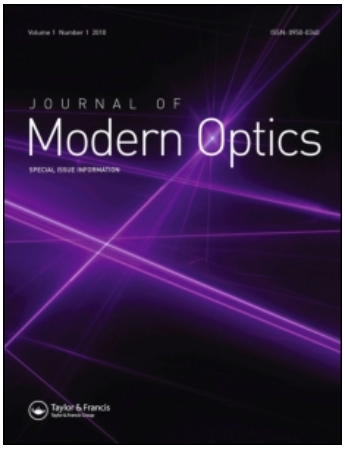

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Number of Degrees of Freedom in Inverse Diffraction

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Number of degrees of freedom in inverse diffraction

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Abstract. The problem of inverse diffraction from plane to plane is considered in the case where a finite aperture exists in the boundary plane. Singular values and singular functions for the problem are introduced, and the number of degrees of freedom is defined in terms of the distribution of the singular values. Numerical computations are presented for the one-dimensional problem, and it is shown that the effect of evanescent waves disappears at a distance of approximately one wavelength from the boundary plane, even when the dimension of the slit is comparable with the wavelength of the diffracted field.

1. Introduction

The problem of inverse diffraction from plane to plane is of some theoretical interest in optics and other fields. When the support of the field on the boundary plane $z=0$ is unrestricted, Fourier techniques can be used [1, 2] and the questions of uniqueness and stability of the solution can easily be investigated [3, 4]. An analysis from the point of view of regularization theory is given by Bertero and De Mol [5].

The problem can be formulated as follows. Consider a monochromatic scalar field

$$V(x, y, z; t) = u(x, y, z) \exp(-i\omega t) \tag{1.1}$$

propagating in the free space $z>0$; here the complex field amplitude $u(x, y, z)$ is a solution of the Helmholtz equation

$$(\Delta + k^2)u = 0, \quad k = \omega/c = 2\pi/\lambda, \tag{1.2}$$

and satisfies the Sommerfeld radiation condition at infinity,

$$\lim_{r\to\infty} \left[r\left(\frac{\partial u}{\partial r} - iku\right)\right] = 0, \quad |\theta| < \pi/2, \tag{1.3}$$

where r and θ are the polar coordinates. If we denote by $u_a(x,y)=u(x,y,a)$ the complex field amplitude on the plane $z=a>0$, generated by a field amplitude $u_0(x,y)$ on the boundary plane $z=0$, then (as is well known [6])

$$u_a(x,y)=\int\int_{-\infty}^{\infty} K_a^{(+)}(x-x',y-y')u_0(x',y')\,dx'dy', \tag{1.4}$$

where the *outgoing wave propagator* $K_z^{(+)}(x,y)$ is given by

$$K_z^{(+)}(x,y)=-\frac{1}{2\pi}\frac{\partial}{\partial z}\frac{\exp(ikr)}{r}$$

$$=\left(\frac{k}{2\pi}\right)^2\int\int_{-\infty}^{\infty}\exp[ik(px+qy+mz)]\,dpdq,$$

$$r=\sqrt{(x^2+y^2+z^2)}, \tag{1.5}$$

and

$$m=\begin{cases}(1-p^2-q^2)^{1/2}, & p^2+q^2\leqslant 1,\\ i(p^2+q^2-1)^{1/2}, & p^2+q^2>1.\end{cases} \tag{1.6}$$

Equation (1.4) gives the solution of the *direct problem*, namely to determine the field amplitude on the plane $z=a>0$, given the field amplitude on the boundary plane $z=0$. The *inverse problem* is then to determine the field amplitude on the boundary plane, given the field amplitude on the plane $z=a>0$.

To solve the inverse problem, we must solve the integral equation (1.4), of the first kind. This is an ill-posed problem, and the ill-posedness appears to be related to the existence of evanescent waves [4, 5]. Indeed, if by means of equations (1.4) and (1.5) we express the field amplitude $u_a(x,y)$ through an angular spectrum of plane waves,

$$u_a(x,y)=\int\int_{-\infty}^{\infty}\exp(ikma)U_0(p,q)\exp[ik(px+qy)]\,dpdq, \tag{1.7}$$

where

$$U_0(p,q)=\left(\frac{k}{2\pi}\right)^2\int\int_{-\infty}^{\infty}\exp[-ik(px+qy)]U_0(x,y)\,dxdy, \tag{1.8}$$

it is clear that the Fourier components of $U_0(p,q)$, corresponding to $p^2+q^2>1$, are attenuated in the propagation process. Conversely, these Fourier components are amplified in the inversion procedure and therefore, in the presence of noise, lead to meaningless results. This is also the origin of the divergent integrals discussed in [1, 2].

Methods for restoring stability in the solution of an ill-posed problem are provided by regularization or optimum filtering [4]; in both cases supplementary constraints expressing some prior knowledge of the solution are required and, if the constraints are sufficiently strong, then an ill-posed problem can be transformed into a well-posed one. In the case of inverse diffraction an example of this procedure is given by Shewell and Wolf [2]. If we assume that the field amplitude on the boundary plane $z=0$ contains only Fourier components corresponding to homogeneous waves, i.e. that $p^2+q^2<1$, then these components are not attenuated in the propagation process. Obviously the presence of noise can introduce fictitious

components of $u_a(x,y)$, corresponding to inhomogeneous (evanescent) waves, but these components can be suppressed by suitable (numerical) filtering. If we still denote by $u_a(x,y)$ the filtered field amplitude on the plane $z=a$, then the solution of the inverse diffraction problem can be expressed as [2]

$$u_0(x,y)=\int\int_{-\infty}^{\infty} K_a^{(-)}(x-x',y-y')u_a(x',y')\,dx'dy', \tag{1.9}$$

where $K_z^{(-)}(x,y)$ denotes the *ingoing wave propagator*

$$\begin{aligned} K_z^{(-)}(x,y) &= -\frac{1}{2\pi}\frac{\partial}{\partial z}\frac{\exp(-ikr)}{r} \\ &= \left(\frac{k}{2\pi}\right)^2 \int\int_{-\infty}^{\infty} \exp\left[ik(px+qy-m^*z)\right] dpdq, \\ & \qquad r=\sqrt{(x^2+y^2+z^2)}, \end{aligned} \tag{1.10}$$

and m is defined as in equation (1.3).

