Imaging: Inverse Problems and Computational Imaging

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A digital image is an array (2D image) or a cube (3D image) of quantized numbers providing a representation of a physical object or a map of some distributed property of the object (density of matter, energy etc.). It can be obtained by digitizing the original image (for instance, a photograph) or can be directly formed by the outputs of the imaging system (digital camera, microscope, telescope, etc.). When visualized on a suitable image display device, a digital image must provide a representation of the physical object which can be easily interpreted and analyzed. In many cases, however, this condition is not satisfied. We give two typical examples.

The first is that of an image degraded by the blurring which can be introduced by defects of the optical systems (such as aberrations or others) or by image motion, image defocusing etc. The representation of the object does not have the quality which should be desirable for its interpretation.

The second case is even more important and arises when the output of the imaging system is not directly related to the physical quantity to be imaged. For instance in X-ray tomography the output of a detector is the attenuation of a X-ray pencil crossing the body while the quantity to be imaged is the density of the body.

In the situations described above the desired image can be obtained by solving a number of mathematical problems. If we denote by f the object, namely the target of the imaging system, and by g the image of f provided by the system, then the problems to be solved can be summarized as follows:

- develop a physical model of the system to obtain a mathematical relationship between the object f and the image g; the computation of g from a given f is usually called the solution of the *direct problem*;

- solve the problem of obtaining the physical quantity f from given outputs g of the system; this second step is usually called the solution of the *inverse problem*.

The digital image of f, obtained by solving an inverse problem, is a computed one, generated by the computer where the algorithm for the solution of the inverse problem has been implemented.

It may be convenient to formulate both the direct and the inverse problem in terms of functions rather than arrays, cubes etc., because such a formulation allows the use of the powerful tools of functional analysis for their investigation. Next the results obtained in the continuous case can be used for understanding the features of the corresponding discrete problem. In most important applications, for instance, the relationship between f and

g is linear so that, in the discrete version of the direct problem one obtains g by applying a suitable matrix A to f while, in the discrete version of the inverse problem, one obtains f from g by solving a linear algebraic system. However the solution of the inverse problem is not so simple as it could appear from this remark and the difficulties can be understood by looking at the corresponding continuous problem.

The basic reason relies on the fact that inverse problems are ill-posed in the sense of Hadamard: the solution may not exist, even if it exists it may not be unique and even if it exists and is unique, it may not depend continuously on the data. The last statement means that a small perturbation of the data, as that caused by noise or any kind of experimental error, can modify completely the solution. These points will be discussed more carefully in the particular case of image deconvolution.

The ill-posedness of inverse problems is generated by the fact that imaging systems do not transmit complete information about the physical object. Therefore the problem is to extract the useful information contained in the data or, in mathematical terms, to look for approximate solutions which are stable against noise. This is the purpose of a mathematical theory introduced in its general form by the Russian mathematicians A. N. Tikhonov and known as *regularization theory*.

We illustrate the general framework outlined above by means of a specific example: *image deconvolution*, also know as image deblurring, image restoration etc.. The object f is the image which should be recorded in the absence of degradation while g is the image corrupted by aberrations or other causes of blurring. Both f and g are functions of 2D (or 3D) space variables $\mathbf{x}$, which are the coordinates of a point in the image domain. If the imaging system is isoplanatic, then it is described by a space-invariant *point spread function*

(PSF) $K(\mathbf{x})$, which is the image of a point source located in the center of the image domain. The PSF provides the response of the system to any point source wherever it is located. On the other hand its Fourier transform, the *transfer function* $\hat{K}(\boldsymbol{\omega})$ (the hat denotes Fourier transform and the variables $\boldsymbol{\omega}$ are the spatial frequencies associated to the space variables $\mathbf{x}$), tells us how a signal of a fixed frequency is propagated through the linear system, so that the blurring can also be viewed as a sort of frequency filtering.

