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On the recovery and resolution of exponential relaxation rates from experimental data. III. The effect of sampling and truncation of data on the Laplace transform inversion

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The method of singular function expansions, which in previous papers in this series was used for the inversion of the Laplace transform for the cases, respectively, of continuous data and continuous solution and discrete data and discrete solution, is extended to cover the case of discrete data and continuous solution.

Two discrete data points distributions are considered: uniform and geometric. For both we prove that the singular values and the singular functions (in the solution space) of the problem with discrete data and continuous solution converge to the singular values and singular functions of the problem with continuous data and continuous solution when the number of points tends to infinity and the distance between adjacents points tends to zero.

Furthermore, we show by means of numerical computations that for geometrically sampled data it is possible to obtain even better approximations of the greatest singular values than in the discrete-to-discrete case using a similar number of data points. Excellent approximations of the continuous-to-continuous case singular functions are also obtained. Implementation of the inversion procedure gives continuous solutions with high computational efficiency.

1. INTRODUCTION

In a previous paper (Bertero *et al.* 1982), hereafter referred to as I, the concepts of 'resolution limit' and of 'number of degrees of freedom', which are fundamental in the theory of band-limited imaging or communication systems, were extended to the problem of the Laplace transform inversion, where one is concerned to recover and resolve exponential relaxation rates. The main purpose of the paper was to quantify the improvement in resolution obtained by using *a priori* knowledge of the support of the unknown solution and to this purpose the singular value decomposition of the Laplace transform inversion was investigated.

In I the data was assumed to be known everywhere, but since the number of experimental data points is necessarily finite, in a further contribution (Bertero *et al.* 1984), hereafter referred to as II, the case of discrete data was considered. This was done using also a model of discrete solution, namely the exponential sampling model of Ostrowsky *et al.* (1981). Optimum choice of experimental data points was defined in II as the choice giving the best conditioned inversion matrix, and the case of a uniform distribution as well as the case of a geometric distribution of data points was investigated from this point of view. One result was that geometrical sampling of data requires a number of sample points much smaller than uniform sampling, for a given ill-conditioning of the inversion matrix.

Modelling the solution by the set of δ-functions of the exponential sampling method provided a practical way for realizing inversions, which has been used in practice, together with a 'sampling theorem' interpolation scheme (Ostrowsky *et al.* 1981) to provide a continuous function for the output of the calculation.

In the present paper we show that the singular function method is sufficiently flexible to accommodate directly transformations between continuous functions and discrete vectors by simple modifications of our previous continuous-to-continuous case analysis and we calculate singular systems for this case. We make some comparisons of our new results with those obtained previously and show that they provide an excellent approach to the problem of Laplace inversion from experimental data.

If f is a function with bounded strictly positive support, we can assume, without loss of generality (see I), that f is supported in $[1, \gamma]$. Then we define the Laplace transform of $f \in L^2(1, \gamma)$ by

$$g(p) = (Kf)(p), \tag{1.1}$$

where

$$(Kf)(p) = \int_1^{\gamma} \mathrm{e}^{-pt} f(t)\,\mathrm{d}t. \tag{1.2}$$

The operator $K: L^2(1, \gamma) \to L^2(0, +\infty)$ is compact and injective with domain $L^2(1, \gamma)$ and range dense in $L^2(0, +\infty)$. We denote by $\{\alpha_k; u_k, v_k\}_{k=0}^{+\infty}$ the singular system of K, namely the set of the solutions of the coupled equations

$$Ku_k = \alpha_k v_k, \quad K^* v_k = \alpha_k u_k. \tag{1.3}$$

As is known, the singular functions u_k are also the eigenfunctions, associated with the eigenvalues α_k^2, of the compact, self-adjoint, positive definite operator $\tilde{K} = K^*K$, whose explicit expression is

$$(\tilde{K}f)(t) = \int_1^{\gamma} \frac{f(s)}{t+s}\,\mathrm{d}s, \quad 1 \leqslant t \leqslant \gamma. \tag{1.4}$$

In I it is proved that all the eigenvalues of $\tilde{K}$ have multiplicity 1, so that the singular values α_k can be ordered in a strictly decreasing sequence. From the compactness of K, it follows that, $\alpha_k \to 0$ when $k \to +\infty$.