In most real situations encountered in optics, the above analysis is adequate since for light waves one may assume, in most cases, a cut-off in spatial-frequency components that is well below the wavenumber of the light. However, in some special cases this condition can be violated, e.g. when in the boundary plane there exists an aperture whose linear dimensions are comparable with the wavelength of the diffracted field. In such a case one needs a formulation of the diffraction problem similar to the description of imaging systems in terms of prolate spheroidal wavefunctions [7]. To this purpose we stress the analogy between equation (1.4) in the far-field region and the equation describing an imaging system; when $a \gg \lambda$, equation (1.4) takes the form

$$u_a(x,y)=\int\int_{-\infty}^{\infty} S_a(x-x',y-y')u_0(x',y')\,dx'dy', \tag{1.11}$$

where

$$S_a(x,y)=\int\int_{p^2+q^2\le 1} \exp\left[ik(px+qy+am)\right] dpdq. \tag{1.12}$$

Equations (1.11) and (1.12) show that the diffraction process is equivalent to an imaging system with a circular pupil affected by phase aberration; the cut-off frequency of the pupil is $\Omega=k=2\pi/\lambda$ and the corresponding Rayleigh distance is $R=1{\cdot}22(\lambda/2)$. Now, for a finite aperture in the boundary plane $z=0$, the integration in equation (1.11) must be restricted to some bounded region $\mathscr{A}$, while the domain of the variables x and y in the plane $z=a$ is unrestricted (in practice, it is very large with respect to the wavelength λ). To describe such an imaging system, where the object and image domains are allowed to differ, one needs a singular value analysis [8, 9] rather than the usual eigenvalue analysis [7]; the singular functions in the object plane are related to the generalized prolate spheroidal functions of Slepian [10], as shown by Bertero and Pike [8] and by De Santis and Palma [11].

In § 2 we introduce singular values and singular functions for the diffraction problem, and show their relation with the generalized prolate spheroidal functions in the far-field limit. In § 3 singular function expansions are used to solve both the direct and inverse diffraction problems, and the number of degrees of freedom is defined assuming, as usual [7, 8, 11, 12], a very simple noise model (additive, uncorrelated,

gaussian white noise). In § 4 we discuss the one-dimensional problem and present the results of numerical computations of the singular values. Finally, in § 5 we discuss the relation between the solution of the inverse diffraction problem presented in this paper and the solution given by the well-known sampling theorem.

2. Singular values and singular functions of the diffraction problem

As specified in the Introduction, we assume now that in the boundary plane $z=0$ there exists a finite aperture $\mathscr{A}$ through an infinite plane screen S, and that the field amplitude u_0 vanishes on S (a perfectly conducting screen in electromagnetics, or perfectly soft screen in acoustics). Then the integral in equation (1.4) has to be restricted to the aperture $\mathscr{A}$.

To present our results in a concise form, we need some notations borrowed from functional analysis. First we denote by X the space of the field amplitudes on the boundary plane $z=0$; if we assume, as usual, that these amplitudes are square-integrable, then X is a Hilbert space with the scalar product

$$(u_0, v_0)_X = \iint_{\mathscr{A}} u_0(x,y) v_0^*(x,y)\, dxdy. \tag{2.1}$$

Next we denote by Y the space of the (noisy) field amplitudes on the plane $z=a$; Y is also a Hilbert space with the scalar product

$$(u_a, v_a)_Y = \iint_{-\infty}^{\infty} u_a(x,y) v_a^*(x,y)\, dxdy. \tag{2.2}$$

The operator A_a which transforms the field amplitude on the boundary plane into the field amplitude on the plane $z=a$ is defined by equation (1.4), the integration now being restricted to the aperture $\mathscr{A}$. Therefore equation (1.4) can be written in the concise form

$$u_a = A_a u_0. \tag{2.3}$$

For an operator like A_a, which maps a function of a space X on to a function of a different space Y, it is impossible to introduce eigenvalues and eigenfunctions. However, we can introduce the *singular values* and *singular functions*. (A short introduction to the subject with applications to Fredholm integral equations of the first kind, is given by Miller [13].)

The first step is to show that the operator A_a in equation (2.3), is an integral operator of the *Hilbert-Schmidt class;* more precisely

$$\begin{aligned}\iint_{-\infty}^{\infty} dxdy \iint_{\mathscr{A}} dx'dy' \,|K_a^{(+)}(x-x', y-y')|^2 &= \iint_{\mathscr{A}} dx'dy' \iint_{-\infty}^{\infty} dxdy\, |K_a^{(+)}(x-x', y-y')|^2 \\ &= \frac{\pi}{\lambda^2}\left[1+\frac{1}{2(ka)^2}\right]\Sigma(\mathscr{A}) < +\infty, \end{aligned} \tag{2.4}$$

where $\Sigma(\mathscr{A})$ denotes the area of the aperture $\mathscr{A}$. Indeed, from the plane wave decomposition of the outgoing wave propagator, equation (1.5), and from Parseval

equality, we obtain

$$\int\int_{-\infty}^{\infty} |K_a^{(+)}(x-x', y-y')|^2\, dxdy = \left(\frac{k}{2\pi}\right)^2 \int\int_{-\infty}^{\infty} \exp\left[-2ka\,\mathrm{Im}\,(m)\right] dpdq$$

$$= \frac{\pi}{\lambda^2}\left\{1+2\int_1^{\infty} \exp\left[-2ka\sqrt{(\rho^2-1)}\right]\rho\, d\rho\right\} = \frac{\pi}{\lambda^2}\left[1+\frac{1}{2(ka)^2}\right]; \quad (2.5)$$

integrating over $\mathscr{A}$ and using the Fubini theorem we recover equation (2.4).