Under these assumptions the image g is the convolution product of object f and PSF K:

$$g(\mathbf{x}) = \int K(\mathbf{x} - \mathbf{x}')f(\mathbf{x}')\ d\mathbf{x}' \tag{1}$$

and the solution of the direct problem is just the computation of a convolution product.

The image g, as given by equation (1), is the so-called noise-free image. The actually recorded image g_r is affected by a noise term and, in general, it can be written in the following form

$$g_r(\mathbf{x}) = g(\mathbf{x}) + n(\mathbf{x}) \tag{2}$$

where $n(\mathbf{x})$ is just a function describing noise contamination. It is a random function and therefore it is unknown even if one can have knowledge of its statistical properties. We point out that equation (2) does not imply that the noise is additive or signal independent. For instance in microscopy and astronomy the contamination of the image is due both to photon noise, which satisfies Poisson statistics, and read-out noise, which is basically Gaussian and white. Therefore the noise term in equation (2) must only be intended as the difference between the noisy and the noise-free image.

In a first attempt of approaching the inverse problem it seems quite natural to ignore the noise term $n(\mathbf{x})$ and to solve equation (1) for $f(\mathbf{x})$ with $g(\mathbf{x})$ replaced by $g_r(\mathbf{x})$. Then, by taking the Fourier transform of both sides of this equation and using the well-known convolution theorem, one finds the following relationship

$$\hat{g}_r(\boldsymbol{\omega}) = \hat{K}(\boldsymbol{\omega})\hat{f}(\boldsymbol{\omega}) \tag{3}$$

which relates the Fourier transform of f and g to the transfer function of the imaging system.

The solution of this equation looks elementary. However a first difficulty is due to the fact that an optical system is in general band-limited; its band Ω is the bounded domain of spatial frequencies where the transfer function is different from zero. Since the noise is not band-limited or, at least, has a band much broader than that of the optical system, the r.h.s. of equation (3) is zero outside Ω while the l.h.s is not; in other words the solution of the deconvolution problem, as formulated above, does not exist because no object f satisfies equation (3) everywhere. In this particular case one can circumvent the difficulty by applying a suitable filter to $g_r(x)$ in order to suppress the out-of-band noise.

The second difficulty is the non-uniqueness of the solution of the problem with the filtered data. This is due to the so-called *invisible objects*, namely objects whose Fourier transform is zero on Ω so that the corresponding images are exactly zero. If we find a solution of the deconvolution problem, by adding an arbitrary invisible object to this solution we find another solution of the same problem. A well defined solution can be obtained by requiring its Fourier transform to be zero outside Ω. In other words, we set to zero what is not transmitted by the imaging system.

The standard way for approaching the previous questions is to look for solutions in the least-squares sense, namely for objects which minimize the functional

$$\epsilon^2(f) = ||K * f - g_r||^2 \tag{4}$$

where the $*$ denotes the convolution product, as defined in equation (1), and the r.h.s. is the square of the L^2-norm (which can also be interpreted as *energy*) of the discrepancy between the recorded image g_r and the computed image $K * f$. If one looks for a least-squares solution with minimal energy then one automatically re-obtains the solution discussed above, namely an object whose Fourier transform in Ω is given by

$$\hat{f}_{inv}(\boldsymbol{\omega}) \;=\; \frac{\hat{g}_r(\boldsymbol{\omega})}{\hat{K}(\boldsymbol{\omega})} \tag{5}$$

and is zero outside Ω. Such a solution is that provided by the so-called *inverse filter*.

