A General Semantics for Evaluation Logic

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Abstract

The original semantics of Evaluation Logic in [Mog93] relies on additional properties of strong monads. This paper extends the original semantics by dropping all additional requirement on strong monads, at the expense of stronger assumptions on the underlying category (consistently with Synthetic Domain Theory). In addition, we investigate several *canonical* interpretations for necessity, criteria for discriminating among them, and their relations with the original semantics of Evaluation Logic and Reyes' topos-theoretic semantics of Modal Logics.

Introduction

Evaluation logic EL_T is a *typed predicate logic* (see [CP92, Pit91]) based on the metalanguage for computational monads ML_T (a typed calculus introduced in [Mog91]), which permits statements about the evaluation of programs to values by the use of *evaluation modalities*. In particular, EL_T might be used for axiomatising computation-related properties of a monad or devising computationally adequate theories (see [Pit91]), and it appears useful when addressing the question of logical principles for reasoning about the behaviour of programs. [Mog93] proposes a semantics for the necessity modality of EL_T (in Higher Order Logic possibility modality and evaluation predicate are definable from necessity), which relies on additional properties of strong monads. More precisely, given a category $\mathcal{C}$ with finite products, a dominion $\mathcal{M}$ and a strong endofunctor (T, t), if T preserves pullbacks of monos in $\mathcal{M}$ along morphisms in $\mathcal{C}$, then interpretation of necessity is

$$\boxed{\begin{array}{lcl} \Gamma, x\colon A \vdash \phi & = & [m] \in \mathcal{M}[\Gamma\times A] \\ \hline \Gamma, c\colon TA \vdash [x\Leftarrow c]\phi & = & \mathrm{t}^*_{\Gamma,A}[Tm] \in \mathcal{M}[\Gamma\times TA] \end{array}}$$

and the interpretation *commutes with substitution* of variables in Γ.

In a left exact category, there are two obvious choices for $\mathcal{M}$: the class of all monos, and the class of regular monos (in a topos they coincide). The latter choice is the *minimal* one for interpreting a LCF-like logic based on conditional equations – in the category **Cpo** of predomains regular monos correspond to inductive subsets. Most computational monads over **Cpo** satisfy the additional properties required in [Mog93], when $\mathcal{M}$ is the class of regular monos. But there are also strong monads (used in Denotational Semantics), which do not satisfy these additional properties:

- continuations $\Sigma^{(\Sigma^X)}$, where Σ is the cpo classifying open subsets;
- Plotkin's powerdomain $P_p(X_\perp)$, where $P_p(X)$ is the free binary semi-lattice over X (similar problems arise with the other powerdomains).

Proposition 0.1 *In* **Cpo** *exists a regular mono m s.t. Tm is not monic, when T is the monad of continuations or Plotkin's powerdomain.*

Proof Let $L \cong (1 + L)_\perp$ be the domain of lazy natural numbers, whose elements are: $s^n(0)$, $s^n(\perp)$ and ∞. The order on L is generated by $s^n(\perp) < s^{n+1}(\perp), s^n(0), \infty$ for every $n \in N$. m is the inductive subset of maximal elements, i.e. the equalizer of $s, s'\colon L \to L$, where $s(s^n(\perp)) = s^{n+1}(\perp)$, $s(s^n(0)) = s'(s^n(0)) = s^{n+1}(0)$ and are the identity otherwise. ∎

In looking for a more general semantics of necessity, we have been motivated by the desire to integrate Evaluation Logic with Synthetic Domain Theory SDT (see [Hyl91, Tay91]) rather than Classical Domain Theory. In line with this objective, we have tried to extend the semantics of necessity (proposed in [Mog93]), by dropping any additional requirement on T, at the expense of making stronger assumptions on $\mathcal{C}$ and $\mathcal{M}$ (consistently with SDT). The main results are:

- an investigation of possible *canonical* interpretations for necessity, and criteria for discriminating among them;
- a study of the relations between these interpretations and those proposed in [Mog93, RZ91];
- simpler definitions for some of these semantics under additional assumptions on $\mathcal{C}$ and $\mathcal{M}$.

If we want to stick to a *standard* semantics for formulas, i.e. formulas over A are interpreted by subobjects of A (this is not the case in [Pit91, RZ91]), then only one among the possible interpretations for necessity seems to work in general.

1 Modalities

This is a technical section, giving the basic definitions and properties regarding modalities in the setting of indexed posets. For the sake of generality, modalities are identified with families $\Box_X : \mathcal{P}_1[X] \to \mathcal{P}_2[X]$ of monotonic maps between indexed posets. However, in the other sections we consider only $\mathcal{P}_i$ of the form $F_i\,;\mathcal{M}$, where $\mathcal{M}$ is a (fixed) indexed poset of subobjects in $\mathcal{C}$ and $F_i : \mathcal{B} \to \mathcal{C}$ are functors.

Definition 1.1 (Modalities) *Given indexed posets $\mathcal{P}_1, \mathcal{P}_2 : \mathcal{C}^{op} \to$ **PoSet**, we say that $\Box$ is a* **modality** *from $\mathcal{P}_1$ to $\mathcal{P}_2$ ($\Box : \mathcal{P}_1 \to \mathcal{P}_2$ for short) if it is an indexed family $\Box_X : \mathcal{P}_1[X] \to \mathcal{P}_2[X]$ of monotonic maps. Given $\Box, \Box' : \mathcal{P}_1 \to \mathcal{P}_2$ and $R \in \{\le, =, \ge\}$, we say that*

- *$\Box$ is a R-**modality** $\stackrel{\Delta}{\iff} \Box_Y(f^*\phi) R f^*(\Box_X \phi)$ for any $f : Y \to X$ in $\mathcal{C}$ and $\phi \in \mathcal{P}_1[X]$*
- *$\Box \le \Box' \stackrel{\Delta}{\iff} \Box_X \phi \le \Box'_X \phi$ for any $X \in \mathcal{C}$ and $\phi \in \mathcal{P}_1[X]$.*