The previous result is fundamental since it implies the existence of singular values and singular functions. In the diffraction problem considered here such a mathematical tool can be introduced as follows. First, we need the adjoint A_a^* of the operator A_a, which is defined by the equation

$$(A_a u, v)_Y = (u, A_a^* v)_X, \quad (2.6)$$

where u is an amplitude in the boundary plane and v an amplitude in the plane $z=a$. From equation (2.6), by changing the order of the integration it is easy to show that:

$$(A_a^* v)(x, y) = \int\int_{-\infty}^{\infty} K_a^{(-)}(x'-x, y'-y) v(x', y')\, dx'dy', \quad (2.7)$$

where $K_a^{(-)}(x, y)$ is the ingoing wave propagator defined in equation (1.10), and the variables x and y are restricted to the aperture $\mathscr{A}$. Therefore A_a^* transforms a function in Y into a function in X and, moreover, this application is continuous.

Next we introduce the operators $A_a^* A_a$ and $A_a A_a^*$. The operator $A_a^* A_a$, acting on a field amplitude u_0, first transforms u_0 into an amplitude on the plane $z=a$ and then takes back this amplitude on to the aperture $\mathscr{A}$ of the boundary plane. A similar interpretation holds for the operator $A_a A_a^*$, the role of the two planes being inverted. Therefore these operators describe a 'cavity' limited by the boundary planes $z=0$ and $z=a$, the modes of which are related to the eigenvalues and eigenfunctions of the operators $A_a^* A_a$ and $A_a A_a^*$. It is also physically clear that, since both operators describe just the same 'cavity', their eigenvalues must coincide and a relation must exist between their eigenfunctions. This fact is a particular case of a general mathematical result which we summarize as follows. If the operator A_a is of the Hilbert–Schmidt class, then the self-adjoint positive definite operators $A_a^* A_a$ and $A_a A_a^*$ are also of the Hilbert–Schmidt class (more precisely of the trace class), and therefore they can be diagonalized. They have the same positive eigenvalues, each with the same finite multiplicity

$$A_a^* A_a \Phi_j = \mu_j^2 \Phi_j, \quad A_a A_a^* \Psi_j = \mu_j^2 \Psi_j, \quad j=0, 1, 2, \ldots \quad (2.8)$$

The eigenvalues μ_j^2, ordered in a non-increasing sequence ($\mu_0^2 \geqslant \mu_1^2 \geqslant \mu_2^2 \geqslant \ldots$), tend to zero for $j \to +\infty$. Finally, it is always possible to choose the eigenfunctions Φ_j and Ψ_j in such a way that

$$A_a \Phi_j = \mu_j \Psi_j, \quad A_a^* \Psi_j = \mu_j \Phi_j, \quad j=0, 1, 2, \ldots \quad (2.9)$$

and $\{\Phi_j\}_{j=0}^{+\infty}$ is an orthonormal system in X, while $\{\Psi_j\}_{j=0}^{+\infty}$ is an orthonormal system in Y. The quantities μ_j are called the *singular values* of the operator A_a, and Φ_j and Ψ_j the corresponding *singular functions*.

The singular values μ_j depend both on the aperture $\mathscr{A}$ and on the distance a between the two planes. We conclude this section by showing that, in the asymptotic

limit $a \to \infty$, the singular values μ_j tend to the square roots of the eigenvalues of the generalized prolate spheroidal functions.

From equations (2.3) and (2.7), it follows that $A_a^* A_a$ is an integral operator

$$(A_a^* A_a u)(x, y) = \iint_{\mathscr{A}} H_a(x-x', y-y') u(x', y')\, dx' dy', \tag{2.10}$$

where x and y are restricted to the aperture $\mathscr{A}$ and the kernel $H_a(x, y)$ is given by

$$H_a(x, y) = \iint_{-\infty}^{\infty} K_a^{(-)}(x', y') K_a^{(+)}(x'+x, y'+y)\, dx' dy'. \tag{2.11}$$

From the plane wave decompositions (1.5) and (1.10), and from Parseval equality, one finds that

$$H_a(x, y) = \left(\frac{k}{2\pi}\right)^2 \iint_{-\infty}^{\infty} \exp[-2ka \operatorname{Im}(m)] \exp[ik(px+qy)]\, dp dq, \tag{2.12}$$

and by simple computation one obtains

$$H_a(x, y) = \frac{k}{2\pi} \frac{J_1(k\rho)}{\rho} + \frac{k^2}{2\pi} \int_0^{\infty} \exp(-2kat) J_0[k\rho\sqrt{(t^2+1)}]\, t\, dt, \quad \rho = \sqrt{(x^2+y^2)}. \tag{2.13}$$

Since $|J_0[k\rho\sqrt{(t^2+1)}]| \leqslant 1$, the integral on the r.h.s. of equation (2.13) is uniformly bounded by $(2ka)^{-1}$, and it can be neglected for $a \gg \lambda$ so that

$$H_a(x, y) \simeq \frac{k}{2\pi} \frac{J_1(k\rho)}{\rho}, \quad a \gg \lambda. \tag{2.14}$$

We conclude that, in the limit $a \to \infty$, the singular functions Φ_j coincide with the generalized prolate spheroidal functions [10], i.e. with the eigenfunctions of an imaging system characterized by an aperture $\mathscr{A}$ and by a circular pupil whose radius is $k = 2\pi/\lambda$.

Moreover, the singular values μ_j coincide with the square roots of the eigenvalues of the generalized prolate spheroidal functions. In the particular case where the aperture $\mathscr{A}$ is a disc, the singular functions Φ_j can be expressed in terms of circular prolate functions [7, 10].

3. Number of degrees of freedom

To expand the boundary field u_0 and the diffracted field u_a as series in the singular functions Φ_j and Ψ_j, respectively, we need a completeness theorem. This follows from general results on Hilbert–Schmidt (more generally compact) operators; indeed, if the operator A_a is injective, then the orthonormal system $\{\Phi_j\}_{j=0}^{+\infty}$ is complete in X.

