot{\cong} \mathring{G}EN_L(\underline{D} \times \underline{X}, \underline{Y})$, for all $\underline{D} \in \mathring{D}om_L$. Therefore, *up to isomorphisms*

$$E_Z = \{\mathring{\Lambda}(f) | \exists \underline{D} \in \mathring{D}om_L . f \in \mathring{G}EN_L(\underline{D} \times \underline{X}, \underline{Y})\}$$

It's immediate from the definition that $E_Z \subset F(L)$ and that the second condition (in *Def* 29) is satisfied provided $Z = \mathring{G}EN_L(\underline{X}, \underline{Y})$. The third condition will follow, if we show that $F_3 \triangleq F_1 \vee F_2 \in E_Z$ whenever F_1 and F_2 are in E_Z.

Let $F_i = \mathring{\Lambda}(f_i)$, then (when i is 1 or 2) $f_i \in \mathring{G}EN_L(\underline{D_i} \times \underline{X}, \underline{Y})$, where D_i is the domain of F_i. Let D_3 be the domain of $\mathring{id}_{D_1} \vee \mathring{id}_{D_2}$, then $\underline{D_3}$ is in $\mathring{D}om_L$ and we want to show that $f_3 \in \mathring{G}EN_L(\underline{D_3} \times \underline{X}, \underline{Y})$, i.e. $f_3 \circ \langle g, e \rangle \in E_Y \circ L$ when $g \in E_{D_3} \circ L = \mathring{id}_{D_3} \circ L$ and $e \in E_X \circ L$

- let $e_i \triangleq f_i \circ \langle (\mathring{id}_{D_i} \vee \mathring{id}_{D_i}) \circ g, e \rangle$, by the assumptions on f_i, $e_i \in E_Y \circ L$
- let h be the partial function s.t.

$$h(a) \triangleq \begin{cases} in_1(a) & \text{if } out_+(g(a)) \in \mathring{i}mg(\mathring{i}nj_1) \\ in_2(a) & \text{if } out_+(g(a)) \in \mathring{i}mg(\mathring{i}nj_2) \\ undefined & \text{otherwise} \end{cases}$$

  by *Prop 25*, h is in L
- $f_3 \circ \langle g, e \rangle = (e_1 \vee e_2) \circ h$, in fact

$$(f_3 \circ \langle g, e \rangle)(a) = F_3(ga)(ea) = \begin{cases} F_1(a_1)(ea) & \text{if } out_+(g(a)) = \mathring{i}nj_1(a_1) \\ F_2(a_2)(ea) & \text{if } out_+(g(a)) = \mathring{i}nj_2(a_2) \end{cases}$$

  and

$$((e_1 \vee e_2) \circ h)(a) = \begin{cases} e_1(a) = f_1(a_1, ea) & \text{if } out_+(g(a)) = \mathring{i}nj_1(a_1) \\ e_2(a) = f_2(a_2, ea) & \text{if } out_+(g(a)) = \mathring{i}nj_2(a_2) \end{cases}$$

and this completes the proof, because $(e_1 \vee e_2) \circ h \in E_Y \circ L$

Similarly one shows that *up to isomorphisms* a partial function space $\underline{Z}$ from $\underline{X}$ to $\underline{Y}$ is $Z = \mathring{P}(\mathring{G}EN_L, \mathcal{M}_L)(\underline{X}, \underline{Y})$ and

$$E_Z = \{\mathring{\Lambda}(f) | \exists \underline{D} \in \mathring{D}om_L . f \in \mathring{P}(\mathring{G}EN_L, \mathcal{M}_L)(\underline{D} \times \underline{X}, \underline{Y})\}$$

∎

The conditions required for L to be acceptable (see *Def* 24) have a counterpart in $\mathring{G}EN_L$:

Proposition 38

- $\underline{A} \times \underline{A} \lhd_{\mathring{p}} \underline{A}$ *(coding of pairs)*
- $\underline{A} + \underline{A} \lhd_{\mathring{p}} \underline{A}$ *(coding of sum)*
- $(\underline{X} + \underline{Y}) \times \underline{Z} \cong (\underline{X} \times \underline{Z}) + (\underline{Y} \times \underline{Z})$ *(definition by cases)*