However there is an additional difficulty which is the most serious one. In general the transfer function tends to zero at the boundary of the band while the Fourier transform of the noise, hence that of g_r, does not or tends to zero in a different way. It follows that $\hat{f}_{inv}(\boldsymbol{\omega})$ is divergent or very large at the boundary of the band. We conclude that its inverse Fourier transform does not exist or, if it exists, is affected by large artifacts due to noise propagation. The latter is the typical situation in the case of digital images. In Figure 1 we give the result of a numerical simulation showing the typical effect produced by the inverse filter. The object is a beautiful picture of a galaxy recorded by the Hubble Space Telescope (HST) while the PSF is a computed one describing the blurring of HST before installation of the optics correction. The blurred image is obtained by convolving the object with the PSF and by adding white noise. The restored image produced by the inverse filter is completely corrupted by noise.

<Figure 1 near here>

The inverse filter provides a solution which fits the data in the best possible way and therefore fits also the noise in the frequency domain where the signal is weakly transmitted by the imaging system. As a result the energy (L^2-norm) of the solution is too large.

This remark suggests that one must search for a reasonable compromise between data fitting and energy of the solution. This is the basic idea of *regularization theory*, at least in its most simple form. The mathematical formulation is obtained by looking for objects which minimize the functional

$$\Phi_\mu(f) = ||K * f - g_r||^2 + \mu||f||^2 \tag{6}$$

where μ, the so-called *regularization parameter*, is a parameter controlling the trade-off between data fitting and energy of the solution: when μ is small, the data fitting is good and the energy is large while, when μ is large, the data fitting is bad and the energy is small.

For a given μ, the function f_μ which minimizes the functional of equation (6) is called a *regularized solution* of the problem. It is easy to show that its Fourier transform is given by

$$\hat{f}_\mu(\boldsymbol{\omega}) = \frac{\hat{K}^*(\boldsymbol{\omega})}{|\hat{K}(\boldsymbol{\omega})|^2 + \mu} \hat{g}_r(\boldsymbol{\omega}) \ . \tag{7}$$

The regularized solutions form a one-parameter family of functions. When

μ is small these functions are strongly contaminated by noise whose effect is gradually reduced when μ increases. For too large values of μ one re-obtains blurred versions of the original object.

<Figure 2 near here>

Such a behavior is illustrated by the sequence of regularized solutions given in Figure 2 and corresponding to the example of Figure 1. The regularization parameter increases starting from left to right and from top to down. Its values are $\mu = 0$ (inverse filter), 0.001, 0.01, 0.05, 0.1 and 0.5 (the noise is of the order of a few percent and the PSF is normalized in such a way that the sum of its value is 1). The sequence makes evident the well-established theoretical result of the existence of an optimal value of the regularization parameter. Such an optimal value can be determined in the case of numerical simulations as that shown in Figure 2. However its estimation in the case of real images is a difficult problem. Several methods for the selection of μ have been proposed. We only mention the so-called *discrepancy principle*. It consists in determining the value of μ such that the value at f_μ of the discrepancy functional defined in equation (4), i. e. $\epsilon^2(f_\mu)$, coincides with an estimate of the energy of the noise. In other words the data are fitted with an accuracy comparable with their uncertainty.

As shown by equation (7), the regularized solution has the same band Ω of the imaging system because $\hat{K}^*(\boldsymbol{\omega})$ is zero outside Ω. As it is known a band-limited function can be represented in terms of its samples taken at a rate which is roughly proportional to the size of the band. This is the content of the famous Shannon sampling theorem (more precisely its 2D extension). On the other hand the sampling distance can be taken as a measure of the resolution provided by the restored image so that one concludes that the restoration method discussed above does not produce an improvement of the resolution of the imaging system. It certainly produces an improvement of the quality of the image which makes possible a visual verification of that resolution.

In many circumstances the term *super-resolution* is used for describing methods which allow an improvement of resolution beyond the limit provided by the sampling theorem. Therefore a super-resolving method must produce a restored image with a band broader than that of the imaging system. In other words super-resolution is related to the problem of out-of-band extrapolation.

Such an approach was already proposed in the sixties when it was observed that the Fourier transform of an object with a finite spatial extent (all objects have this property !) is analytic; then the theorem of unique analytic continuation implies that the Fourier transform of the object can be determined everywhere from its values on the band of the imaging system.