It is quite easy to prove that both operators A_a and A_a^* are injective, i.e. that the equations $A_a u = 0$ and $A_a^* v = 0$ have only the trivial solutions $u = 0$ and $v = 0$. Consider first the equation $A_a u = 0$ which, written explicitly, gives

$$\iint_{\mathscr{A}} K_a^{(+)}(x-x', y-y') u(x', y')\, dx' dy' = 0. \tag{3.1}$$

If we denote by $U(p,q)$ the Fourier transform (defined as in equation (1.8)) of the function which coincides with $u(x,y)$ on $\mathscr{A}$ and is zero on S, then from equations (3.1) and (1.5) we obtain

$$\exp(ikam)U(p,q)=0. \tag{3.2}$$

It follows that $U(p,q)=0$, and also that $u(x,y)=0$.

For the operator A_a^*, the equation $A_a^*v=0$ implies that

$$\iint_{\mathscr{A}}\int_{-\infty}^{\infty} K_a^{(-)}(x'-x,y'-y)v(x',y')\,dx'dy'=0, \tag{3.3}$$

the variables x and y being restricted to the aperture $\mathscr{A}$. But the l.h.s. of equation (3.3) is an analytic function of x and y—as follows from its representation by a Fourier integral—and therefore if it is zero on $\mathscr{A}$, it is zero everywhere. Then, from equations (3.3) and (1.10), we obtain

$$\exp(-ikam^*)V(p,q)=0. \tag{3.4}$$

It follows that $V(p,q)=0$, and also that $v(x,y)=0$.

From the completeness in X of the singular functions Φ_k, and from equations (2.3) and (2.9), we can expand the diffracted field in terms of the singular functions Ψ_k as

$$u_a(x,y)=\sum_{j=0}^{\infty}\mu_j(u_0,\Phi_j)_X\Psi_j(x,y). \tag{3.5}$$

Equation (3.5) also provides a route for solving the inverse diffraction problem. If u_a is given, its components with respect to the basis $\{\Psi_j\}_{j=0}^{+\infty}$ can be computed, and the components of the unknown boundary field amplitude u_0 with respect to the basis $\{\Phi_j\}_{j=0}^{+\infty}$ can be derived from equation (3.5); in such a way the solution of the inverse diffraction problem is obtained as

$$u_0(x,y)=\sum_{j=0}^{\infty}\frac{(u_a,\Psi_j)_Y}{\mu_j}\Phi_j(x,y). \tag{3.6}$$

Since $\mu_j\to 0$ when $j\to+\infty$, as stated in § 2, the series (3.6) converges only when the diffracted field u_a satisfies some rather particular conditions, which are always satisfied in the absence of noise. On the other hand, the series (3.6) does not usually converge when the diffracted field u_a is affected by noise or experimental errors; the ill-posedness of the inverse problem manifests itself through the clustering at zero of the μ_j.

In such a situation one can expect to obtain only approximate solutions of the inverse problem, like the truncated singular function expansions

$$u_0(x,y)=\sum_{j=0}^{M}\frac{(u_a,\Psi_j)_Y}{\mu_j}\Phi_j(x,y). \tag{3.7}$$

The main problem then is to determine the number of terms to be taken in equation (3.7). That can be done in the framework of standard optimum filtering theory [4] by making suitable statistical assumptions on the noise and on the boundary field amplitude.

The simplest assumption is to take the diffracted field to be given by

$$u_a=A_au_0+n, \tag{3.8}$$

where n denotes additive noise. Moreover, as usual [7, 8, 11, 12], one can assume that n is a realization of a white-noise gaussian process (power spectrum ε^2) and that u_0 is also a realization of a white-noise gaussian process (power spectrum E^2). Then the best truncated singular function expansion is obtained, in the sense of optimum filtering theory, when in equation (3.7) only those terms are retained such that

$$\mu_j \geqslant \frac{\varepsilon}{E}. \tag{3.9}$$

Since the singular values are ordered in a non-increasing sequence, equation (3.9) is satisfied by all the values of j up to some maximum value M_0. The quantity $N_0 = M_0 + 1$ is the *number of degrees of freedom* for the problem of inverse diffraction. In the limit $a \gg \lambda$, the number of degrees of freedom defined here coincides with the number of degrees of freedom of the equivalent imaging system, as defined in [8].

4. The one-dimensional problem—numerical results

If the aperture in the boundary plane $z=0$ is a slit (of width $2b$), and if the field amplitude is invariant with respect to translations in the direction of the axis of the slit (say the y axis), then the diffraction problem is one-dimensional and the operator A_a takes the form

$$(A_a u_0)(x) = \int_{-b}^{b} K_a^{(+)}(x-x') u_0(x')\, dx', \tag{4.1}$$

where

$$K_z^{(+)}(x) = \frac{k}{2\pi} \int_{-\infty}^{\infty} \exp\left[ik(px+mz)\right] dp \tag{4.2}$$

and

$$m = (1-p^2)^{1/2}, \quad p^2 \leqslant 1, \quad m = i(p^2-1)^{1/2}, \quad p^2 > 1. \tag{4.3}$$

The singular value analysis of §§ 2 and 3 can also be applied to this one-dimensional problem, the singular values now being the square roots of the eigenvalues of the integral operator

$$(A_a^* A_a u)(x) = \int_{-b}^{b} H_a(x-x') u(x')\, dx', \tag{4.4}$$

where

$$\begin{aligned} H_a(x) &= \frac{k}{2\pi} \int_{-\infty}^{\infty} \exp\left[-2ka\,\mathrm{Im}\,(m)\right] \exp\,(ikpx)\, dp \\ &= \frac{\sin\,(kx)}{\pi x} + \frac{k}{\pi} \int_{1}^{\infty} \exp\left[-2ka\sqrt{(p^2-1)}\right] \cos\,(kpx)\, dp. \end{aligned} \tag{4.5}$$

The singular functions $\Phi_j(x)$ are the (normalized) eigenfunctions of the operator (4.4), and the singular functions $\Psi_j(x)$ can be obtained through the relation

$$\Psi_j(x) = \frac{1}{\mu_j} \int_{-d}^{d} K_a^{(+)}(x-x') \Phi_j(x')\, dx'. \tag{4.6}$$

For a numerical analysis of the problem, it is convenient to introduce the dimensionless variables

$$\alpha=\frac{a}{\lambda}, \quad \beta=\frac{b}{\lambda}, \quad t=\frac{x}{d}, \tag{4.7}$$

2β being the width of the slit measured in wavelengths of the radiation field, and α the distance between the two planes in the same units.