When they exist, we write

- *$\Box^R$ for the biggest R-modality s.t. $\Box^R \le \Box$*
- *${}^R\Box$ for the smallest R-modality s.t. $\Box \le {}^R\Box$.*

We define the following constructions of modalities

- *if $F : \mathcal{B} \to \mathcal{C}$ is a functor, then $(\Box_F) : F\,;\mathcal{P}_1 \to F\,;\mathcal{P}_2$ is given by* $(\Box_F)_X(\phi) \stackrel{\Delta}{=} \Box_{FX}(\phi)$ *for any X in $\mathcal{B}$ and $\phi \in \mathcal{P}_1[FX]$*
- *if $\sigma : G \dot{\to} F : \mathcal{B} \to \mathcal{C}$ is a natural transformation, then $\mathcal{P}_1[\sigma] : F\,;\mathcal{P}_1 \to G\,;\mathcal{P}_1$ is given by* $(\mathcal{P}_1[\sigma])_X(\phi) \stackrel{\Delta}{=} \sigma^*_X \phi$ *for any X in $\mathcal{B}$ and $\phi \in \mathcal{P}_1[FX]$*
- *if $\Diamond : \mathcal{P}_2 \to \mathcal{P}_3$ is a modality, then $(\Diamond\Box) : \mathcal{P}_1 \to \mathcal{P}_3$ is given by* $(\Diamond\Box)_X(\phi) \stackrel{\Delta}{=} \Diamond_X(\Box_X(\phi))$ *for any X in $\mathcal{C}$ and $\phi \in \mathcal{P}_1[X]$.*

Lemma 1.2 *Given a functor F, a natural transformation σ and modalities $\Box$ and $\Diamond$ as in Definition 1.1, the following closure properties hold:*

- *$\mathcal{P}_1[\sigma]$ is an $=$-modality*
- *if $\Box$ is an R-modality, then so is $\Box_F$*
- *if $\Box$ and $\Diamond$ are R-modalities, then so is $\Diamond\Box$*

Given $R \in \{\leq, =, \geq\}$, when the expressions involved are defined, the following inequalities hold:

- $\Box^{=} \leq \Box^{R} \leq \Box \leq {}^{R}\Box \leq {}^{=}\Box$
- $(\Box^{R})_F \leq (\Box_F)^R$ *and* ${}^{R}(\Box_F) \leq ({}^{R}\Box)_F$
- $(\Diamond^R)(\Box^R) \leq (\Diamond\Box)^R$ *and* $(\Diamond\Box)^R \leq (\Diamond^R)(\Box^R)$
- $\Box^R \leq \Diamond^R$ *and* ${}^{R}\Box \leq {}^{R}\Diamond$*, when* $\Box \leq \Diamond$

The study of $=$-modalities is greatly simplified, when $\mathcal{C}$ has *enough points* for $\mathcal{P}$.

Definition 1.3 *Given a category $\mathcal{C}$ with a terminal object 1 and $\mathcal{P}: \mathcal{C}^{op} \to$ **PoSet**, we say that $\mathcal{C}$ has* **enough points** *for $\mathcal{P}$ iff for every $X \in \mathcal{C}$ and $\phi, \psi \in \mathcal{P}[X]$*

- $\phi \leq \psi$ *in* $\mathcal{P}[X] \iff \forall x: 1 \to X. x^*\phi \leq x^*\psi$ *in* $\mathcal{P}[1]$.

Lemma 1.4 *Given a category $\mathcal{C}$ with finite products and $\mathcal{P}: \mathcal{C}^{op} \to$ **PoSet**, if $\mathcal{C}$ has enough points for $\mathcal{P}$, then $\mathcal{C}$ has enough points for $\mathcal{P}_A: \mathcal{C}^{op} \to$ **PoSet**, where $\mathcal{P}_A[X] \stackrel{\Delta}{=} \mathcal{P}[X \times A]$ (for any $A \in \mathcal{C}$).*

Proof Given $X \in \mathcal{C}$ and $\phi, \psi \in \mathcal{P}_A[X]$ we have the following equivalences:

- $\phi \leq \psi$ in $\mathcal{P}_A[X] \stackrel{\Delta}{\iff}$
- $\phi \leq \psi$ in $\mathcal{P}[X \times A] \iff$ because $\mathcal{C}$ has enough points for $\mathcal{P}$
- $\forall x: 1 \to X, a: 1 \to A. \langle x, a\rangle^*\phi \leq \langle x, a\rangle^*\psi$ in $\mathcal{P}[1] \iff$
  because $\mathcal{C}$ has enough points for $\mathcal{P}$ and $\langle x, a\rangle = \langle \mathrm{id}_1, a\rangle\, ;\, (x \times \mathrm{id}_A)$
- $\forall x: 1 \to X. (x \times \mathrm{id}_A)^*\phi \leq (x \times \mathrm{id}_A)^*\psi$ in $\mathcal{P}[1 \times A] \stackrel{\Delta}{\iff}$
- $\forall x: 1 \to X. x^*\phi \leq x^*\psi$ in $\mathcal{P}_A[1]$.

∎

Lemma 1.5 *If $\mathcal{C}$ has enough points for $\mathcal{P}_2$ and $\Box: \mathcal{P}_1 \to \mathcal{P}_2$ is an $=$-modality, then*

- $(\Box_X\phi)R\psi \iff \forall x: 1 \to X. \Box_1(x^*\phi)R(x^*\psi)$

for any $X \in \mathcal{C}$, $\phi \in \mathcal{P}_1[X]$, $\psi \in \mathcal{P}_2[X]$ and $R \in \{\leq, =, \geq\}$.

Proof The claim is a consequence of the following equivalences:

- $(\Box_X\phi)R\psi \iff$ because $\mathcal{C}$ has enough points for $\mathcal{P}$
- $\forall x: 1 \to X. x^*(\Box_X\phi)R(x^*\psi) \iff$ because $\Box$ is a $=$-modality
- $\forall x: 1 \to X. \Box_1(x^*\phi)R(x^*\psi)$.