Unfortunately this problem is ill-posed, more precisely very badly ill-posed, so that out-of-band extrapolation is hardly feasible in practice. Only recently it has been recognized that a considerable amount of super-resolution is possible when the spatial extent of the object is not much greater than the resolution distance of the instrument. For example super-resolving methods could be used for detecting unresolved binary stars in astronomical images.

The important point is that a super-resolving method must implement explicitly the finite extent of the object: the domain of the object must be estimated and the method must search for a solution which is zero outside this domain. This remark introduces an important question in object restoration and, more generally, in the theory of inverse problems, namely the design of regularized solutions satisfying additional conditions (constraints). This question is also important from the theoretical point of view: the ill-posedness of the problem is due to insufficient information on the object as transmitted by the imaging system; the use of additional constraints reduces the class of the solutions which are compatible with the data and therefore can improve the restoration. This is the use of *a priori information* in the solution of inverse problems.

A constraint which has been widely investigated and used is the non-negativity of the solution, i.e. all the values of the restored image must be positive or zero. The physical meaning of the constraint is obvious. Its beneficial effect is to reduce ringing artifacts which affect the regularized solutions previously discussed in cases where the object contains bright spots over a black background or sharp intensity variations from zero to some positive value. The ringing is essentially related to the well-known Gibbs effect in the truncation of the Fourier series.

The constraint of non-negativity implies a particular lower bound on the values of the solution. Therefore one can also consider more general constraints consisting in given lower and/or upper bounds on the values of the solution, possibly combined with a constraint on the domain to produce, for instance, a super-resolved non-negative solution. The most general form of these constraints consists in requiring that the solution belongs to a given closed and convex set $\mathcal{C}$ in some functional space. A quite general method producing regularized solutions in a convex set is an iterative method which is essentially a gradient method for the minimization of the discrepancy functional of equation (4) with a projection on the set $\mathcal{C}$ at each iteration. In the case of object deconvolution the method has the following form: if f_k is the result of the k-th iteration, then f_{k+1} is given by

$$f_{k+1} = P_{\mathcal{C}} \left[f_k + \tau K^T * (g_r - K * f_k) \right] \tag{8}$$

where $P_{\mathcal{C}}$ is the convex projection onto the set $\mathcal{C}$, τ is a relaxation parameter and $K^T(\mathbf{x}) = K(-\mathbf{x})$. In general the algorithm, known as *projected Landweber method*, is initialized with $f_0 = 0$. An important property is that it has a regularization effect in the sense that iterations must not be pushed to convergence but conveniently stopped to avoid excessive noise contamination. In Figure 3 we compare in a specific example the result obtained by means of the regularised solution of equation (7) and that obtained by means of the projected iterative method with a lower and upper bound on the solution. The reduction of ringing is evident.

<Figure 3 near here>

Among the methods producing non-negative solutions we must mention the most popular one in Astronomy, the so-called *Richardson-Lucy method*, which is an iterative method for Maximum- Likelihood estimation. If the PSF has been normalized in such a way that the sum of all its values is one, it takes the following form

$$f_{k+1} = f_k \left(K^T * \frac{g_r}{K * f_k} \right) \quad . \tag{9}$$

The method is in general initialized with a uniform image and, as the projected Landweber method, must not be pushed to convergence to avoid noise amplification. Another algorithm suggested by statistical arguments and producing positive solution is the so-called *Maximum Entropy Method* (MEM).

The image restoration methods described above have wide applications both in microscopy and astronomy. For instance the minimization of the functional (6) with the additional constraint of non-negativity has been used for deconvolving images in microscopy to reduce the effect of the missing cone. On the other hand the Richardson-Lucy method was widely used for deconvolving the images of HST before installation of corrective optics.