Proof

- it's an immediate consequence of coding of pairs
- it's an immediate consequence of coding of sum
- $\_ \times \underline{Z}$ is left-adjoint of $\underline{Z} \rightarrow \_$. Therefore, it preserves coproducts (more geneally colimits), which means exactly the distributivity of binary products over coproducts

∎

The generalized L-numbered sets don't have a uniform coding for their elements (as in the case of L-numbered sets), so we want to remain in $\mathring{E}N_L$ as far as possible. This aim is not incompatible with *working in* $\mathring{G}EN_L$, provided $\mathring{E}N_L$ is embedded fully and faithfully in $\mathring{G}EN_L$ (which has been already established) and every *categorical construction* that can be performed in $\mathring{E}N_L$ can be performed in $\mathring{G}EN_L$ and yields the same result.

Theorem 39 *The embedding of $\mathring{E}N_L$ in $\mathring{G}EN_L$ preserves limits and (partial) function spaces*

Proof

The proof makes essential use of *Lemma 36*. We consider only the case of binary products, because the other cases are similar. Let $\underline{Z}$ be the product of $\underline{X}$ and $\underline{Y}$ in $\mathring{E}N_L$, then for any $\underline{D} \in \mathring{D}om_L$

$\mathring{G}EN_L(D, \underline{Z}) = \mathring{E}N_L(D, \underline{Z}) \dot{\cong}$
$\mathring{E}N_L(D, \underline{X}) \times \mathring{E}N_L(D, \underline{Y}) = \mathring{G}EN_L(D, \underline{X}) \times \mathring{G}EN_L(D, \underline{Y}) \dot{\cong}$
$\mathring{G}EN_L(D, \underline{X} \times \underline{Y})$

therefore, $\underline{Z} \cong \underline{X} \times \underline{Y}$ ∎

The embedding of $\mathring{E}N_L$ in $\mathring{G}EN_L$, unlike the Yoneda embedding of a category in the corresponding category of presheaves, preserves also some colimits:

Theorem 40 *$\mathring{E}N_L$ has binary coproducts and coequalizers and they are preserved by the embedding of $\mathring{E}N_L$ in $\mathring{G}EN_L$*

Proof

We already know how to compute colimits in $\mathring{G}EN_L$, since $\mathring{E}N_L$ is a full sub-category of $\mathring{G}EN_L$, the claim amounts to showing that the colimit (object) under consideration is in $\mathring{E}N_L$

- $\underline{X} + \underline{Y} = (X + Y, \mathring{e}_X \vee \mathring{e}_Y)$
- $coeq(f, g) = (Z, h \circ \mathring{e}_Y)$, where $f, g: \underline{X} \to \underline{Y}$ and $h: Y \to Z$ is the coequaliser of f and g in $\mathring{S}ET$

∎

However, the initial object of $\mathring{E}N_L$ (when it exists) is not preserved.

3 Generalized Banach-Mazur Functionals

The category $\mathring{G}EN_L$ of *generalized L-numbered sets* provides an alternative characterization of the type structure $\{\mathring{L}_n | n \in \omega\}$ of the *Hereditary Partial Effective Functionals* over an acceptable set L, defined in [Lon82]:

Definition 41 *(Longo) Hereditary Partial Effective Functionals*

Let $L \subseteq A \rightharpoonup A$ be a monoid of partial functions, then the $HPEF$ over L are defined as follows:

- $\mathring{L}_0 = A$
- $\mathring{L}_1 = L$
- $\mathring{L}_{n+1.5} = \{g: \mathring{L}_n \to \mathring{L}_{n+1} | \exists f \in \mathring{L}_{n+1}. \forall x, y \in \mathring{L}_n . g(x)(y) = f(\langle x, y \rangle_n)\}$

  where $\langle \_, \_ \rangle_n: \mathring{L}_n \times \mathring{L}_n \to \mathring{L}_n$ is an effective coding of pairs for $\mathring{L}_n$ w.r.t. $\mathring{L}_{n+1}$ (see Def 24)
- $\mathring{L}_{n+2} = \{f: \mathring{L}_{n+1} \to \mathring{L}_{n+1} | f \circ \mathring{L}_{n+1.5} \subseteq \mathring{L}_{n+1.5}\}$