The problem of inverting (1.1) is ill-posed and therefore one needs regularization techniques. Since the singular values drop to zero extremely fast, the various regularized solutions are practically equivalent and therefore one can use the most simple one, namely a singular function truncated expansion

$$\tilde{f}(t) = \sum_{k=0}^{J-1} \frac{g_k}{\alpha_k} u_k(t), \tag{1.5}$$

where

$$g_k = \int_0^{+\infty} g(p)\, v_k(p)\, \mathrm{d}p \tag{1.6}$$

and J is the number of singular values greater than some threshold value (see I). In most cases J is rather small: for instance, when $\gamma = 5$, we have only 5 singular values greater than 10^{-3}.

In the absence of noise, $\tilde{f}(t)$ is a 'smoothed' version of the unknown function f. Indeed, if we put $g = Kf$ in (1.6) and we use the second equation (1.3) we obtain

$$\tilde{f}(t) = \int_1^{\gamma} M(t,s) f(s)\, \mathrm{d}s, \tag{1.7}$$

where

$$M(t,s) = \sum_{k=0}^{J-1} u_k(t)\, u_k(s). \tag{1.8}$$

Since it is possible to extract from continuous data only a small number of components of f it is reasonable to argue that the same goal can be obtained by using only a small number of conveniently placed data points. Results in this direction were obtained in II, using the exponential sampling model of Ostrowsky *et al.* (1981). In that paper it was shown that if data points are geometrically distributed and optimally placed, then, roughly speaking, the number of data points required for inversion is not much greater than the 'number of degrees of freedom' J.

As in II we will consider mainly two data point distributions: a set of N equidistant points:

$$p_n = c + d(n-1); \quad n = 1, 2, \ldots, N, \tag{1.9}$$

characterized by two parameters, the position c of the first point and the distance d between adjacent points, and a set of N points forming a geometric progression

$$p_n = c\Delta^{n-1}; \quad n = 1, 2, \ldots, N, \tag{1.10}$$

characterized again by two parameters, the position c of the first point and the dilation factor Δ, giving the ratio between adjacent data points. We will call the choice (1.9) uniform sampling and the choice (1.10) geometrical sampling of the data. Remark that geometrical sampling is a uniform sampling in the variable $\ln p$ and, for this reason, geometrical sampling seems to be more natural than uniform sampling, since using this variable the Laplace transformation can be written as a convolution integral. Arguments founded on the sampling theorem are also given in Pike *et al.* (1983).

The paper is organized as follows. In §2 we give a general outline of the singular value decomposition of the Laplace transform inversion with discrete data.

In §§3, 4 we investigate the behaviour of the singular values and singular functions when the number of points tend to infinity, both for uniform sampling and for geometrical sampling. In §5 the analytical investigations of the singular values and singular functions developed in the previous sections are completed by numerical computations. It is shown by an example that geometrical sampling gives a much better reproduction of the relevant singular values and of the corresponding singular functions, with a much smaller number of data points than uniform sampling. Finally, in §6 the problem of resolution limits, which was the main aim of I, is reinvestigated to show explicitly the intrinsic limitations of Laplace transform inversion and the beneficial effect of the knowledge of the support of the unknown function.

2. Singular value decomposition of the Laplace transform inversion with discrete data

Let us assume that the Laplace transform g of the function $f \in L^2(1,\gamma)$ is given at the points $p_1, p_2, \ldots p_N$ without specifying, for the moment, the distribution of these points. Then we call K_N the operator that transforms the function f into the vector whose components are the values of g at the points p_n:

$$(K_N f)(p_n) = \int_1^\gamma e^{-p_n t} f(t)\,dt; \quad n = 1, \ldots, N. \tag{2.1}$$

The operator K_N is an operator from $X = L^2(1,\gamma)$ into the N-dimensional euclidean space $Y = E^N$, in which we introduce the scalar product

$$(g,h)_Y = \sum_{n=1}^{N} w_n\, g(p_n)\, h(p_n). \tag{2.2}$$

The problem of the choice of the weights w_n, which must be introduced when the distribution of the data points is not uniform, will be discussed in the next sections.

The adjoint operator K_N^*, which transforms a vector of Y into a function of X, has the expression

$$(K_N^* g)(t) = \sum_{n=1}^{N} w_n\, g(p_n)\, e^{-p_n t}. \tag{2.3}$$

The operator K_N is (trivially) compact and we will denote by $\{\alpha_{N,k};\, u_{N,k}, v_{N,k}\}_{k=0}^{N-1}$ the singular system of K_N, namely the set of the solutions of the coupled equations

$$K_N u_{N,k} = \alpha_{N,k} v_{N,k}, \quad K_N^* v_{N,k} = \alpha_{N,k} u_{N,k}. \tag{2.4}$$