Nowadays image deconvolution is becoming more and more important for the ground-based telescopes equipped with adaptive optics. Indeed, even if adaptive optics is able to provide a considerable compensation of the atmospheric blur, a further improvement can be obtained by deconvolving the detected images. The PSF is provided by the image of a suitable guide star.

<Figure 4 near here>

Finally it is worthy of note the Large Binocular Telescope, in construction on the Mount Graham in Arizona, because this instrument requires image restoration methods for a full exploitation of its imaging properties. The telescope (a picture is shown in Figure 4) consists of two $8.4m$ mirrors on a common mount: both mirrors are equipped with adaptive optics and are combined interferometrically in a Fizeau mode to reach the resolution of a $22.8m$ mirror in the direction of the baseline. The telescope can be rotated to record images of the same astronomical target with different orientations of the baseline, i.e. the line joining the centers of the two mirrors. Finally the images must be processed by means of multiple images deconvolution methods to obtain a unique high-resolution image, equivalent to that produced by a $22.8m$ mirror.

The applications of Inverse Problems with major social impact are in the area of medical imaging. The spectacular success of X-ray Computerized Tomography (CT), invented by G. H. Hounsfield at the beginning of the seventies, has stimulated the development of other imaging modalities, based essentially on the same principle, such as *positron emission tomography* (PET) and *single photon emission computerized tomography* (SPECT). Computed images are also those provided by Magnetic Resonance Imaging (MRI). In such a case however the computational problem is rather simple since it consists essentially in Fourier transform inversion.

<Figure 5 near here>

The basic principle of X-ray CT is the following. A finely collimated source S (see Figure 5a) emits a pencil of X-rays which propagates through the body along a straight line L up to a receiver R. If we know both the intensity I_0 of the source and the intensity I detected by the receiver, the logarithm of the ratio I_0/I is the integral of the linear attenuation function (roughly proportional to the density function of the body) along the line L. If the source and the receiver are moved along two parallel lines, having direction $\boldsymbol{\theta}$ and defining the plane Π (in practice a planar slice of the body under consideration), one obtains the integrals of the linear attenuation function $f(\mathbf{x})$ along all parallel lines orthogonal to $\boldsymbol{\theta}$. An angular scanning is obtained by rotating the source-receiver system.

The scanning variables in the plane Π are defined in Figure 5b: φ is the angle formed by the unit vector $\boldsymbol{\theta}$ with the x_1-axis and s is the signed distance between the origin and the integration line L. The integrals along all parallel lines orthogonal to $\boldsymbol{\theta}$ define a function of s which is called the *projection* of $f(\mathbf{x})$ in the direction $\boldsymbol{\theta}$

$$P_{\boldsymbol{\theta}}(s) = \int_{-\infty}^{+\infty} f(s\cos\varphi - u\sin\varphi, s\sin\varphi + u\cos\varphi)\; du \quad . \tag{10}$$

The set of the projections of $f(\mathbf{x})$ for all possible $\boldsymbol{\theta}$ is called the *Radon* *transform* of $f(\mathbf{x})$, in honor of the mathematician Johann Radon who first investigated the problem of recovering a function of two variables from its line integrals. Nowadays such a problem is known as *Radon transform inversion* or also *object reconstruction from projections.* Sampled values of the Radon transform, i. e. sampled values of the projections for a number of possible directions between 0 and π are the outputs of the acquisition system of the medical equipments known as CT-scanners (a picture of one of them is shown in Figure 6). Their data-processing system contains the implementation of an algorithm for Radon transform inversion and the final result is a set of computed images providing maps of slices of the human body.

<Figure 6 near here>

<Figure 7 near here>

The plot of the Radon transform of f in the (s, φ)-plane, obtained by representing its values as grey levels, is called the *sinogram* of f. In Figure 7 we give the picture of the sinogram of a brain slice. The horizontal rows provide representations of the projections of the phantom. In the picture the angle φ defining the direction θ takes value between 0 and 2π while the variable s takes values between $-a$ and a, if a is the radius of a disc containing the phantom. As it is evident, the sinogram is symmetric with respect to the point $(0, \pi)$. Each point in the rectangle corresponds to a straight line crossing the phantom and the straight lines through a fixed point of the phantom describe a sinusoidal curve in the (s, φ)-plane. This property has given raise to the name sinogram.