The key idea in the definition above is that the functions in $\mathring{L}_{n+1.5}$ *gödelize* $\mathring{L}_{n+1}$ by $\mathring{L}_n$, and the coding of pairs $\langle \_, \_ \rangle_n$ is used to define the $n+1$ level in the same way as the effective coding of pairs for ω is used to define the Banach-Mazur Functionals (see [Rog67]). In general the $\langle \_, \_ \rangle_n$ may not exist. However, under the assumption that L is acceptable, all codings of pairs required in the definition of $\{\mathring{L}_n | n \in \omega\}$ exist (see *Lemma 48*). In this section we show that the Hereditary Partial Effective Functionals over L actually *live* in $\mathring{G}EN_L$.

Definition 42 *Partial Functionals*

Let C be a pCCC and a an object of C, then the type structure $\{E_\sigma|\sigma \in T\}$ of partial functionals over a in C is defined by induction on the functional types T ($T ::= 0|T \to T$) as follows:

- $E_0 = a$
- $E_{\sigma\to 0} = E_\sigma \rightharpoonup a$
- $E_{\sigma\to\tau} = E_\sigma \to E_\tau$ *if* $\tau \neq 0$

In the last case we use a function space, but if $\tau \neq 0$, then $E_\tau \cong x \rightharpoonup a$ for some x (the proof is by induction on $\tau \in T$), therefore $E_\sigma \to E_\tau \cong (E_\sigma \times x) \rightharpoonup a$.

We will consider the Partial Functionals in $\mathring{G}EN_L$ and show that (over the integer types $n+1 = n \to n$) they correspond to the Hereditary Partial Effective Functionals (see *Def* 41). In fact, the definition of $\mathring{L}_{n+2}$ in terms of $\mathring{L}_{n+1.5}$ is very similar to the definition of $\mathring{G}EN_L(\underline{X},\underline{Y})$ in terms of E_X and E_Y. We have the following general result, which relates $\{\mathring{L}_n|n \in \omega\}$ with the $\{E_n|n \in \omega\}$ on $\underline{A}$ in $\mathring{G}EN_L$ (see [LM84a]):

Theorem 43 *Main Theorem*

(n) $\mathring{L}_n = \Gamma\, E_n$

$(n+1.5)$ $\mathring{L}_{n+1.5} = \mathring{G}EN_L(E_n, E_{n+1})$

where Γ is the forgetful functor from $\mathring{G}EN_L$ to $\mathring{S}ET$ (see Def 32)

Although $\mathring{E}N$ is not cartesian closed, in [Ers75] Ershov shows that the Partial Functionals over $\underline{\omega}$ are well-defined in $\mathring{E}N$, and calls them the **Partial Computable Functionals**. In [Ers74] and [Ers77] these functionals are related to the Hereditary Effective Operations (HEO) and to the Countable Functionals (see [Kle59] and [Nor80]). By *Theor 39* the Partial Computable Functionals and the Partial Functionals over $\underline{\omega}$ in $\mathring{G}EN$ are the same, therefore, *Theor 43* implies the main result in [LM84b], namely the equivalence between the Hereditary Partial Effective Functionals on $\mathring{P}R$ and the Partial Computable Functionals of Ershov.

In order to prove *Theor 43* we must define when a generalized numbered set can be *numbered* by another generalized numbered set.