The eigenvalue problem for the operator (4.4) now becomes

$$\int_{-1}^{1} H_{\alpha,\beta}(t-s)\Phi_j(s)\,ds=\mu_j^2\Phi_j(t), \tag{4.8}$$

where

$$H_{\alpha,\beta}(t)=\frac{\sin(2\pi\beta t)}{\pi t}+2\beta\int_1^{\infty}\exp[-4\pi\alpha\sqrt{(p^2-1)}]\cos(2\pi\beta tp)\,dp. \tag{4.9}$$

The following properties of the eigenvalues μ_j^2 can easily be proved.

(1) When $\alpha\to+\infty$, $\mu_j^2\to\lambda_j$, the eigenvalues of the linear prolate spheroidal functions $\Psi_j(c,x)$ with $c=2\pi\beta$ (c is defined as in [7]); the proof is similar to that given at the end of § 2.

(2) Each eigenvalue μ_j^2 is non-degenerate; this result follows from [14], since the Fourier transform of $H_{\alpha,\beta}(t)$ is non-increasing.

As a consequence of (2) the eigenvalues μ_j^2 can be ordered in a decreasing sequence converging to zero: $\mu_0^2>\mu_1^2>\mu_2^2>\ldots$, and, as in § 3, the number of degrees of freedom is given by $N_0=M_0+1$, where M_0 is the greatest of the values of the index j satisfying the condition (3.9).

The problem of the numerical computation of the eigenvalues of the integral operator (4.8), (4.9) is not easy since the kernel $H_{\alpha,\beta}(t)$ is not in a closed form. In the Appendix we show that the integral on the r.h.s. of equation (4.9) is analytic in a strip $4\alpha/\beta$ wide, symmetric with respect to the real axis. Therefore, if $\alpha<\beta/2$, $H_{\alpha,\beta}(t)$ can be computed over the whole interval $[-1,1]$ by means of a power-series expansion. In the Appendix it is also shown that the coefficients of the series expansion can be obtained by a very simple recurrence relation. Otherwise, the kernel $H_{\alpha,\beta}(t)$ must be computed by means of suitable quadrature formulae.

Another possibility is to Fourier-transform the problem (4.8) into an eigenvalue problem with a kernel having a closed form. A preliminary point is that, thanks to the analyticity of the kernel $H_{\alpha,\beta}(t)$, the eigenfunctions $\Phi_j(t)$, which in principle are defined only in the interval $[-1,1]$, can be continued to functions (still denoted by $\Phi_j(t)$) defined along the whole real axis. Then we define the Fourier transform of these functions as

$$\hat{\Phi}_j(p)=\int_{-\infty}^{\infty}\Phi_j(t)\exp[-i(2\pi\beta)pt]\,dt, \tag{4.10}$$

and by taking the Fourier transform of both sides of equation (4.8) we obtain

$$\hat{H}_{\alpha,\beta}(p)\int_{-\infty}^{\infty}\frac{\sin[2\pi\beta(p-q)]}{\pi(p-q)}\hat{\Phi}_j(q)\,dq=\mu_j^2\hat{\Phi}_j(p), \tag{4.11}$$

where, as follows from equation (4.9) and from the definition (4.10) of the Fourier transform,

$$\hat{H}_{\alpha,\beta}(p)=\begin{cases}1, & |p|\leqslant 1\\ \exp[-4\pi\alpha\sqrt{(p^2-1)}], & |p|>1.\end{cases} \tag{4.12}$$

The eigenvalue problem (4.11) can be expressed in a symmetric form by introducing the functions

$$\hat{\chi}_j(p)=\hat{\Phi}_j(p)/\sqrt{(\hat{H}_{\alpha,\beta}(p))}. \tag{4.13}$$

It then becomes

$$\int_{-\infty}^{\infty}\sqrt{(\hat{H}_{\alpha,\beta}(p)}\frac{\sin[2\pi\beta(p-q)]}{\pi(p-q)}\sqrt{(\hat{H}_{\alpha,\beta}(q))}\hat{\chi}_j(q)\,dq=\mu_j^2\hat{\chi}_j(p). \tag{4.14}$$