The sinogram is, in a sense, the image of the body slice under investigation as provided directly by a CT-scanner. It is obvious that an inspection of this image does not easily provide information about the body. Therefore it must be processed to obtain a more significant image. This is just the problem of Radon transform inversion which is a nice example of inverse problem. It is also ill-posed and therefore its solution must be treated with a special care.

The crucial point in Radon transform inversion is the *Fourier slice theorem* whose content is the following: the Fourier transform of the projection in the direction $\boldsymbol{\theta}$ of the function f is the Fourier transform of f on the straight line passing through the origin and having direction $\boldsymbol{\theta}$, i. e.

$$\hat{P}_{\boldsymbol{\theta}}(\omega) = \hat{f}(\omega\boldsymbol{\theta}) \quad . \tag{11}$$

This theorem clarifies the information content of CT data: each projection provides the Fourier transform of the function f along a well-defined straight line in the plane of the spatial frequencies. It is also the starting point for deriving the basic algorithm of data inversion in tomography, the *filtered back-projection* (FBP). It is essentially obtained from Fourier transform inversion in polar coordinates and is a two-step algorithm: the first step is a filtering of the projections by means of a ramp-filter; the second is the back-projection of the filtered sinogram. More precisely the two steps are the following:

- Filtering: for each $\boldsymbol{\theta}$ the projection $P_{\boldsymbol{\theta}}(s)$ is replaced by a filtered projection given by

$$Q_{\boldsymbol{\theta}}(s) = \int_{-\infty}^{+\infty} |\omega| \hat{P}_{\boldsymbol{\theta}}(\omega) e^{i\omega s} \, d\omega \ , \tag{12}$$

where the multiplication by the ramp filter $|\omega|$ derives from the Jacobian of polar coordinates;

- Back-projection: this operation is the dual of the projection operation: indeed the projection assigns to a straight line L with coordinates (s, φ), hence to a point in the domain of definition of the Radon transform, the integral of f along L; on the other hand the back-projection assigns the value of the Radon transform at (s, φ) to all points of the straight line L with the same coordinates (a pictorial representation of projection and back-projection is given in Figure 8); as a consequence the back-projection operator $R^{\#}$ assigns to each point $\mathbf{x}$ the sum (integral) of all values of the Radon transform corresponding to straight lines passing through $\mathbf{x}$

$$R^{\#}P_{\boldsymbol{\theta}}(\mathbf{x}) = \int_0^{2\pi} P_{\boldsymbol{\theta}}(x\cos\varphi + y\sin\varphi)\ d\varphi \quad . \tag{13}$$

<Figure 8 near here>

The back-projection operator, when applied directly to the projections as in equation (13), provides a blurred image of f; in order to get a satisfactory restoration it must be applied to the filtered projections $Q_{\boldsymbol{\theta}}(s)$. This point is illustrated in Figure 9.

<Figure 9 near here>

We point out that the ramp filter introduced in equation (12) amplifies the high frequency components of the projections hence the noise corrupting these components. This remark clarifies the ill-posedness of Radon transform inversion which also requires some kind of regularization. This is obtained by introducing an additional low-pass filter which reduces the effect of the high frequencies. The most frequently used combination of ramp and low-pass filter is the so-called *Shepp-Logan filter*.

X-ray tomography is also called *transmission computerized tomography* (TCT) because the image is obtained by detecting the X-rays transmitted by the body. It provides information about anatomical details of human organs because the map of the linear attenuation function is essentially the map of the density of the tissues.