Definition 44 $\underline{X}$ **factorizes** $\underline{Y} \overset{\Delta}{\iff} E_Y \subseteq \mathring{G}EN_L(\underline{X},\underline{Y}) \circ E_X \circ L$

Proposition 45 *If $\underline{X}$ factorizes $\underline{Y}$, then*

$$f \in \mathring{G}EN_L(\underline{Y},\underline{Z}) \iff f \circ \mathring{G}EN_L(\underline{X},\underline{Y}) \subseteq \mathring{G}EN_L(\underline{X},\underline{Z})$$

for any $\underline{Z} \in \mathring{G}EN_L$ and set-theoretic function $f : Y \to Z$

Proof

The implication from left to right is obvious, because morphisms are closed w.r.t. composition. For the other implication, we have to show that $f \circ E_Y \subseteq E_Z \circ L$:
$f \circ E_Y \subseteq$ by $\underline{X}$ factorizes $\underline{Y}$
$f \circ \mathring{G}EN_L(\underline{X}, \underline{Y}) \circ E_X \circ L \subseteq$ by hypothesis
$\mathring{G}EN_L(\underline{X}, \underline{Z}) \circ E_X \circ L \subseteq$ by definition of $\mathring{G}EN_L$
$E_Z \circ L \circ L = E_Z \circ L$ ∎

The following is a sufficient condition that implies factorization, and will be used to show that E_n factorizes E_{n+1}:

Proposition 46 *If* $\underline{A} \lhd_{\mathring{p}} \underline{X}$ *and* $\underline{Y}$ *is complete, then* $\underline{X}$ *factorizes* $\underline{Y}$

Proof

Let $\underline{A} \lhd_{\mathring{p}} \underline{X}$ via (in, out). Since $\underline{Y}$ is complete, then for all $e \in E_Y$ (e is also a partial L-effective morphism from $\underline{A}$ to $\underline{Y}$) there exists an extension $f: \underline{X} \to \underline{Y}$ of the partial L-effective morphism $e \circ out$. An easy check shows that $e = f \circ (in \circ \mathring{id}_{\mathring{dom}(e)}) \in E_X \circ L$ ∎

In *Prop 38* it is shown that the coding of pairs w.r.t. L *corresponds* to the existence of a retraction in $\mathring{G}EN_L$. In general in a concrete category (with domains) a retraction $a \times a \lhd a$ ($a \times a \lhd_{\mathring{p}} a$) corresponds to an effective coding of pairing w.r.t. a suitable L. We state this correspondence in the case of $\mathring{G}EN_L$:

Proposition 47 *If* $\underline{X} \times \underline{X} \lhd \underline{X}$ *(*$\underline{X} \times \underline{X} \lhd_{\mathring{p}} \underline{X}$*) via* (in, out)*, then* (in, out) *is an effective coding of pairs (for* X*) w.r.t.* $\mathring{G}EN_L(\underline{X}, \underline{X})$ *(*$\mathring{P}(\mathring{G}EN_L, \mathcal{M}_L)(\underline{X}, \underline{X})$*)*

Lemma 48

1. E_{n+1} *is complete*
2. $\underline{A} \lhd_{\mathring{p}} E_{n+1}$
3. $E_{n+1} \times E_{n+1} \lhd E_{n+1}$

Proof

- By the remark after *Def* 42 $E_{n+1} \cong \underline{X} \rightharpoonup \underline{A}$ for some $\underline{X}$, therefore E_{n+1} is complete (by *Prop 21*)
- Since $\underline{A} \lhd_{\mathring{p}} A_\perp$, it's enough to show that $\underline{A}_\perp \lhd E_{n+1}$ (by induction on n):

  (1) $\underline{A}_\perp = \mathring{1} \rightharpoonup \underline{A} \lhd$ (by $\mathring{1} \lhd \underline{A}$)
  $\underline{A} \rightharpoonup \underline{A} = E_1$

$(n+1)$ $\underline{A}_\perp \cong \mathring{1} \to \underline{A}_\perp \lhd$ (by $\mathring{1} \lhd E_n$ and inductive hypothesis)
$E_n \to E_n = E_{n+1}$

- Also in this case the proof is by induction on n:

(1) $E_1 \times E_1 \cong (\underline{A} \rightharpoonup \underline{A}) \times (\underline{A} \rightharpoonup \underline{A}) \cong$ (by the following general fact: $(a+b) \rightharpoonup c \cong (a \rightharpoonup c) \times (b \rightharpoonup c)$)
$\underline{A} + \underline{A} \rightharpoonup \underline{A} \lhd$ (by $\underline{A} + \underline{A} \lhd_{\mathring{p}} \underline{A}$ and the following general fact: if $a \lhd_{\mathring{p}} b$, then $a \rightharpoonup c \lhd b \rightharpoonup c$)
$\underline{A} \rightharpoonup \underline{A}$

$(n+1)$ $E_{n+1} \times E_{n+1} = (E_n \to E_n) \times (E_n \to E_n) \cong$ (by the following general fact: $(a \to b) \times (a \to c) \cong a \to (b \times c)$)
$E_n \to (E_n \times E_n) \lhd$ (by the inductive hypothesis and the following general fact: if $b \lhd c$, then $a \to b \lhd a \to c$)
$E_n \to E_n \cong E_{n+1}$

∎

The third part of *Lemma 48* (together with *Theor 43*) means that there is an effective coding of pairs (for $\mathring{L}_n$) w.r.t. $\mathring{L}_{n+1}$.

Corollary 49 *E_n factorizes E_{n+1}*

Proof

It follows from (1) and (2) in *Lemma 48*, and *Prop 46* ∎

We can now prove *Theor 43*:
Proof

The proof is by induction, where the base steps are (0), (1) and (1.5) and the inductive steps are $(n+1.5)$ (which uses (n) and $(n+1)$) and $(n+2)$ (which uses $(n+1)$ and $(n+1.5)$):

(0) trivial

(1) trivial

(1.5) we have a retraction $\underline{A} \times \underline{A} \lhd_{\mathring{p}} \underline{A}$ (see *Prop 38*) which is also an effective coding of pairs (for $A = \mathring{L}_0$) w.r.t. $\mathring{P}(\mathring{G}EN_L, \mathcal{M}_L)(\underline{A}, \underline{A}) = L$ (use *Lemma 47*, (0) and (1) above), so that
$\mathring{L}_{1.5} =$
$\{f \mid \exists g \in L. \forall x, y \in A. f(x)(y) = g(\langle x, y\rangle)\} =$
$\{f \mid \exists g \in \mathring{P}(\mathring{G}EN_L, \mathcal{M}_L)(\underline{A}, \underline{A}). \forall x, y \in A. f(x)(y) = g(\langle x, y\rangle)\} =$
$\{f \mid \exists g \in \mathring{P}(\mathring{G}EN_L, \mathcal{M}_L)(\underline{A} \times \underline{A}, \underline{A}). \forall x, y \in A. f(x)(y) = g(x, y)\} =$
$\mathring{G}EN_L(\underline{A}, \underline{A} \rightharpoonup \underline{A}) =$
$\mathring{G}EN_L(E_0, E_1)$

$(n+2)$ let $f\colon \mathring{L}_{n+1} \to \mathring{L}_{n+1}$:
$f \in \mathring{L}_{n+2} \iff$
$f \circ \mathring{L}_{n+1.5} \subseteq \mathring{L}_{n+1.5} \iff$ by $(n+1.5)$
$f \circ \mathring{G}EN_L(E_n, E_{n+1}) \subseteq \mathring{G}EN_L(E_n, E_{n+1}) \iff$ by *Corr 49*
$f \in \mathring{G}EN_L(E_{n+1}, E_{n+1}) = \Gamma\, E_{n+2}$

$(n+2.5)$ similar to case (1.5), but using (3) of *Lemma 48* instead of *Prop 38*