The singular functions $u_{N,k}$ are also the eigenfunctions, associated with the eigenvalues $\alpha_{N,k}^2$, of the finite rank integral operator $\tilde{K}_N = K_N^* K_N$ given by

$$(\tilde{K}_N f)(t) = \int_1^\gamma T_N(t+s) f(s)\,ds, \tag{2.5}$$

where

$$T_N(t) = \sum_{n=1}^{N} w_n\, e^{-p_n t}. \tag{2.6}$$

As in the problem with continuous data, we assume that the singular values $\alpha_{N,k}$ are ordered in a decreasing sequence. The normal solution (namely the solution of smallest norm) of the problem: given $g(p_n)$ find $f \in L^2(1,\gamma)$ such that $g(p_n) = (K_N f)(p_n)$; $n = 1, \ldots, N$, can be expressed in terms of the singular system of K_N. The solution is extremely ill-conditioned even when N is moderately large and therefore we must use regularization or filtering techniques as for the problem (1.1). If we truncate the expansion and we take the same number of terms as in (1.5), we obtain

$$\tilde{f}_N(t) = \sum_{k=0}^{J-1} \frac{g_{N,k}}{\alpha_{N,k}} u_{N,k}(t), \tag{2.7}$$

where

$$g_{N,k} = \sum_{n=1}^{N} w_n g(p_n) v_{N,k}(p_n), \tag{2.8}$$

$v_{N,k}(p_n)$ being the nth component of the singular vector $v_{N,k}$.

The problem of investigating the effect of sampling and truncation of data on the Laplace transform inversion is just the problem of estimating the difference between the truncated solutions $\tilde{f}_N$ and $\tilde{f}$. Furthermore, since the function $\delta\tilde{f} = \tilde{f} - \tilde{f}_N$ represents the error due to the discretization of the data, we can say that a set of data points is acceptable when $\delta\tilde{f}$ has the same magnitude (for instance in the sense of the norm of X) as the noise contribution to $\tilde{f}$.

In the absence of noise, we can find for $\tilde{f}_N$ an expression similar to (1.9). Indeed, if we put $g = K_N f$ in (2.8) we obtain

$$\tilde{f}_N(t) = \int_1^{\gamma} M_N(t,s) f(s)\, \mathrm{d}s, \tag{2.9}$$

where

$$M_N(t,s) = \sum_{k=0}^{J-1} u_{N,k}(t)\, u_{N,k}(s). \tag{2.10}$$

By comparing (2.9), (2.10) with (1.7), (1.8) we see that, in the absence of noise, $\tilde{f}_N$ can be an accurate estimate of $\tilde{f}$ if the singular functions $u_{N,k}$ $(k = 0, 1, \ldots, J-1)$ provide accurate approximations of the corresponding singular functions u_k. In the presence of noise there is an extra term in (2.9) as well as in (1.7). This term, depending on the noise, contains explicitly the singular values. Therefore it is also necessary to require that the singular values $\alpha_{N,k}$ $(k = 0, 1, \ldots, J-1)$ provide accurate approximations of the corresponding singular values α_k.

Now, if we remark that u_k is the eigenfunction of the operator $\tilde{K}$ (equation (1.4)), associated with the eigenvalue α_k^2 and that, analogously, $u_{N,k}$ is the eigenfunction of the operator $\tilde{K}_N$ (equation (2.5)) associated with the eigenvalue $\alpha_{N,k}^2$, we see that we can consider the problem with discrete data as a linear perturbation of the problem with continuous data, the linear perturbation being described by the integral operator $\tilde{R}_N = \tilde{K} - \tilde{K}_N$, whose expression is

$$(\tilde{R}_N f)(t) = \int_1^{\gamma} \rho_N(t+s) f(s)\, \mathrm{d}s, \tag{2.11}$$

where
$$\rho_N(t) = \frac{1}{t} - \sum_{n=1}^{N} w_n \, \mathrm{e}^{-p_n t}. \tag{2.12}$$

Then, from the Weyl–Courant lemma (Riesz & Nagy 1955) it follows that

$$|\alpha_k^2 - \alpha_{N,k}^2| \leqslant \| \tilde{R}_N \| \tag{2.13}$$

(where $\| \tilde{R}_N \|$ denotes the usual operator norm of a bounded operator in $L^2(1, \gamma)$) and also that

$$|\alpha_k^2 - \alpha_{N,k}^2| \leqslant \left(\int_1^\gamma \mathrm{d}t \int_1^\gamma \mathrm{d}s \, | \rho_N(t+s) |^2 \right)^{\frac{1}{2}}, \tag{2.14}$$

since the operator norm is smaller than the L^2-norm of the kernel. Therefore $\alpha_{N,k}^2$ is certainly a good approximation of α_k^2 when the L^2-norm of the function $\rho_N(t+s)$, defined by (2.12), is much smaller than α_k^2.