∎

We are particularly interested in modalities of the form $\Box^{=}$. The following result give sufficient conditions for the existence of ${}^{R}\Box$ and $\Box^R$, but no *effective* way of computing them. However, under additional assumptions on $\mathcal{P}$ amd $\Box$, there is an effective way of computing $\Box^{=}$.

Definition 1.6 *Given $\mathcal{P}: \mathcal{C}^{op} \to$ **PoSet**, we say that*

- *$\mathcal{P}$ is closed under $\bigwedge$ iff each fibre has arbitrary meets and they are preserved by substitution;*

- *$\mathcal{P}$ is closed under $\forall$ iff ($\mathcal{C}$ has finite limits and) for every $f{:}\,U \to X$ in $\mathcal{C}$ exists the right adjoint $\forall_f$ to substitution f^* and it satisfies the Beck-Chevalley condition*

$$\begin{tikzcd} Y \arrow[r, "g"] & X \\ V \arrow[u, "h"] \arrow[r, "k"'] \arrow[ur, phantom, very near start, "\urcorner"] & U \arrow[u, "f"'] \end{tikzcd} \quad \textit{implies} \quad \begin{tikzcd} \mathcal{P}[X] \arrow[r, "g^*"] & \mathcal{P}[Y] \\ \mathcal{P}[U] \arrow[u, "\forall_f"] \arrow[r, "k^*"'] & \mathcal{P}[V] \arrow[u, "\forall_h"'] \end{tikzcd}$$

The properties "$\mathcal{P}$ is closed under $\bigvee/\exists$" are defined in the dual way.

Lemma 1.7 *Given (categories $\mathcal{B}$ and $\mathcal{C}$ with finite limits,) a functor $F{:}\,\mathcal{B} \to \mathcal{C}$ and an indexed poset $\mathcal{P}{:}\,\mathcal{C}^{op} \to$ **PoSet**, the following implications hold:*

- *if $\mathcal{P}$ is closed under $\bigwedge/\bigvee$, then so is $F \,;\mathcal{P}$*
- *if F preserves pullbacks and $\mathcal{P}$ is closed under $\forall/\exists$, then so is $F\,;\mathcal{P}$.*

Lemma 1.8 *If $\mathcal{P}_2$ is closed under $\bigwedge$, then modalities/R-modalities from $\mathcal{P}_1$ to $\mathcal{P}_2$ are closed under arbitrary meets, namely $(\bigwedge_{i\in I} \Box_i)_X\phi = \bigwedge_{i\in I}(\Box_{iX}\phi)$, and ${}^R\Box$ exists for any $\Box{:}\,\mathcal{P}_1 \to \mathcal{P}_2$.*
Dual results hold when $\mathcal{P}_2$ is closed under $\bigvee$.

Theorem 1.9 (Stabilisation) *If ($\mathcal{C}$ is a category with finite limits,) $\mathcal{P}_2{:}\,\mathcal{C}^{op} \to$ **PoSet** is closed under $\bigwedge$ and $\forall$, and $\Box{:}\,\mathcal{P}_1 \to \mathcal{P}_2$ is a $\leq$-modality, then $\Box^=$ exists and is given by*

$$\Box^=_X\phi = \Box^s_X\phi \stackrel{\Delta}{=} \bigwedge_{f:Y\to X} \forall_f(\Box_Y(f^*\phi))$$

Only to prove that $\Box^s$ is a $\leq$-modality, we use that $\Box$ is a $\leq$-modality (and substitution commutes with arbitrary meets and universal quantification).
Also the dual result ${}^=\Box_X\phi = \bigvee_{f:Y\to X}\exists_f(\Box_Y(f^*\phi))$ holds, but for our purposes the dual version is not applicable.

Proof It is immediate from the definition that $\Box^s$ is a modality and that $\Box \leq \Diamond$ implies $\Box^s \leq \Diamond^s$. Therefore, to prove that $\Box^= = \Box^s$ it is enough to establish the following facts:

- $\Box^s \leq \Box$. In fact
  $\Box^s_X\phi \stackrel{\Delta}{=} \bigwedge_{f:Y\to X}\forall_f(\Box_Y(f^*\phi)) \leq$ by taking $f = \mathrm{id}_X$
  $\forall_{\mathrm{id}_X}(\Box_X(\mathrm{id}^*_X\phi)) = \Box_X\phi$
- $\Box$ $=$-modality implies $\Box \leq \Box^s$. This is a consequence of the following equivalences:

  - $\Box_X\phi \leq \Box^s_X\phi \stackrel{\Delta}{\iff}$
  - $\Box_X\phi \leq \bigwedge_{f:Y\to X}\forall_f(\Box_Y(f^*\phi)) \iff$ by definition of $\bigwedge$
  - $\Box_X\phi \leq \forall_f(\Box_Y(f^*\phi))$ for every $f{:}\,Y \to X \iff$ by $f^* \dashv \forall_f$
  - $f^*(\Box_X\phi) \leq \Box_Y(f^*\phi)$ for every $f{:}\,Y \to X$
    which is the inequality saying that $\Box$ is a $\geq$-modality

- $\Box^s$ is a $\geq$-modality, i.e. $\Box^s_Y(f^*\phi) \geq f^*(\Box^s_X\phi)$ for any $f{:}\,Y \to X$ and $\phi \in \mathcal{P}_1[X]$. This is a consequence of the following equivalences:

  - $f^*(\Box^s_X\phi) \leq \Box^s_Y(f^*\phi) \stackrel{\Delta}{\iff}$

- $f^*(\Box^s_X \phi) \leq \bigwedge_{g:Z \to Y} \forall_g(\Box_Z((g\,;f)^*\phi)) \iff$ by definition of $\bigwedge$
- $f^*(\Box^s_X \phi) \leq \forall_g(\Box_Z((g\,;f)^*\phi))$ for every $g: Z \to Y \iff$ by $f^* \dashv \forall_f$
- $\Box^s_X \phi) \leq \forall_{g;f}(\Box_Z((g\,;f)^*\phi))$ for every $g: Z \to Y$
  which is true by definition of $\Box^s_X \phi$.