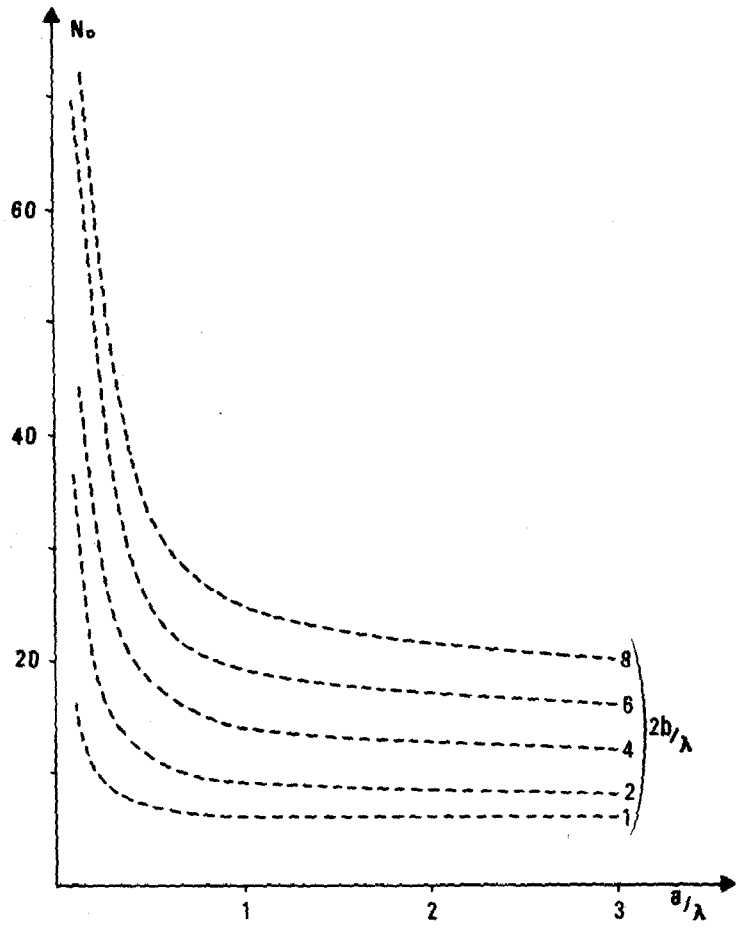

Figure 1. Number of degrees of freedom N_0, corresponding to $\varepsilon/E=10^{-2}$, as a function of the distance $\alpha=a/\lambda$ between the two planes, for various values of the slit width $2\beta=2b/\lambda$.

It is obvious that the eigenvalues of the problem (4.14) coincide with the eigenvalues of the problem (4.8); the relation between eigenfunctions is given by

$$\Phi_j(t)=\frac{\sqrt{\beta}}{\mu_j}\int_{-\infty}^{\infty}\sqrt{(\hat{H}_{\alpha,\beta}(p))}\hat{\chi}_j(p)\exp\left[i(2\pi\beta)pt\right]dp. \tag{4.15}$$

If the eigenfunctions $\hat{\chi}_j(p)$ are orthonormal, then the corresponding eigenfunctions $\Phi_j(t)$ are also orthonormal over the interval $[-1,1]$ since, by means of equations (4.15) and (4.14), one can obtain the relation

$$\int_{-1}^{1}\Phi_j(t)\Phi_i^*(t)\,dt=\frac{\mu_j}{\mu_i}\int_{-\infty}^{\infty}\hat{\chi}_j(p)\hat{\chi}_i^*(p)\,dp. \tag{4.16}$$

The eigenvalues μ_j^2 have been computed by solving both equations (4.8) and (4.14) for values of α in the range $0{\cdot}1<\alpha<10$ and for values of β in the range $0{\cdot}25<\beta<3$ ($\beta=0{\cdot}25$ corresponds to a slit of width $\lambda/2$).

In the case of equation (4.8), the eigenvalues have been determined by means of a method developed by Hämmerlen and Schumaker [15]: the kernel is approximated by means of spline functions, and the values of the kernel at the knots are computed by a gaussian quadrature formula (the accuracy of the result could be checked, for $\alpha>\beta/2$, by means of the power-series expansion discussed in the Appendix). This method is not appropriate for equation (4.14) since the kernel is not smooth and it cannot be accurately approximated by spline functions. For this reason, equation

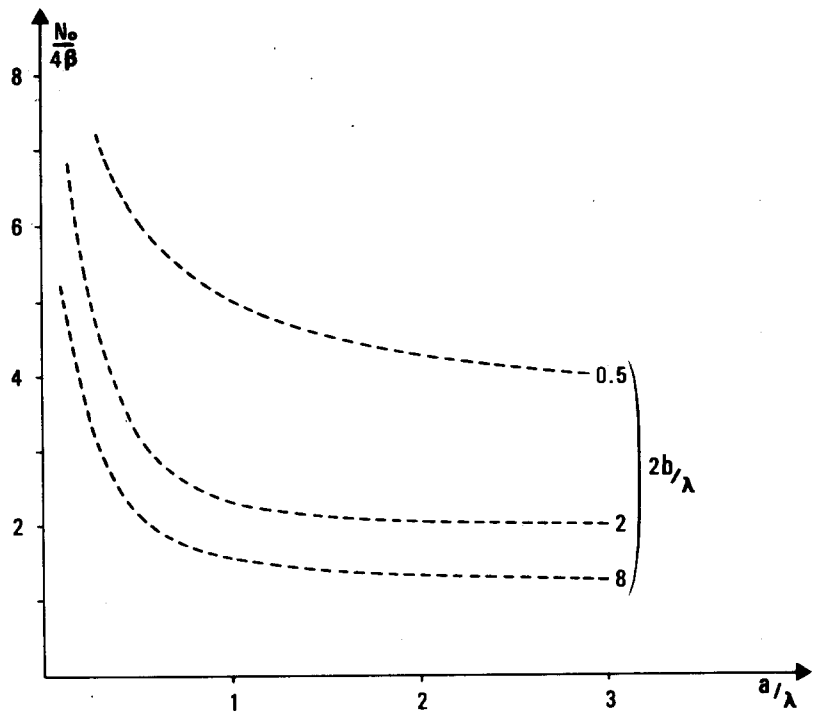

Figure 2. Number of resolution elements ($N_0/(2b)$) $R=N_0/(4\beta)$ within the Rayleigh distance $R=\lambda/2$, as a function of the distance $\alpha=a/\lambda$ between the two planes, for various values of the slit width $2\beta=2b/\lambda$ ($\varepsilon/E=10^{-2}$).

(4.14) has been discretized by means of gaussian quadrature formulae using a large number of points (~ 400), and the eigenvalues have been computed by a computer library program, assuming that they are all positive and less than one. The results obtained by the two methods are in good agreement. The number of degrees of freedom, corresponding to $\varepsilon/E = 10^{-2}$, and for various values of α and β are shown in figure 1. In figure 2 are shown the number of degrees of freedom contained in the Rayleigh distance $R = \lambda/2$, i.e. $(N_0/2b)R = N_0/(4\beta)$. It is obvious that the number of degrees of freedom tends to infinity when the distance between the two planes tends to zero.

5. Concluding remarks

As is clear from figures 1 and 2, the effect of evanescent waves on the number of degrees of freedom is already very small when the distance between the planes is approximately equal to λ, even if the width of the slit is of the order of λ. As indicated in figure 2, superresolution—i.e. resolution beyond the Rayleigh limit $R = \lambda/2$—is possible, but this possibility is a consequence of the effect discussed by Bertero and Pike [8] and not a consequence of evanescent waves.