By straightforward techniques of perturbation theory (Kato 1966) it is also possible to show that $\| u_{N,k} - u_k \|$ is bounded by the norm of $\tilde{R}_N$.

3. Uniform sampling

The problem of Laplace transform inversion with equidistant data points, which is related to the Hausdorff moment problem, is one of the most investigated since the pioneering work of Papoulis (1956). A well known result is the following (Doetsch 1943): the Laplace transform $g(p)$ is uniquely specified when its values $g(p_n)$ at the points p_n, (1.9) with $N = \infty$, are given. The case of equidistant points has also been of great interest in some applications of the Laplace transform inversion (Cummins & Pike 1974; Pike *et al.* 1983).

To apply the method of §2, we must specify the weights w_n. The simplest choice for equidistant points is

$$w_n = d, \tag{3.1}$$

d being the distance between adjacent points. Then we will denote by $\tilde{K}_N(c, d)$ the operator $\tilde{K}_N$ and by $\alpha_{N,k}(c, d)$ the singular values $\alpha_{N,k}$; analogously, for $c = 0$, we will denote by $\tilde{K}_N(d)$ the operator $\tilde{K}_N$ and by $\alpha_{N,k}(d)$ the singular values $\alpha_{N,k}$.

The kernel $T_N(c, d; t+s)$ of the operator $\tilde{K}_N(c, d)$ is given by (see (2.6), (1.9) and (3.1))

$$T_N(c, d; t) = d\mathrm{e}^{-ct} \frac{1 - \mathrm{e}^{-(Nd)t}}{1 - \mathrm{e}^{-dt}} \tag{3.2}$$

and therefore the following relation holds between the kernel of $\tilde{K}_N(c, d)$ and the kernel of $\tilde{K}_N(c', d)$:

$$T_N(c, d; t+s) = \mathrm{e}^{-(c-c')t} \, T_N(c', d; t+s) \, \mathrm{e}^{-(c-c')s}. \tag{3.3}$$

This relation can be used to investigate the dependence of the singular values $\alpha_{N,k}(c, d)$ on the position c of the first data point and the following result can be easily proved by using the minimax characteristics of the eigenvalues. For fixed

N, k and d, $\alpha_{N,k}(c,d)$ is a decreasing function of c and the following inequalities hold true:

$$e^{-\gamma c}\alpha_{N,k}(d) \leqslant \alpha_{N,k}(c,d) \leqslant e^{-c}\alpha_{N,k}(d); \tag{3.4}$$

therefore $\alpha_{N,k}(c,d)$ tends to $\alpha_{N,k}(d)$ from below, when $c \to 0$.

From this result it follows that the best choice for c is $c = 0$. If in a practical experiment it is not possible to have $c = 0$, one has to take c as small as possible, since the singular values decrease rather rapidly when c increases. But, if c is small, from the inequalities (3.4) it follows that the difference $\alpha_{N,k}(d) - \alpha_{N,k}(c,d)$ is, at most, of the order of $(\gamma c)\,\alpha_{N,k}(d)$. In the following we will only consider the case $c = 0$.

To investigate the dependence of the singular values $\alpha_{N,k}(d)$ on the number N of data points, for fixed k and d, let us firstly remark that the operator

$$\tilde{R}_{NN'}(d) = \tilde{K}_{N'}(d) - \tilde{K}_N(d) \quad (N' > N) \tag{3.5}$$

is positive definite, since

$$(\tilde{R}_{NN'}(d)f, f)_X = d\sum_{n=N}^{N'-1}\left(\int_1^{\gamma} e^{-(nd)t} f(t)\,dt\right)^2 \geqslant 0 \tag{3.6}$$

and therefore, from the Weyl–Courant lemma (Riesz & Nagy 1955) it follows that

$$\alpha^2_{N,k}(d) \leqslant \alpha^2_{N',k}(d) \quad (N < N'); \tag{3.7}$$

namely, for fixed k and d, the singular value $\alpha_{N,k}(d)$ is an increasing function of the number N of data points.