A quite different type of information is obtained by the so-called *emission computerized tomography* (ECT) which is based on the administration of radio-nuclide-labelled agents known as radio-pharmaceutical . Their distribution in the body of the patient depends on factors such as blood flow, metabolic processes, etc. Then a map of this distribution is obtained by detecting the γ-rays produced by the decay of the radio-nuclides. Therefore

ECT yields functional information, in the sense that the images produced by ECT show the function of the tissues of the organs.

Two different modalities of ECT are usually considered:

- *single photon emission computerized tomography* (SPECT) which makes use of radio-isotopes where a single γ-ray is emitted per nuclear disintegration;
- *positron emission tomography*, which makes use of β^+-emitters, where the final result of a nuclear disintegration is a pair of γ-rays, propagating in opposite directions, produced by the annihilation of the emitted positron in the tissues.

In both cases it is necessary to detect the γ-rays coming from well-defined regions of the body and, to this purpose, different methods are used in the two cases.

In SPECT the discrimination of the γ-rays is obtained by means of a collimator which is a large slab covering the crystal detector face, consisting of holes separated by lead septa. In PET the collimation is obtained by means of pairs of detectors in coincidence. This technique, which is also called electronic collimation, is more accurate than the physical collimation used in SPECT and allows the design of systems with a great efficiency.

The basic reconstruction method in TCT, namely FBP, is often used in the reconstruction of SPECT and PET data. However several corrections are necessary in practice. The principal ones are due to the collimator blur and to the scattering of photons in the body. If an accurate model of data acquisition is developed by taking into account these effects then the reconstruction problem implies the inversion of a very large matrix which is sparse and ill-conditioned. To this purpose iterative regularization methods are used such that the basic operation required at each step is matrix-vector multiplication. Other imaging modalities which have been proposed for medical applications are electrical impedance and microwave tomography. The underlying inverse problems are basically non-linear so that the computational cost of the methods for their solution is, in general, too high for clinical applications. However the research in these areas is very active and therefore the situation could be improved in the near future.

References

[1] Bertero M., 1989, Linear inverse and ill-posed problems, in *Advances in Electronics and Electron Physics*, ed. P. W. Hawkes, Volume 75, pp. 1-120, Academic Press, New York.

[2] Bertero M. and Boccacci P., 1998, *Introduction to Inverse Problems in Imaging*, IOP Publishing, Bristol.

[3] Bertero M. and De Mol C., 1996, Super-resolution by data inversion, in *Progress in Optics*, ed. E. Wolf, Volume XXXVI, pp. 129-178, Elsevier, Amsterdam.

[4] Engl H. W., Hanke M. and Neubauer A., 1996, *Regularization of Inverse Problems*, Kluwer, Dordrecht.

[5] Groetsch C. W., 1993, *Inverse Problems in Mathematical Sciences*, Vieweg, Braunsschweig.

[6] Herman G. T., (ed.), 1979, *Image Reconstruction from Projections*, Springer, Berlin.

[7] Herman G. T., 1980, *Image Reconstruction from Projections. The fundamentals of Computerized Tomography*, Academic Press, New York.

[8] Huang T. S. (ed.), 1975, *Picture Processing and Digital Filtering*, Springer, Berlin.

[9] Kak A. C. and Slaney M., 1988, *Principles of Computerized Tomographic Imaging*, IEEE Press, New York.

[10] Krestel E. (ed.), *Imaging Systems for Medical Diagnostics*, Siemens Aktiengesellschaft, Berlin.

[11] Natterer F., 1986, *The Mathematics of Computerized Tomography*, Wiley, New York.

[12] Tikhonov A. N. and Arsenin V. Y., 1977, *Solutions of Ill-Posed Problems*, Winston/Wiley, Washington.