- $\Box$ $\leq$-modality implies $\Box^s$ $\leq$-modality, i.e. $\Box^s_Y(f^*\phi) \leq f^*(\Box^s_X \phi)$ for any $f: Y \to X$ and $\phi \in \mathcal{P}_1[X]$. This is a consequence of the following equivalences:

- $\Box^s_Y(f^*\phi) \leq f^*(\Box^s_X \phi) \overset{\Delta}{\iff}$
- $\Box^s_Y(f^*\phi) \leq f^*(\bigwedge_{g:U \to X} \forall_g(\Box_U(g^*\phi))) \iff$ because f^* preserves $\bigwedge$
- $\Box^s_Y(f^*\phi) \leq f^*(\forall_g(\Box_U(g^*\phi)))$ for every $g: U \to X \iff$ by Beck-Chevalley applied to

$$\begin{tikzcd} U \arrow[r, "g"] & X \\ V \arrow[u, "h"] \arrow[r, "k"'] \arrow[ur, phantom, "\urcorner", very near start] & Y \arrow[u, "f"'] \end{tikzcd}$$

- $\Box^s_Y(f^*\phi) \leq \forall_k h^*(\Box_U(g^*\phi))$ for every $\begin{tikzcd} U \arrow[r, "g"] & X \\ V \arrow[u, "h"] \arrow[r, "k"'] \arrow[ur, phantom, "\urcorner", very near start] & Y \arrow[u, "f"'] \end{tikzcd}$

  which is true by the following sequence of inequalities:
  $\Box^s_Y(f^*\phi) \leq$ by definition of $\Box^s_Y \phi$
  $\forall_k(\Box_V((k\,;f)^*\phi)) =$ because $k\,;f = h\,;g$
  $\forall_k(\Box_V((h\,;g)^*\phi)) \leq$ because $\Box$ is a $\leq$-modality
  $\forall_k h^*(\Box_U(g^*\phi))$.

∎

Theorem 1.10 *Under the assumptions of Theorem 1.9, if $\mathcal{C}$ has enough points for $\mathcal{P}_2$, then*

- $(\Box^s_X \phi) R \psi \iff \forall x: 1 \to X. \Box_1(x^*\phi) R (x^*\psi)$

for any $X \in \mathcal{C}$, $\phi \in \mathcal{P}_1[X]$, $\psi \in \mathcal{P}_2[X]$ and $R \in \{\leq, =, \geq\}$.

Proof Because of Lemma 1.5, it is enough to prove that $\Box^s_1 \phi = \Box_1 \phi$. By Theorem 1.9 we know already that $\Box^s_1 \phi \leq \Box_1 \phi$, so we need to prove only that $\Box_1 \phi \leq \Box^s_1 \phi$. This is a consequence of the following equivalences:

- $\Box_1 \phi \leq \Box^s_1 \phi \overset{\Delta}{\iff}$
- $\Box_1 \phi \leq \bigwedge_{!: X \to 1} \forall_!(\Box_X(!^*\phi)) \iff$ by definition of $\bigwedge$
- $\Box_1 \phi \leq \forall_!(\Box_X(!^*\phi))$ for every $!: X \to 1 \iff$ by $f^* \dashv \forall_f$
- $!^*(\Box_1 \phi) \leq \Box_X(!^*\phi)$ for every $!: X \to 1 \iff$ by $\mathcal{C}$ has enough points for $\mathcal{P}_2$
- $(x;!)^*(\Box_1 \phi) \leq x^*(\Box_X(!^*\phi))$ for every $!: X \to 1$ and $x: 1 \to X \iff$ by $x;! = \mathrm{id}_1$
- $\Box_1(x^*(!^*\phi)) \leq x^*(\Box_X(!^*\phi))$ for every $!: X \to 1$ and $x: 1 \to X$
  which is true because $\Box_X$ is a $\leq$-modality.

∎

A similar result should hold when 1 is replaced by a full sub-category $\mathcal{B}$ of generators, and $\Box_1$ is replaced by the stabilisation of $\Box$ over $\mathcal{B}$.

2 A general semantics of necessity

Throughout this section we fix: a category $\mathcal{C}$ with finite limits, a factorisation system $(\mathcal{E}, \mathcal{M})$ with the unique fill-in property s.t. $\mathcal{M}: \mathcal{C}^{op} \to \mathbf{PoSet}$ is closed under $\bigwedge$ and $\forall_f$, a strong endofunctor (T, t) over $\mathcal{C}$. Our aim is to associate to (T, t) (and every $A \in \mathcal{C}$) an $=$-modality $\Box^s_{\_,A}: \mathcal{M}_A \to \mathcal{M}_{TA}$ for interpreting necessity.

Definition 2.1 *Under the assumptions above, we can define the following modalities:*

- $\Box: \mathcal{M} \to T\,; \mathcal{M}$ *s.t.* $\Box_X([m])$ *is the image of* Tm, *i.e.* $\Box_X([m: X' \to X]) \stackrel{\Delta}{=} [m']$, *where* (e', m') *is the factorisation of* $Tm: TX' \to TX$ *in* $(\mathcal{E}, \mathcal{M})$
- $\Box_{\_,A}: \mathcal{M}_A \to \mathcal{M}_{TA}$ *is* $\Box_{X,A}\phi \stackrel{\Delta}{=} \mathrm{t}^*_{X,A}(\Box_{X\times A}\phi)$
- $\Box^s_{\_,A}: \mathcal{M}_A \to \mathcal{M}_{TA}$ *is the stabilisation of* $\Box_{\_,A}$ *(see Theorem 1.9).*