It is also possible to investigate the limit of the singular values when $N \to \infty$, since the function $T_N(d;t) = T_N(0,d;t)$ has as a limit the function

$$T(d;t) = d/(1 - e^{-dt}) \tag{3.8}$$

(the convergence is uniform over the bounded interval $2 \leqslant t \leqslant 2\gamma$). Then $T(d;t+s)$ is the kernel of an integral operator, which we denote by $\tilde{K}(d)$ and which corresponds to the problem of Laplace transform inversion for an infinite set of equidistant data points. We will denote by $\alpha_k(d)$ the singular values of this problem. From the uniqueness result stated at the beginning of this section, it follows that the operator $\tilde{K}(d)$ is injective. Remark that the difference $\alpha_k - \alpha_k(d)$ (where the α_k are the singular values of the original operator (1.2)) can be taken as a measure of the error due to the (uniform) *sampling of data*, while the difference $\alpha_k(d) - \alpha_{N,k}(d)$ can be taken as a measure of the error due to the *truncation of data*. We first estimate the latter.

The operator

$$\tilde{R}_N(d) = \tilde{K}(d) - \tilde{K}_N(d), \tag{3.9}$$

whose kernel $R_N(d;t+s)$ is given by

$$R_N(d;t) = d\,e^{-Ndt}/(1 - e^{-dt}), \tag{3.10}$$

is positive definite. Furthermore, if we remark that, for any $d \leqslant 1$ and any $t \geqslant 2$ we have $d[1 - \exp(-dt)]^{-1} < \frac{4}{3}$, we get $R_N(d;t) < (\frac{4}{3})\exp(-Ndt)$, so that, from (2.14) we obtain

$$0 \leqslant \alpha^2_k(d) - \alpha^2_{N,k}(d) \leqslant (Nd)^{-1}\,e^{-2Nd}. \tag{3.11}$$

We conclude that the singular values $\alpha_{N,k}(d)$ tend to $\alpha_k(d)$ from below when $N \to \infty$, the convergence being exponentially fast with respect to N, for fixed k and d. As a consequence in numerical computations, using necessarily a finite arithmetic, the limit $N = \infty$ is already reached with a moderate number of data points, at least for the greatest singular values, and when the distance d between adjacent points is not too small.

As concerns the effect of the sampling of data, it can also be described as a linear perturbation of the operator $\tilde{K}$ (equation (1.4)) if we put $\tilde{R}(d) = \tilde{K}(d) - \tilde{K}$. It is interesting to remark that $\tilde{R}(d)$ is an analytic perturbation (Kato 1966), as follows from the Taylor expansion of the kernel $\rho(d; t+s)$ of $\tilde{R}(d)$:

$$\rho(d;t) = \frac{d}{1-\mathrm{e}^{-dt}} - \frac{1}{t} = \tfrac{1}{2}d - d \sum_{m=1}^{+\infty} (-1)^m \frac{B_m}{(2m)!} (dt)^{2m-1} \tag{3.12}$$

(where the B_ms are the Bernoulli numbers), which converges when $dt < 2\pi$. Since when $t \leqslant 2\gamma$, $d \leqslant \frac{1}{2}\pi/\gamma$, the following inequality holds:

$$|\rho(d;t)| \leqslant \tfrac{1}{2}d + d \sum_{m=1}^{+\infty} \frac{B_m}{(2m)!} \pi^{2m-1} = \left(\frac{1}{2} + \frac{1}{\pi}\right) d < d; \tag{3.13}$$

from the inequality (2.14) we obtain

$$|\alpha_k^2(d) - \alpha_k^2| \leqslant d(\gamma - 1). \tag{3.14}$$

By combining the inequalities (3.11) and (3.14) we get

$$|\alpha_{N,k}^2(d) - \alpha_k^2| \leqslant d(\gamma-1) + (Nd)^{-1}\,\mathrm{e}^{-Nd}, \tag{3.15}$$

where the first term is the error due to sampling and the second term is the error due to truncation. Furthermore, we may see that a similar estimate holds for the norm of the difference $u_{N,k} - u_k$. Therefore we can summarize our results as: if $d \to 0$ and $N \to \infty$ in such a way that $Nd \to \infty$, then, for any fixed k, $\alpha_{N,k}(d) \to \alpha_k$ and $u_{N,k} \to u_k$, the last convergence being the strong convergence in $L^2(1,\gamma)$.

As a final remark we point out that, for fixed N, the error given by the right side of (3.15), has a minimum as a function of d. Therefore we expect that for fixed N, there exists an optimum distance between adjacent points, in the sense that it gives the best approximation of (some) singular values of the problem with continuous data. We will investigate this point numerically in §5.

4. Geometrical sampling

The data point distribution (1.10) has been suggested by Pike *et al.* (1983) using arguments founded on the sampling theorem. The aim was to find a distribution giving the same results as a uniform distribution, but using a much smaller number of data points. The same arguments suggest the choice of the weights

$$w_n = (\ln \Delta)\, p_n. \tag{4.1}$$

We will justify this choice.