Figure captions

Figure 1 - Illustrating the effect of the inverse filter: a) Image of the Circinus galaxy taken on April 10, 1999 with the Wide Field Planetary Camera 2 of the Space Telescope; b) the PSF used for blurring the RGB components of the image (here we represent the red one); c) the blurred image obtained by convolving the components of the image in a) with the corresponding PSFs and adding noise; d) the restoration obtained by applying the inverse filter to the three components of c).

Figure 2 - Illustrating the effect of the regularization parameter. The blurred image is that shown in Figure 1 c). The sequence is obtained by increasing the value of the regularization parameter: a) $\mu = 0$ (inverse filter); b) $\mu = 0.001$; c) $\mu = 0.01$; d) $\mu = 0.05$; e) $\mu = 0.1$; f) $\mu = 0.5$. It is evident that the best restoration is that shown in c).

Figure 3 - Illustrating the effect of constraints on image restoration: a) Image of the Earth taken during Apollo 11 mission; b) Blurred image of a) obtained by assuming out-of-focus blur and adding noise: c) optimal restoration provided by the regularization method; the ringing around the boundary of the Earth is evident; d) optimal restoration obtained by means of the projected Landweber method with a lower and an upper bound on the solution at each iteration.

Figure 4 - Picture of the Large Binocular Telescope (LBT) in construction on the Mount Graham in Arizona. It consists of two mirrors on the same mount. The images of the two mirrors are combined interferometrically to produce a high resolution image in the direction of the line joining the centers of the mirrors (baseline).

Figure 5 - a) Scheme of the scanning procedure in X-ray tomography: the source(S)-receiver(R) pair is moved along a direction $\boldsymbol{\theta}$ orthogonal to

the line S-R, defining the plane Π to be imaged; when the scanning has been completed the system is rotated by a certain angle, the scanning procedure is repeated and so on. b) Definition of the variables used in the representation of the Radon transform; φ is the angle formed by the S-R line with the x_1-axis; s is the signed distance between the origin and the integration line (orthogonal to $\boldsymbol{\theta}$) and u is the coordinate on the integration line.

Figure 6 - Picture of a scanner for X-ray tomography. The annular part contains the acquisition and scanning systems.

Figure 7 - a) Picture of a brain slice; b) the corresponding sinogram. As explained in the text the sinogram is a gray-level representation of the Radon transform in the s, φ plane. Each horizontal line in b) corresponds to a projection of a).

Figure 8 - In a) the effect of the projection operation is shown by drawing the projections of a circular phantom corresponding to three directions. In b) we show the result obtained by applying the back-projection operation to the three projections given in a). It is evident that the picture provided by this operation has a maximum on the domain of the circular phantom.

Figure 9 - Illustrating the effect of the filtered back-projection. In this figure we show: a slice of the so-called Shepp-Logan phantom; the corresponding sinogram (indicated by the arrow *CT scanner*); the filtered sinogram (indicated by the arrow *filtering*; the restorations obtained by applying the back-projection operation both to the original sinogram and to the filtered sinogram.

Keywords

Inverse problems, ill-posed problems, image blurring, image restoration, image deconvolution, regularization methods, super-resolution, medical imaging, tomography, Radon transform, emission tomography

Nomenclature

Digital image

Direct problem

Inverse problem

Ill-posed problem

Image deconvolution

Image deblurring

Image restoration

Point spread function

Fourier transform

Transfer function

Noise-free image

Noisy image

Invisible object

Least-squares problem

Inverse filter

Regularization theory

Regularized solution

Regularization parameter

Discrepancy principle

Super-resolution

Out-of-band extrapolation

Positivity

Projected Landweber method

Richardson-Lucy method

Maximum Entropy method

Large Binocular Telescope (LBT)

Medical imaging

X-ray computerized tomography

Projection (of a function)

Radon transform

Radon transform inversion

Object restoration from projections

Fourier slice theorem

Filtered back-projection

Ramp filter

Back-projection

Transmission computerized tomography

Emission computerized tomography

Single photon emission computerized tomography (SPECT)

Positron emission tomography