∎

4 Numbered Sets and Sheaves

Instead of introducing $\mathring{G}EN_L$ we could have used a more canonical cartesian closed extension of $\mathring{E}N_L$, namely the topos of presheaves over $\mathring{E}N_L$ (see [Sco80]). Moreover, the Yoneda embedding preserves limits and (partial) function spaces, so *Theor 39* is for free. A more elaborate construction, that sometimes makes it possible to preserve even more structure (e.g. colimits), is the topos of sheaves for a *subcanonical Grothendieck topology* (for details, see [Joh77] Section 0.3 and [MR77] Section 1.1). The latter approach is used in [Mul81] to define the recursive topos $\mathcal{R}$, in which $\mathring{E}N$ is embedded fully and faithfully and the embedding preserves limits, finite colimits and function spaces. The relation between $\mathring{G}EN$ and the recursive topos is investigated in [Ros86], where it is shown that $\mathring{G}EN$ is the *quasitopos* of *separated* objects in $\mathcal{R}$ for the *double-negation topology.* This characterization of $\mathring{G}EN$ *relativizes* (somehow) to $\mathring{G}EN_L$. More precisely, we define a topos $\mathcal{R}_L$ and show that $\mathring{G}EN_L$ is the quasitopos of separated objects in $\mathcal{R}_L$ for a suitable *topology.* When L is an acceptable set of total functions, this topology has a *logical nature*, namely is the *double-negation* topology (see [Mul81]). This characterization of $\mathring{G}EN_L$, gives us a lot of information. For instance, that $\mathring{G}EN_L$ has limits, colimits and function spaces. Moreover, one can relate the internal logics in $\mathcal{R}_L$, $\mathring{G}EN_L$ and $\mathring{S}ET$, as described in [Hyl82] and [Ros86].

In the following we define a topology J_{can} on $\mathring{G}EN_L$ and prove that it is the *canonical topology* on $\mathring{G}EN_L$. It's important to note that the canonical topologies on $\mathring{G}EN_L$, $\mathring{E}N_L$ and $\mathring{D}om_L$ (see *Def* 35) may *induce* different toposes (e.g. when L is the set of all total endofunctions on ω), although these toposes are actually equivalent when L is the set of total recursive functions.

Definition 50 *A sieve R over $\underline{X}$ is in $J_{can}(\underline{X}) \overset{\Delta}{\iff} E_X$ is included in the ideal generated by $\bigcup\{f \circ E_Y \mid f\colon \underline{Y} \to \underline{X} \in R\}$*

Proposition 51 *J_{can} is a Grothendieck topology on $\mathring{G}EN_L$*

Proof

1. there is a *topologically generating set* (see [MR77] Definition 1.1.1), namely the objects of $\mathring{D}om_L$

2. the maximum sieve over $\underline{X}$ is in $J_{can}(\underline{X})$
3. If $R \in J_{can}(\underline{Y})$, $f: \underline{Y} \to \underline{X}$ is a morphism in $\mathring{G}EN_L$, then $f^{-1}(R) \triangleq \{g | f \circ g \in R\} \in J_{can}(\underline{X})$. More explicitly, for any $e \in E_Y$ we have to find $W \subseteq_{fin} f^{-1}(R)$ and $e_g \in E_Z$ (for any $g: \underline{Z} \to \underline{Y} \in W$) s.t. $e \leq \bigvee\{g \circ e_g | g \in W\}$. The proof of this makes essential use of *Prop 25*. Since $f \circ e$ is in $E_X \circ L$ and R is in $J_{can}(\underline{X})$, there exist $V \subseteq_{fin} R$ and $e_g \in E_Z$ (for any $g: \underline{Z} \to \underline{X} \in V$) s.t. $f \circ e \leq \bigvee\{g \circ e_g | g \in V\}$. For simplicity we assume that V has only two elements, g_1 and g_2, s.t.
   - $g_i: \underline{D_i} \to \underline{X}$
   - $f \circ e = (g_1 \vee g_2) \circ h$ for some $h \in L$
   - there exist a_i and b_i s.t. $f(a_i) = g_i(b_i)$