As a preliminary remark we point out that the Laplace transform $g(p)$ of a function $f \in L^2(1, \gamma)$ is uniquely specified when its values are given at points forming a geometric progression. To prove this result it is sufficient to remark that if the set of the data points is given by

$$p_n = q\Delta^n; \quad n = 0, \pm 1, \pm 2, \ldots, \tag{4.2}$$

then these points accumulate to $p = 0$. Since $g(p)$ is an entire function, the points p_n as well as the limiting point $p = 0$ belong to the analyticity domain of $g(p)$. The well known properties of the zeros of an analytic function imply that if $g(p_n) = 0$ for any p_n, then $g(p) = 0$ everywhere and the uniqueness is proved.

In the following we will assume that the set of data points is a subset of (4.2), say corresponding to $n = -N_1, -N_1+1, \ldots, 0, \ldots, N_2-1, N_2$. Then the relation between the parameter c of (1.10) and the parameter q of (4.2) is $q = c \exp(N_1 \ln \Delta)$. Furthermore we have $N = N_1 + N_2 + 1$.

We introduce the function

$$T_N(q, \Delta; t) = \sum_{n=-N_1}^{N_2} w_n \, \mathrm{e}^{-p_n t} \quad (N = N_1 + N_2 + 1), \tag{4.3}$$

w_n and p_n being given by (4.1) and (4.2) respectively, and also the function

$$T(q, \Delta; t) = \sum_{n=-\infty}^{+\infty} w_n \, \mathrm{e}^{-p_n t}. \tag{4.4}$$

Accordingly we will denote by $\tilde{K}_N(q, \Delta)$ the operator whose kernel is $T_N(q, \Delta; t+s)$ and by $\alpha_{N,k}(q, \Delta)$ the corresponding singular values. We will also denote by $\tilde{K}(q, \Delta)$ the operator with kernel $T(q, \Delta; t+s)$ and by $\alpha_k(q, \Delta)$ the corresponding singular values. From the uniqueness result, stated at the beginning of this section, it follows that the operator $\tilde{K}(q, \Delta)$ is injective.

As in §3, the difference $\alpha_k - \alpha_k(q, \Delta)$ can be taken as a measure of the error due to the (geometrical) sampling of data, while the difference $\alpha_k(q, \Delta) - \alpha_{N,k}(q, \Delta)$ can be taken as a measure of the error due to the truncation of data.

We investigate first the effect of the truncation of data. From (4.3) and (4.4) we have the relation

$$T(q, \Delta; t) = T_N(q, \Delta; t) + R_{N_1}^{(-)}(q, \Delta; t) + R_{N_2}^{(+)}(q, \Delta; t) \tag{4.5}$$

if we put

$$R_{N_1}^{(-)}(q, \Delta; t) = \sum_{N=-\infty}^{-(N_1+1)} w_n \, \mathrm{e}^{-p_n t} \tag{4.6}$$

and

$$R_{N_2}^{(+)}(q, \Delta; t) = \sum_{n=N_2+1}^{+\infty} w_n \, \mathrm{e}^{-p_n t}. \tag{4.7}$$

As in §3 it is easy to prove that, for fixed k, q, Δ, the singular value $\alpha_{N,k}(q, \Delta)$ is an increasing function of the number N of data points, if we increase the number of data points on both sides of the 'central point' $p_0 = q$; furthermore $\alpha_{N,k}(q, \Delta) \leqslant \alpha_k(q, \Delta)$.

To prove that $\alpha_{N,k}(q,\Delta)\to\alpha_k(q,\Delta)$ when both N_1 and N_2 tend to infinity, first remark that

$$
\begin{aligned}
R_{N_1}^{(-)}(q,\Delta;t) &\leqslant \sum_{n=-\infty}^{-(N_1+1)} w_n = q(\ln\Delta)\sum_{m=0}^{+\infty}\Delta^{-(m+N_1+1)}\\
&= q\,\frac{\ln\Delta}{\Delta-1}\,\mathrm{e}^{-(N_1+1)\ln\Delta} = \frac{\ln\Delta}{\Delta-1}\,p_{-(N_1+1)}.
\end{aligned}
\tag{4.8}
$$