Theorem 2.2 $\Box$ *and* $\Box_{\_,A}$ *are* $\leq$*-modalities, while* $\Box^s_{\_,A}$ *is an* $=$*-modality.*

Proof If $\Box$ is a $\leq$-modality, the other claims follow by Lemma 1.2 and Theorem 1.9.
To prove that $\Box$ is a $\leq$-modality, we show that for every pullback square

$$\begin{tikzcd} B \arrow[r, "f"] & A \\ B' \arrow[u, hook, "m'"] \arrow[r, "f'"'] \arrow[ur, phantom, "\lrcorner", very near start] & A' \arrow[u, hook, "m \in \mathcal{M}"'] \end{tikzcd}$$

if (e, n) and (e', n') are the factorisations of Tm and Tm', then exists (unique) g s.t. $n'\,; Tf = g\,; n$

- $Tm'\,; Tf = Tf'\,; Tm$, because $m'\,; f = f'\,; m$
- $e'\,; (n'\,; Tf) = (Tf'\,; e)\,; m$, because $Tm' = e'\,; n'$ and $Tm = e\,; n$
- exists unique g s.t. $n'\,; Tf = g\,; n$ and $Tf'\,; e = e'\,; g$, by the unique fill-in property.

∎

Corollary 2.3 *If* $\mathcal{C}$ *has enough points for* $\mathcal{M}$, *then for any* $\phi \in \mathcal{M}[X\times A]$ *and* $\psi \in \mathcal{M}[X\times TA]$

$$(\Box^s_{X,A}\phi)R\psi \iff \forall x: 1 \to X.(\Box_A\phi_x)R\psi_x$$

where $\phi_x \stackrel{\Delta}{=} \langle !\,; x, \mathrm{id}_A\rangle^*\phi$ *when* $!: A \to 1$, $x: 1 \to X$ *and* $\phi \in \mathcal{M}[X\times A]$.

Proof By Lemma 1.4 and Theorem 1.10 we know that for any $\phi \in \mathcal{M}[X\times A]$ and $\psi \in \mathcal{M}[X\times TA]$

$$(\Box^s_{X,A}\phi)R\psi \iff \forall x: 1 \to X.\Box_{1,A}((x\times\mathrm{id}_A)^*\phi)R(x\times\mathrm{id}_{TA})^*\psi$$

Therefore the claim is a consequence of the following equivalences:

- $\Box_{1,A}((x\times\mathrm{id}_A)^*\phi)R(x\times\mathrm{id}_{TA})^*\psi \iff$
  by $\mathcal{M}[1\times A] \cong \mathcal{M}[A]$ via $\phi \mapsto \langle !, \mathrm{id}_A\rangle^*\phi$ and by definition of ψ_x
- $\langle !, \mathrm{id}_{TA}\rangle^*(\Box_{1,A}((x\times\mathrm{id}_A)^*\phi))R\psi_x \iff$
  by definition of $\Box_{X,A}$ and $\langle !, \mathrm{id}_{TA}\rangle\,; \mathrm{t}_{1,A} = T\langle !, \mathrm{id}_A\rangle$
- $(T\langle !, \mathrm{id}_A\rangle)^*(\Box_{1\times A}((x\times\mathrm{id}_A)^*\phi))R\psi_x \iff$
  since $T\langle !, \mathrm{id}_A\rangle: TA \to T(1\times A)$ is iso and $\Box_A$ is a $\leq$-modality
- $\Box_A(\langle !\,; x, \mathrm{id}_A\rangle^*\phi)R\psi_x \iff$ by definition of ϕ_x
- $\Box_A(\phi_x)R\psi_x$.

∎

2.1 A test case for necessity

In the setting of Section 2, there are several ways of constructing a =-modality from $\mathcal{M}_A$ to $\mathcal{M}_{TA}$ starting from a modality $\Box{:}\,\mathcal{M} \to T\,;\mathcal{M}$. Since $\mathcal{M}{:}\,\mathcal{C}^{op} \to \mathbf{PoSet}$ is closed under $\bigwedge$ and $\bigvee$, we can construct =-modalities by applying either ${}^{=}(\_)$ or $(\_)^{=}$ (see Lemma 1.8). In particular, the =-modality $\Box^s_{\_,A}$ of Definition 2.1 is given by $(\Diamond\Box_F)^{=}$, where $F{:}\,\mathcal{C} \to \mathcal{C}$ is the functor $(\_\times A)$, and $\Diamond_X{:}\,\mathcal{M}[T(X\times A)] \to \mathcal{M}[X\times TA]$ is the =-modality s.t. $\Diamond_X(\phi) = \mathrm{t}^*_{X,A}\phi$. By applying $(\_)^{=}$ at a different stage and/or by replacing it with ${}^{=}(\_)$, we obtain six (possibly different) =-modalities:

$$\Diamond(\Box^{=})_F \le \Diamond(\Box_F)^{=} \le (\Diamond\Box_F)^{=} = \Box^s_{\_,A} \le \Box_{\_,A} = \Diamond\Box_F \le {}^{=}(\Diamond\Box_F) \le \Diamond^{=}(\Box_F) \le \Diamond({}^{=}\Box)_F$$

Here we consider a specific choice of $\mathcal{C}$, $\mathcal{M}$ and (T,t):

- $\mathcal{C}$ is the category **PoSet** of posets and monotonic maps, **PoSet** is complete and cartesian closed;
- $\mathcal{M}$ is the class of regular monos, regular monos and surjective (monotonic) maps form a stable factorisation system, and the regular subobjects of $\underline{X} = (X, \le_X)$ are in one-one correspondence with the subsets of X (with the induced partial order);
- (T,t) is the strong functor given by $T(X) = 2^{(2^X)}$.

For this specific choice, we show that all =-modalities above, apart from $(\Diamond\Box_F)^{=}$, are *trivial* (i.e. they are families of constant functions). We take for granted the following facts about **PoSet**.