   Then we can define the following functions:
   $$e_i(x) \triangleq \begin{cases} e(x) & \text{if } h(x) = in_i(y) \text{ for some } y \\ undefined & \text{if } h(x) \text{ is undefined} \\ a_i & \text{otherwise} \end{cases}$$
   $$h_i(x) \triangleq \begin{cases} y & \text{if } h(x) = in_i(y) \\ undefined & \text{if } h(x) \text{ is undefined} \\ b_i & \text{otherwise} \end{cases}$$
   $$h'(x) \triangleq \begin{cases} in_1(x) & \text{if } h(x) = in_1(y) \text{ for some } y \\ in_2(x) & \text{if } h(x) = in_2(y) \text{ for some } y \\ undefined & \text{otherwise} \end{cases}$$
   It's easy to check that $f \circ e_i = g_i \circ h_i$, i.e. $e_i \in f^{-1}(R)$, and $e \leq (e_1 \vee e_2) \circ h'$, i.e. e is in the ideal generated by $f^{-1}(R)$
4. If $R \in J_{can}(\underline{X})$ and S is a sieve over $\underline{X}$ s.t. $f^{-1}(S) \in J_{can}(\underline{Y})$ for any $f: \underline{Y} \to \underline{X} \in R$, then $S \in J_{can}(\underline{X})$. The proof is straightforward and is left to the reader

∎

Theorem 52 *J_{can} is the canonical topology on $\mathring{G}EN_L$*

Proof

It is easy to verify that J_{can} is subcanonical. To complete the proof we show that if a sieve R over $\underline{X}$ is not in $J_{can}(\underline{X})$, then there exists $\underline{Y}$ s.t. the functor $\mathring{G}EN_L(\_, \underline{Y})$ is not a sheaf for the least topology containing R.

Let $Y \triangleq X$ and E_Y be the ideal generated by $\{f \circ E_Z | f: \underline{Z} \to \underline{X} \in R\}$. By assumption on R, $E_X \not\subseteq E_Y \circ L$. Therefore, $\mathring{id}_X$ is not an L-effective morphism from $\underline{X}$ to $\underline{Y}$. On the other hand, the R-indexed family $\{f: \underline{Z} \to \underline{Y} | f: \underline{Z} \to \underline{X} \in R\}$ is compatible ∎

Since $\mathring{D}om_L$ is a topologically generating set, by a general result in topos theory (see [MR77] Theorem 1.3.16 or [Ros86] Theorem 1.5.6), the topos $\mathring{S}h(\mathring{G}EN_L, J_{can})$

of J_{can}-sheaves over $\mathring{G}EN_L$ is equivalent to the topos $\mathring{S}h(\mathring{D}om_L, J)$, where J is the restriction of J_{can} to $\mathring{D}om_L$, i.e.

$$R \in J(\underline{D}) \overset{\Delta}{\iff} \exists S \in J_{can}(\underline{D}).R = \{f : \underline{D'} \to \underline{D} | \underline{D'} \in \mathring{D}om_L \wedge f \in S\}$$

J in general is not the canonical topology on $\mathring{D}om_L$.

We write $\mathcal{R}_L$ for $\mathring{S}h(\mathring{D}om_L, J)$. The embedding $\_'$ (see *Def* 35) of $\mathring{G}EN_L$ in $\mathring{D}om_L^{op} \to \mathring{S}ET$ factors through $\mathcal{R}_L$, because J_{can} is subcanonical. Therefore, there is a full and faithful embedding of $\mathring{G}EN_L$ in $\mathcal{R}_L$, which preserves limits and function spaces. So far we haven't proved anything special about $\mathring{G}EN_L$, to get some really useful information we have to give a *topos-theoretic characterization* of $\mathring{G}EN_L$ as a sub-category of $\mathcal{R}_L$. We introduce a topology J' on $\mathring{D}om_L$, s.t. $\mathring{G}EN_L$ is equivalent to the full sub-category of $\mathcal{R}_L$, whose objects are J'-separated:

Definition 53 *A sieve R over $\underline{D}$ is in $J'(\underline{D}) \overset{\Delta}{\iff}$ the set D is equal to $\bigcup\{\mathring{i}mg(f) | f : \underline{Y} \to \underline{D} \in R\}$*

Proposition 54 *J' is a Grothendieck topology on $\mathring{D}om_L$*

Proof

A sieve R over $\underline{D}$ is in $J'(\underline{D})$ iff $\{\mathring{K}_d : \underline{A} \to \underline{D} | d \in D\}$ is included in R. From this observation it follows easily that J' is a Grothendieck topology. ∎

Theorem 55 *$\mathring{G}EN_L$ is equivalent to the quasitopos of J'-separated objects in $\mathcal{R}_L$*

Proof

It's easy to show that $\underline{X}'$ is J'-separated, for any generalized numbered set $\underline{X}$. We prove that for any J'-separated presheaf F there exist a set X and a functor G s.t.