Concerning $R_{N_2}^{(+)}(q,\Delta;t)$, if we introduce the functions

$$
p(x) = q\Delta^x,\quad w(x) = p'(x) = (\ln\Delta)\,p(x)
\tag{4.9}
$$

we have

$$
\begin{aligned}
R_{N_2}^{(+)}(q,\Delta;t) &= \sum_{n=N_2+1}^{+\infty} w_n\,\mathrm{e}^{-p_n t}\\
&\leqslant \int_{N_2}^{+\infty} p'(x)\,\mathrm{e}^{-tp(x)}\,\mathrm{d}x = \int_{p_{N_2}}^{+\infty}\mathrm{e}^{-pt}\,\mathrm{d}p\\
&\leqslant \tfrac{1}{2}\exp(-tp_{N_2})\quad (t\geqslant 2).
\end{aligned}
\tag{4.10}
$$

From the inequalities (4.8), (4.10) and from the Weyl–Courant lemma (equation (2.14)), it follows that

$$
\alpha_k^2(q,\Delta)\geqslant\alpha_{N,k}^2(q,\Delta)\geqslant\alpha_k^2(q,\Delta)-\epsilon^2(q,\Delta;N_1,N_2),
\tag{4.11}
$$

where

$$
\epsilon^2(q,\Delta;N_1,N_2) = q(\gamma-1)\,\frac{\ln\Delta}{\Delta-1}\,\Delta^{-N_1}+\frac{\Delta^{-N_2}}{2q}\exp(-2q\Delta^{N_2}).
\tag{4.12}
$$

We conclude that, for fixed k,q,Δ, $\alpha_{N,k}(q,\Delta)$ converges exponentially fast to $\alpha_k(q,\Delta)$ when N_1 and N_2 tend to infinity.

It is also interesting to remark that the 'error' $\epsilon^2(q,\Delta;N_1,N_2)$ has a minimum as a function of q, for fixed Δ, N_1, N_2: therefore we can take it that, for given values of the dilation factor Δ and of the number N of data points there are optimum values of q (or equivalently of the position c of the first data point) in the sense that we can minimize the truncation error on the singular values. This point will be confirmed by our numerical computations (see §5).

Finally we will estimate the effect of the sampling of data. We write

$$
\rho(q,\Delta;t) = \frac{1}{t}-\sum_{n=-\infty}^{+\infty} w_n\,\mathrm{e}^{-p_n t}
\tag{4.13}
$$

and we observe that, using the change of variable defined by (4.9), by putting $a(x) = w(x)\exp(-tp(x))$, we have

$$
\begin{aligned}
\rho(q,\Delta;t) &= \int_0^{+\infty}\mathrm{e}^{-pt}\,\mathrm{d}p-\sum_{n=-\infty}^{+\infty} w_n\,\mathrm{e}^{-p_n t}\\
&= \sum_{n=-\infty}^{+\infty}\int_n^{n+1}\left(\int_n^x a'(y)\,\mathrm{d}y\right)\mathrm{d}x\\
&= \sum_{n=-\infty}^{+\infty}\int_n^{n+1}(n+1-y)\,a'(y)\,\mathrm{d}y.
\end{aligned}
\tag{4.14}
$$

Since the function $a(x)$ has only one maximum at the point $x_0 = -\ln(qt)/\ln \varDelta$, so that $a'(y) \geqslant 0$ for $y \leqslant x_0$ and $a'(y) \leqslant 0$ for $y \geqslant x_0$, and using also the fact that $n+1-y \leqslant 1$ on the interval $[n, n+1]$, we obtain $-a(x_0) \leqslant \rho(q, \varDelta; t) \leqslant a(x_0)$ and, computing $a(x_0)$:

$$|\rho(q, \varDelta; t)| \leqslant (\ln \varDelta)/et. \tag{4.15}$$

From (2.14) we obtain

$$|\alpha_k^2 - \alpha_k^2(q, \varDelta)| \leqslant \frac{1}{e} \ln\left[\frac{(\gamma+1)^2}{4\gamma}\right] \ln \varDelta. \tag{4.16}$$

It follows that the effect of the sampling of data tends to zero when $\varDelta \to 1$.

Collecting the inequalities (4.11) and (4.16) we conclude that for fixed k and q, if $\varDelta \to 1$ and $N_1, N_2 \to \infty$ in such a way that $N_1 \ln \varDelta \to \infty$ and $N_2 \ln \varDelta \to \infty$, then $\alpha_{N,k}(q, \varDelta) \to \alpha_k$. A similar result holds for the singular functions, i.e. $u_{N,k} \to u_k$, the convergence being the strong convergence in $L^2(1, \gamma)$.