Proposition 2.4 **PoSet** *and* **Set** *are related via the following adjunctions*

$$\begin{tikzcd} \mathbf{Set} \arrow[r, hook, "\Delta"] & \mathbf{PoSet} \arrow[l, bend right, "\pi"'] \arrow[l, bend left, "U"] \end{tikzcd}$$

U is the forgetful functor mapping a poset $\underline{X}$ to its underlying set, Δ is the full embedding mapping a set X to the discrete order on X, π is the functor mapping $\underline{X}$ to the set of its connected components. **Set** *is a full reflective sub-category of* **PoSet** *and the reflection π preserves finite products, therefore* **Set** *is an exponential ideal of* **PoSet** *and T factors through Δ, namely $T(X) \cong \Delta(2^{(2^{\pi(X)})})$.*

Lemma 2.5 $(\Box_F)^{=}_X(\phi) = \top \in \mathcal{M}[T0]$ *when $A = 0$, and* $(\Box_F)^{=}_X(\phi) = \bot \in \mathcal{M}[T(X\times A)]$ *otherwise, for every $X \in \mathcal{C}$ and $\phi \in \mathcal{M}[X\times A]$.*

Proof Given $X \in \mathcal{C}$ and $\phi \in \mathcal{M}[X\times A]$, we have the following pullback squares

$$\begin{tikzcd} X\times A \arrow[r, "\eta^{\perp}_X\times A"] & X_\perp\times A & 1\times A \arrow[l, "\perp_X\times A"'] \\ \phi \arrow[u, hook] \arrow[r] \arrow[ur, phantom, "\lrcorner", very near start] & \phi \arrow[u, hook] & 0 \arrow[l] \arrow[u, hook] \arrow[ul, phantom, "\llcorner", very near start] \end{tikzcd}$$

Since $(\Box_F)^{=}$ is an =-modality, then we have the following pullback squares

$$\begin{tikzcd} T(X\times A) \arrow[r, "T(\eta^{\perp}_X\times A)"] & T(X_\perp\times A) & T(1\times A) \arrow[l, "T(\perp_X\times A)"'] \\ (\Box_F)^{=}_X(\phi) \arrow[u, hook] \arrow[r] \arrow[ur, phantom, "\lrcorner", very near start] & (\Box_F)^{=}_{X_\perp}(\phi) \arrow[u, hook] & (\Box_F)^{=}_1(0) \arrow[l] \arrow[u, hook] \arrow[ul, phantom, "\llcorner", very near start] \end{tikzcd}$$

$T(\perp_X \times A) \colon T(1 \times A) \to T(X_\perp \times A)$ is an isomorphism, because $X_\perp$ has one connected component, moreover $(\Box_F)^{=}_{X}(\phi)$ does not depend on ϕ, because

$$\begin{tikzcd}
T(X \times A) \arrow[r, "T(!_X \times A)"] & T(1 \times A) \\
(\Box_F)^{=}_{X}(\phi) \arrow[u, hook] \arrow[r] \arrow[ur, phantom, "\urcorner", very near start] & (\Box_F)^{=}_{1}(0) \arrow[u, hook]
\end{tikzcd}$$

Let $\psi \stackrel{\Delta}{\equiv} (\Box_F)^{=}_{1}(0)$. If $A = 0$, then $T(X \times A) = T0$ and clearly $\psi = \top$.
In the case $A \neq 0$, we prove that $(\Box_F)^{=}_{X}(\phi) = \perp$ by showing that $\psi = \perp$. We have that $\psi \leq (\Box_F)_1(0) = \{\lambda k \colon 2^{(1 \times A)}.r | r \in 2\}$, because $(\Box_F)^{=} \leq \Box_F$. Therefore, there are only four possible choices for ψ: $\emptyset \subset \{\lambda k \colon 2^{(1 \times A)}.r\} \subset \{\lambda k \colon 2^{(1 \times A)}.r | r \in 2\}$. To prove that $\psi = \emptyset$, we show that the choice $\psi = \{\lambda k \colon 2^{(1 \times A)}.r\}$ is *too big*, i.e. $(T(!_2 \times A))^* \psi \not\leq (\Box_F)_2(\emptyset)$:

- $(\Box_F)_2(\emptyset) \stackrel{\Delta}{\equiv} \{\lambda k \colon 2^{(2 \times A)}.r | r \in 2\}$
- $c \in (T(!_2 \times A))^* \psi \stackrel{\Delta}{\Longleftrightarrow} \forall k \colon 2^A.c(\lambda x \colon 2, a \colon A.ka) = r$
- fix an $a \in A$ and let $c \in T(2 \times A)$ be $c(k) \stackrel{\Delta}{\equiv} \begin{cases} r & \text{if } k(0,a) = k(1,a) \\ k(r,a) & \text{otherwise} \end{cases}$
- $c \in (T(!_2 \times A))^* \psi$, by definition
- $c \notin (\Box_F)_2(\emptyset)$, because $c(k) \neq c(k')$ when $k(x,y) \stackrel{\Delta}{\equiv} x$ and $k'(x,y) \stackrel{\Delta}{\equiv} \neg x$.

∎

Lemma 2.6 *${}^{=}(\Diamond\Box_F)_X(\phi) = \top \in \mathcal{M}[X \times TA]$, for every $X \in \mathcal{C}$ and $\phi \in \mathcal{M}[X \times A]$.*

Proof Since $\mathcal{C}$ has enough points for $\mathcal{M}$ and ${}^{=}(\Diamond\Box_F)$ is an $=$-modality, then (by Lemma 1.5) the claim follows from ${}^{=}(\Diamond\Box_F)_1(\phi) = \top$ for every $\phi \in \mathcal{M}[1 \times A]$.
Given $\phi \in \mathcal{M}[1 \times A]$, let $\psi \in \mathcal{M}[\Sigma \times A]$ be $(\{*\} \times \phi) \cup (\{\perp\} \times A)$, then ${}^{=}(\Diamond\Box_F)_1(\phi) = \top$ because of the following facts:

- $\phi = (\eta_1^\perp \times A)^* \psi$, by definition of ψ
- $(\Diamond\Box_F)_\Sigma(\psi) = \top$, because $\Box_{\Sigma \times A}(\psi) = \top$. In fact, let $m \colon \psi \hookrightarrow \Sigma \times A$ be the *canonical* mono representing ψ, then πm and Tm are iso, because each connected component of $\Sigma \times A$ has one element in ψ (of the form $\langle \perp, a \rangle$)
- $\top = (\Diamond\Box_F)_\Sigma(\psi) \leq {}^{=}(\Diamond\Box_F)_\Sigma(\psi)$, by definition of ${}^{R}(\_)$
- ${}^{=}(\Diamond\Box_F)_1(\phi) = (\eta_1^\perp \times TA)^*({}^{=}(\Diamond\Box_F)_\Sigma(\psi)) = \top$, because ${}^{=}(\Diamond\Box_F)$ is an $=$-modality.

∎

Proposition 2.7 $\Diamond(\Box^{=})_F = \Diamond(\Box_F)^{=} = \perp$ *(if $A \neq 0$) and* $\top = {}^{=}(\Diamond\Box_F) = \Diamond^{=}(\Box_F) = \Diamond({}^{=}\Box)_F$.

Proof The first equalities follow from $\Diamond(\Box^{=})_F \leq \Diamond(\Box_F)^{=}$ and Lemma 2.5. The second equalities follow from ${}^{=}(\Diamond\Box_F) \leq \Diamond^{=}(\Box_F) \leq \Diamond({}^{=}\Box)_F$ and Lemma 2.6. ∎

In **Cpo** regular monos are not closed under $\bigvee$.

3 Related approaches

In the setting of [Mog93], where T preserves pullbacks of monos in $\mathcal{M}$ along morphisms in $\mathcal{C}$, the modality $\Box$ of Definition 2.1 is given by $\Box_X([m]) = [Tm]$ and is already an =-modality. Therefore, the six =-modalities considered in Section 2.1 are all equal to $\Box_{\_,A}$.

The approach of [RZ91] defines necessity in terms of a geometric morphism $\mathcal{S} \underset{\longleftarrow \Gamma \longrightarrow}{\overset{\longrightarrow \Delta \longrightarrow}{\perp}} \mathcal{E}$ where Δ is left exact (i.e. preserves finite limits). The geometric morphism induces a left exact comonad $G = \Gamma\,;\Delta$ over $\mathcal{E}$ and a left exact (and therefore strong) monad $T = \Delta\,;\Gamma$ over $\mathcal{S}$. Since Δ and Γ preserves pullbacks, the adjunction $\Delta \dashv \Gamma$ lifts to a fibred adjunction

$$
\begin{tikzcd}
\mathcal{M}_{\mathcal{S}} \arrow[r, shift left, "\tilde{\Delta}"] \arrow[r, phantom, "\perp"] \arrow[d, "\pi"'] & \mathcal{M}_{\mathcal{E}} \arrow[l, shift left, "\tilde{\Gamma}"] \arrow[d, "\pi"] \\
\mathcal{S} \arrow[r, shift left, "\Delta"] \arrow[r, phantom, "\perp"] & \mathcal{E} \arrow[l, shift left, "\Gamma"]
\end{tikzcd}
$$

where $\pi: \mathcal{M}_{\mathcal{C}} \to \mathcal{C}$ is the posetal fibration (we freely switch between indexed posets and posetal fibrations) of subobjects of $\mathcal{C}$, when $\mathcal{C}$ is left exact. This fibred adjunction induces two =-modalities: $\Box^T: \mathcal{M}_{\mathcal{S}} \to T\,;\mathcal{M}_{\mathcal{S}}$ and $\Box^G: \mathcal{M}_{\mathcal{E}} \to G\,;\mathcal{M}_{\mathcal{E}}$. The interpretation of necessity proposed in [RZ91] is the =-modality $\Box: \mathcal{P} \to \mathcal{P}$ defined as follows:

- $\mathcal{P} = \Delta\,;\mathcal{M}_{\mathcal{E}}: \mathcal{S}^{op} \to \mathbf{PoSet}$
- $\Box_X(\phi) = (\Delta\eta_X)^*(\Box^G_{\Delta X}\phi)$, for any $X \in \mathcal{S}$ and $\phi \in \mathcal{P}[X] = \mathcal{M}_{\mathcal{E}}[\Delta X]$

where $\eta_X: X \to TX$ is the unit of the adjunction $\Delta \dashv \Gamma$, and $\Box^G_{\Delta X}\phi \in \mathcal{P}[TX]$ because $\Delta;G = T\,;\Delta$.

Remark 3.1 The approach of [RZ91] is related to that in [Mog93]. In fact, $\Box^G$ is obtained from G as proposed in [Mog93], when $\mathcal{C} = \mathcal{E}$ and $\mathcal{M} = \mathcal{M}_{\mathcal{E}}$, since G is left exact. However, there is one important difference in the logical setting: [Mog93] sticks to a *standard* semantics of formulas (i.e. formulas are interpreted as subobjects in the base category), while [RZ91] adopts a non standard semantics based on $\mathcal{P}$. It may be worth investigating whether a combination of [Mog93] and [RZ91] can place the non-standard semantics of Evaluation Logic proposed in [Pit91] into a wider context.

3.1 A semantics of necessity based on fibred adjunctions

For interpreting necessity the setting of [RZ91] is excessive. In fact, a fibred adjunction

$$
\begin{tikzcd}
\mathcal{M}_1 \arrow[r, shift left, "\tilde{F}"] \arrow[r, phantom, "\perp"] \arrow[d, "\pi"'] & \mathcal{M}_2 \arrow[l, shift left, "\tilde{U}"] \arrow[d, "\pi"] \\
\mathcal{C}_1 \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{C}_2 \arrow[l, shift left, "U"]
\end{tikzcd}
$$

is all we need to define (mutatis mutandis) $\mathcal{P}$ and $\Box: \mathcal{P} \to \mathcal{P}$ as sketched above. Also the semantics proposed in [Mog93] can be recast in terms of a fibred adjunction.