It could be objected that our analysis is exceedingly involved since, in the case of a slit, the solution of the inverse diffraction problem can be obtained by means of the sampling theorem. However, we shall show, by considering the simple noise model described in § 3, that the equivalence between the solution given by the singular function expansions and the solution given by the sampling theorem disappears in the presence of noise, and that singular function expansions provide the most appropriate method.

By using the sampling theorem for the Fourier transform of $u_0(x)$, which is zero outside the interval $[-b, b]$, it is easy to show that the solution of the inverse diffraction problem can be written as

$$u_0(x) = \sum_{j=-\infty}^{+\infty} \frac{\pi}{kb} \exp[-ikam(p_j)] U_a(p_j) \exp(ikxp_j), \tag{5.1}$$

where $U_a(p)$ denotes the Fourier transform of the diffracted field $u_a(x)$,

$$U_a(p) = \frac{k}{2\pi} \int_{-\infty}^{\infty} \exp(-ikpx) u_a(x)\, dx. \tag{5.2}$$

Here the p_j are the sampling points,

$$p_j = \frac{\pi j}{kb} = \frac{j}{2\beta}, \tag{5.3}$$

and the $m(p_j)$ are the values of the function $m = m(p)$, defined in equation (4.3), at the points p_j. Indeed, if we represent $u_0(x)$ as a Fourier series in the interval $[-b, b]$, then the Fourier coefficients of $u_0(x)$ coincide with the values of $U_0(p)$ at the sampling points p_j, and $U_0(p)$ is related to $U_a(p)$ through equations (4.1) and (4.2).

In the absence of noise, due to the uniqueness of the solution of the inverse diffraction problem, the Fourier expansion (5.1) coincides with the singular function expansion

$$u_0(x) = \sum_{j=0}^{\infty} \frac{u_{a,j}}{\mu_j} \Phi_j(x), \tag{5.4}$$

where

$$u_{a,j}=(u_a, \Psi_j)_Y=\int_{-\infty}^{\infty} u_a(x)\Psi_j^*(x)\,dx. \tag{5.5}$$

However, the equivalence between the two methods disappears in the presence of noise, in the sense that one can find a precise criterion for the truncation of the series (5.4), but not for the truncation of the series (5.1).

If we take, for instance, the noise model used in § 3, then the variances of the coefficients $u_{a,j}$ of the expansion (5.4) are given by

$$\langle |u_{a,j}|^2\rangle = \mu_j^2 E^2 + \varepsilon^2 \tag{5.6}$$

and, when condition (3.9) is satisfied, the variance $\mu_j^2 E^2$ of the 'signal' is greater than the variance ε^2 of the noise; for those components one can extract signal from noise and therefore it is precisely those components that must be retained in the expansion (5.4).

On the other hand, if we look for a similar criterion for the truncation of the series (5.1), we must compute the correlation functions of $U_a(p)$, which is given by:

$$\langle U_a(p)U_a^*(p')\rangle = E^2\frac{k}{2\pi}\frac{\sin[kb(p-p')]}{\pi(p-p')}\exp[ika(m-m')]+\varepsilon^2\frac{k}{2\pi}\delta(p-p'). \tag{5.7}$$

The contribution of the noise to the variance of $U_a(p)$ is infinite, while the contribution of u_0 is finite. In such a situation it is not possible to give a prescription for the truncation of the series (5.1), and therefore it is not possible to give a definition of the number of degrees of freedom.

Appendix

Let us denote by $I_{\alpha,\beta}(t)$ the integral on the r.h.s. of equation (4.9):

$$I_{\alpha,\beta}(t)=\int_1^{\infty}\exp[-4\pi\alpha\sqrt{(p^2-1)}]\cos(2\pi\beta tp)\,dp. \tag{A 1}$$

This function is also defined for complex values of the argument, say $t+i\sigma$, and it is analytic in the strip $|\sigma|<2\alpha/\beta$. Indeed, for large values of p we have the approximation

$$|\exp[-4\pi\sqrt{(p^2-1)}]\cos[2\pi\beta(t+i\sigma)p]|\lesssim\exp[-2\pi(2\alpha-\beta|\sigma|)p], \tag{A 2}$$

and therefore the integral (A 1) is uniformly convergent in any bounded region within the strip $|\sigma|<2\alpha/\beta$. As a consequence, if we expand $I_{\alpha,\beta}(t)$ as a power series in the neighbourhood of $t=0$, the radius of convergence of this series is $2\alpha/\beta$. It follows that the series converges over the whole interval $[-1,1]$ only if $2\alpha/\beta>1$.

The Taylor expansion of $I_{\alpha,\beta}(t)$ can be obtained from the Taylor expansion of $\cos(z)$ as

$$I_{\alpha,\beta}(t)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(2\pi\beta t)^n I_n, \tag{A 3}$$

where

$$I_n=\int_1^{\infty}\exp[-4\pi\alpha\sqrt{(p^2-1)}]\,p^{2n}dp. \tag{A 4}$$

The integrals I_n can be determined, as we shall prove, from the recurrence relation

$$I_n=\frac{2n(2n-1)}{(4\pi\alpha)^2}I_{n-1}-\frac{(2n-1)(2n-3)}{(4\pi\alpha)^2}+\frac{1}{(4\pi\alpha)^2}, \tag{A 5}$$

so that only I_0 and I_1 need to be computed. If we introduce the coefficients

$$c_n=\frac{(4\pi\alpha)^{2n}}{(2n)!}I_n, \tag{A 6}$$

then we have the computational scheme

$$\left.\begin{aligned} I_{\alpha,\beta}(t)&=\sum_{n=0}^{\infty}(-1)^n c_n\left(\frac{\beta t}{2\alpha}\right)^{2n},\\ c_n&=c_{n-1}-\frac{(4\pi\alpha)^2}{2n(2n-2)}c_{n-2}+\frac{(4\pi\alpha)^{2n-2}}{(2n)!},\\ c_0&=\int_0^{\infty}\frac{q\exp(-4\pi\alpha q)}{\sqrt{(q^2+1)}}dq,\quad c_1=(4\pi\alpha)^2\int_0^{\infty}q\sqrt{(q^2+1)}\exp(-4\pi\alpha q)\,dq. \end{aligned}\right\} \tag{A 7}$$

Note that the series (A 7) is an alternating series and that, when it converges, $c_n[\beta t/(2\alpha)]^{2n}$ is a decreasing sequence (at least when n is large); therefore the truncation error is bounded by the first neglected term.