- G is isomorphic to F
- $G(\underline{D})$ is a set of functions from D to X, for any $\underline{D} \in \mathring{D}om_L$
- $G(f) : g \mapsto g \circ f$, for any morphism f of $\mathring{D}om_L$

Let $X \overset{\Delta}{=} \{a \in F(\underline{A}) | \forall f : \underline{A} \to \underline{A}.F(f)(a) = a\}$, for every $\underline{D} \in \mathring{D}om_L$ and $a \in F(\underline{D})$ let $g_a : D \to X$ be the set-theoretic function s.t. $g_a(d) = F(\mathring{K}_d)(a)$, and take G to be the functor s.t.

$$G(\underline{D}) = \{g_a | a \in F(\underline{D})\} \text{ and } G(f : \underline{D'} \to \underline{D}) : g \mapsto g \circ f$$

$\eta_{\underline{D}} : a \in F(\underline{D}) \mapsto g_a \in G(\underline{D})$ is clearly a natural transformation from F to G. To show that $F \cong G$ (i.e. $\eta_{\underline{D}}$ is an isomorphism, for any $\underline{D}$), we prove that if $g_a = g_b$, then $a = b \in F(\underline{D})$.

Consider the J'-cover $\{\mathring{K}_d: \underline{A} \to \underline{D} | d \in D\}$ of $\underline{D} \in \mathring{D}om_L$, then

$$F(\mathring{K}_d)(a) \stackrel{\Delta}{=} g_a(d) = g_b(d) \stackrel{\Delta}{=} F(\mathring{K}_d)(b)$$

but F is J'-separated, so a and b must be equal

Finally, we have to show that, when G is in $\mathcal{R}_L$, then $G = \underline{X}'$, for some generalized numbered set $\underline{X}$. But, if $G \in \mathcal{R}_L$, then $E_X \stackrel{\Delta}{=} \bigcup\{G(\underline{D}) | \underline{D} \in \mathring{D}om_L\}$ is an ideal (this is left to the reader), therefore $G = \underline{X}'$ ∎

Proposition 56 *If L is an acceptable set of total functions, then J' is the double-negation topology on $\mathring{D}om_L$*

Proof

Since $\mathring{D}om_L$ is the monoid $(L, \circ)$, a sieve over $\underline{A}$ (which is the only object of $\mathring{D}om_L$) is just an $R \subseteq L$ s.t. $R \circ L = R$, and the *negation* operation $\neg$ maps a sieve R to the sieve $\{f \in L | \forall g \in L. f \circ g \notin R\}$. By definition of double-negation topology $J_{\neg\neg}$, a sieve R is in $J_{\neg\neg}(\underline{A})$ iff $\neg\neg R = L$, therefore:
$R \in J_{\neg\neg}(\underline{A}) \iff \forall f \in L. f \in \neg\neg R \iff$
$\forall f \in L. \forall g \in L. f \circ g \notin \neg R \iff$
$\forall f \in L. \forall g \in L. \exists h \in L. f \circ g \circ h \in R \iff$ we can assume that h is a constant function, because $R = R \circ L$
$\forall f \in L. \forall g \in L. \exists a \in A. f \circ g \circ \mathring{K}_a \in R \iff$
$\forall a \in A. \mathring{K}_a \in R \stackrel{\Delta}{\iff} R \in J'(\underline{A})$ ∎

When there is a partial function in L (and therefore also the everywhere undefined function $\emptyset$), J' is not the double-negation topology $J_{\neg\neg}$ on $\mathring{D}om_L$, in fact $R \in J_{\neg\neg}(\underline{D})$ iff $\emptyset \in R$.

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