Definition 3.2 *Given a category $\mathcal{C}$, a dominion $\mathcal{M}$ and a monad T, we write $\mathcal{M}^T$ for the dominion over the category $\mathcal{C}^T$ of Eilenberg-Moore T-algebras s.t. $m\colon \alpha' \to \alpha \in \mathcal{M}^T \overset{\Delta}{\iff}$*

$$\begin{tikzcd} TA \arrow[r, "\alpha"] & A \\ TA' \arrow[u, "Tm"] \arrow[r, "\alpha'"'] & A' \arrow[u, hook, "m \in \mathcal{M}"'] \end{tikzcd}$$

$\mathcal{M}^T$ is closed under pullbacks, because $U\colon \mathcal{C}^T \to \mathcal{C}$ creates limits.

Proposition 3.3 *The forgetful functor $U\colon \mathcal{C}^T \to \mathcal{C}$ lifts to a fibred functor from $\mathcal{M}^T$ to $\mathcal{M}$.*

In fact, U preserves limits and maps monos in $\mathcal{M}^T$ into $\mathcal{M}$ (by definition of $\mathcal{M}^T$).

Remark 3.4 Consider $\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{C}^T \arrow[l, shift left, "U"] \end{tikzcd}$ where $FA = (\mu_A\colon T^2A \to TA)$ and $U(\alpha\colon TA \to A) = A$. When T preserves pullbacks of monos in $\mathcal{M}$, then F lifts to a fibred functor from $\mathcal{M}$ to $\mathcal{M}^T$, which is left adjoint to the lifting of U

$$\begin{tikzcd} \mathcal{M} \arrow[r, shift left, "\tilde{F}"] \arrow[r, phantom, "\perp"] \arrow[d, "\pi"'] & \mathcal{M}^T \arrow[l, shift left, "\tilde{U}"] \arrow[d, "\pi"] \\ \mathcal{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{C}^T \arrow[l, shift left, "U"] \end{tikzcd}$$

In this case the $=$-modality $\Box\colon \mathcal{M} \to T\,;\mathcal{M}$ of Definition 2.1 is simply $\tilde{F}\,;\tilde{U}$.

In general, a fibred left adjoint $(F, \tilde{F})$ to $(U, \tilde{U})$ may not exists, but when it does, it induces an $=$-modality $\Box' \overset{\Delta}{\equiv} \tilde{F}\,;\tilde{U}\colon \mathcal{M} \to T\,;\mathcal{M}$. The following proposition relates $\Box'$ to the modalities introduced in Definition 2.1.

Proposition 3.5 *Under the assumptions of Definition 2.1, if $\Box'$ is defined, then $\Box \leq \Box'$.*

Proof $\Box_A([m\colon A' \hookrightarrow A])$ is the image of Tm (w.r.t. $\mathcal{M}$) and $\Box'_A([m]) \in \mathcal{M}[TA]$ by definition, therefore $\Box_A([m]) \leq \Box'_A([m])$ iff Tm factors through $\Box'_A([m])$. Moreover, from the definition of $\Box'$ one can easily prove that $\Box'_A([m\colon A' \hookrightarrow A]) = \tilde{F}_A([m]) \in \mathcal{M}^T[\mu_A\colon T^2A \to TA]$, and that $\Box'_A([m])$ enjoys the following universal property

$$[m] \leq f^*[n] \iff \Box'_A([m]) \leq ((Tf)\,;\beta)^*[n]$$

for every $f\colon A \to B$ and $n \in \mathcal{M}^T[\beta\colon TB \to B]$. To prove that Tm factors through $\Box'_A([m])$ consider the following commuting squares

$$\begin{tikzcd} A \arrow[r, "\eta_A"] & TA \\ \cdot \arrow[u, hook, "m"] \arrow[r, dashed] & \cdot \arrow[u, hook, "\Box'_A m"'] \end{tikzcd} \qquad \begin{tikzcd} T^2A \arrow[r, "\mu_A"] & TA \\ \cdot \arrow[u, "T(\Box'_A m)"] \arrow[r, dashed] & \cdot \arrow[u, hook, "\Box'_A m"'] \end{tikzcd}$$

The first square commutes because of ($T\eta_A\,;\mu_A = \mathrm{id}_{TA}$ and) the universal property applied to $f = \eta_A$, $\beta = \mu_A$ and $[n] = \Box'_A([m])$, while the second amounts to $\Box'_A m \in \mathcal{M}^T[\mu_A\colon T^2A \to TA]$.

By applying T to the first square, and then composing the result with the second square we get

$$\begin{tikzcd} TA \arrow[r, "T\eta_A"] & T^2A \arrow[r, "\mu_A"] & TA \\ \cdot \arrow[u, "Tm"] \arrow[r, dashed] & \cdot \arrow[u, "T(\Box'_A m)"] \arrow[r, dashed] & \cdot \arrow[u, hook, "\Box'_A m"'] \end{tikzcd}$$

which amounts to say that Tm factors through $\Box'_A([m])$, since $T\eta_A \,;\, \mu_A = \mathrm{id}_{TA}$. ∎

In conclusion, when $\Box$ is not an $=$-modality, the interpretation of necessity based on $\Box'$ differs from that using $\Box^s$, and (when $\Box'$ is defined) it has the same problems (established in Section 2.1) of the three semantics of necessity defined using $^=(\_)$.

Further Research

There are three research directions we would like to pursue:

- To find logical principles sound and complete for this extended semantics of necessity. Unfortunately, the techniques in [Mog93] do not seem immediately applicable.
- To extend our treatment to indexed monads. This should provide a semantic understanding of Evaluation Logic in the presence of dependent types.
- To fit into a wider context the *non-standard* semantics of Evaluation Logic proposed in [Pit91].

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