Finally, we give the proof of the recurrence relation (A 5). By partial integration we obtain the relation

$$\begin{aligned} I_n=\frac{2n-1}{4\pi\alpha}\int_1^{\infty}\sqrt{(p^2-1)}p^{2n-2}\exp[-4\pi\alpha\sqrt{(p^2-1)}]\,dp\\ +\frac{1}{4\pi\alpha}\int_1^{\infty}\frac{p^{2n}}{\sqrt{(p^2-1)}}\exp[-4\pi\alpha\sqrt{(p^2-1)}]\,dp, \end{aligned} \tag{A 8}$$

which can also be written as

$$\begin{aligned} I_n=\frac{2n}{4\pi\alpha}\int_1^{\infty}\sqrt{(p^2-1)}p^{2n-2}\exp[-4\pi\alpha\sqrt{(p^2-1)}]\,dp\\ +\frac{1}{4\pi\alpha}\int_1^{\infty}\frac{p^{2n-2}}{\sqrt{(p^2-1)}}\exp[-4\pi\alpha\sqrt{(p^2-1)}]\,dp. \end{aligned} \tag{A 9}$$

A second partial integration on both terms of equation (A 9) gives

$$\begin{aligned} I_n=\frac{2n}{(4\pi\alpha)^2}\int_1^{\infty}[(2n-1)p^{2n-2}-(2n-3)p^{2n-4}]\exp[-4\pi\alpha\sqrt{(p^2-1)}]\,dp\\ +\frac{1}{(4\pi\alpha)^2}+\frac{1}{(4\pi\alpha)^2}\int_1^{\infty}(2n-3)p^{2n-4}\exp[-4\pi\alpha\sqrt{(p^2-1)}]\,dp, \end{aligned} \tag{A 10}$$

which implies the recurrence relation (A 5).

We note that the convergence of the series can be accelerated by means of Padé-approximant techniques [16], which could also be used for computing $I_{\alpha,\beta}(t)$ outside the circle of convergence by means of the expansion (A 3).

Le problème de la diffraction inverse de plan à plan est étudié dans le cas où il existe une ouverture finie dans le plan limite. Des valeurs singulières et des fonctions singulières pour le problème sont introduites et le nombre de degrés de liberté est défini d'après la répartition des valeurs singulières. Des calculs numériques sont présentés dans le cas du problème unidimensionnel et on montre que l'effet des ondes évanescentes disparait à une distance d'approximativement une longueur d'onde à partir du plan limite même lorsque la dimension de la fente est comparable à la longueur d'onde du champ diffracté.

Es wird das Problem der inversen Beugung von einer Ebene zur nächsten für den Fall betrachtet, daß in der Grenzebene eine endliche Apertur existiert. Singuläre Werte und singuläre Funktionen werden eingeführt und die Anzahl der Freiheitsgrade wird durch die Verteilung der singulären Werte definiert. Numerische Rechnungen werden für den eindimensionalen Fall präsentiert und es wird gezeigt, daß die Auswirkungen von quergedämpften Wellen in einem Abstand von etwa einer Wellenlänge von der Grenzebene verschwinden und zwar auch dann, wenn die Spaltabmessung vergleichbar ist mit der Wellenlänge des Beugungsfelds.

References

[1] SHERMAN, G. C., 1967, *J. opt. Soc. Am.*, **57,** 1490.
[2] SHEWELL, J. R., and WOLF, E., 1968, *J. opt. Soc. Am.*, **58,** 1596.
[3] HOENDERS, B. J., 1978, *Inverse Source Problems in Optics*, edited by H. P. Baltes, Topics in Current Physics Vol. 9 (Berlin, Heidelberg, New York: Springer-Verlag), p. 41.
[4] BERTERO, M., DE MOL, C., and VIANO, G. A., 1980, *Inverse Scattering Problems in Optics*, edited by H. P. Baltes, Topics in Current Physics Vol. 20 (Berlin, Heidelberg, New York: Springer-Verlag), p. 161.
[5] BERTERO, M., and DE MOL, C., 1981, *I.E.E.E. Trans. Antennas Propag.*, **29,** 368.
[6] LUNEBURG, R. K., 1964, *Mathematical Theory of Optics* (Berkeley: University of California Press), p. 1.
[7] FRIEDEN, B. R., 1971, *Progress in Optics*, Vol. 9, edited by E. Wolf (Amsterdam: North-Holland), p. 311.
[8] BERTERO, M., and PIKE, E. R., 1982, *Optica Acta*, **29,** 727.
[9] BERTERO, M., BOCCACCI, P., and PIKE, E. R., 1982, *Optica Acta*, **29,** 1599.
[10] SLEPIAN, D., 1964, *Bell Syst. tech. J.*, **43,** 3009.
[11] DE SANTIS, P., and PALMA, C., 1976, *Optica Acta*, **23,** 743.
[12] BENDINELLI, M., CONSORTINI, A., RONCHI, L., and FRIEDEN, B. R., 1974, *J. opt. Soc. Am.*, **64,** 1498.
[13] MILLER, G. F., 1974, *Numerical Solution of Integral Equations*, edited by L. M. Delves and J. Walsh (Oxford: Clarendon Press), p. 175.
[14] GORI, F., and PALMA, C., 1975, *J. Phys.* A, **8,** 1709.
[15] HÄMMERLIN, G., and SCHUMAKER, L. L., 1980, *Numer. Math.* **34,** 125.
[16] BAKER, G. A., and GAMMEL, J. L. (editors), 1970, *The Padé Approximants in Theoretical Physics* (New York: Academic Press), p. 311.