5. Numerical results

The singular system of the operator K_N (equation (2.1)), or, more precisely, the singular subsystem generated by the greatest singular values, can be easily computed by solving the eigenvalue problem for the $N \times N$ matrix $K_N K_N^*$. As follows from (2.1), (2.3) the matrix elements of $K_N K_N^*$ are given by

$$t_{nm}^{(N)} = \frac{w_m}{p_n + p_m} [\mathrm{e}^{-(p_n+p_m)} - \mathrm{e}^{-\gamma(p_n+p_m)}]. \tag{5.1}$$

This matrix is not symmetric (in the usual sense) when the weights w_n are not constant, but it has the same eigenvalues as the symmetric matrix

$$\bar{t}_{n,m}^{(N)} = \left(\frac{w_n}{w_m}\right)^{\frac{1}{2}} t_{n,m}^{(N)}. \tag{5.2}$$

Furthermore, if $\bar{v}_{N,k}$ is the eigenvector of the matrix (5.2) associated with the eigenvalue $\alpha_{N,k}^2$ and normalized to 1 with respect to the usual euclidean norm, then the corresponding singular function $u_{N,k}$ is given by

$$u_{N,k}(t) = \frac{1}{\alpha_{N,k}} \sum_{n=1}^{N} w_n^{\frac{1}{2}} v_{N,k}(p_n)\, \mathrm{e}^{-p_n t}. \tag{5.3}$$

We report now a few numerical examples that show how the method, developed in the previous sections, can be put in practice.

5.1. *Uniform sampling*

We have mainly considered the case $\gamma = 5$ and, as in II, we have done computations with $N = 32$, 64 and 128.

As illustrated in table 1, the convergence of the singular values when N grows is rather fast, in agreement with the theoretical estimate (3.11). Furthermore, the convergence is faster with greater values of d. It follows that, if we take $N = 64$, the error due to the truncation of the data is rather small (a small percentage) and therefore we will take this value in the following.

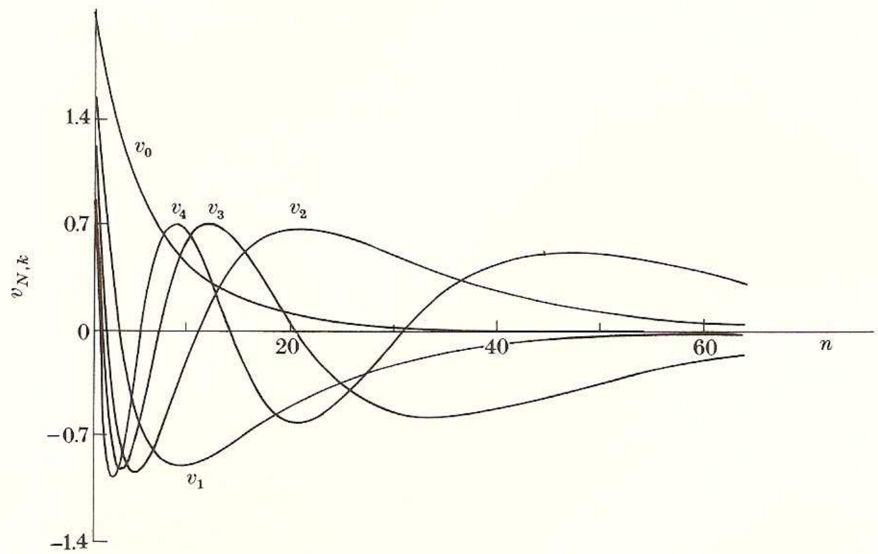

FIGURE 1. Singular vectors $v_{N,k}$, $k = 0, 1, \ldots, 4$ (denoted by v_k in the figure) in the case $\gamma = 5$, $N = 64$ and $d = 0.07$. The values of the components have been joined by a continuous line.

TABLE 1. BEHAVIOUR OF THE GREATEST SINGULAR VALUES AS A FUNCTION OF N, IN THE CASE $\gamma = 5$ AND FOR TWO VALUES OF THE DISTANCE d BETWEEN ADJACENT POINTS

$d = 0.04$	$N = 32$	$N = 64$	$N = 128$
$\alpha_{N,0}(d)$	0.9159098	0.9190460	0.9191318
$\alpha_{N,1}(d)$	0.1900254	0.2024905	0.2035560
$\alpha_{N,2}(d)$	0.2789200×10^{-1}	0.3889461×10^{-1}	0.4014776×10^{-1}
$d = 0.06$	$N = 32$	$N = 64$	$N = 128$
$\alpha_{N,0}(d)$	0.9411925	0.9416686	0.9416719
$\alpha_{N,1}(d)$	0.2054506	0.2083106	0.2083400
$\alpha_{N,2}(d)$	0.3670656×10^{-1}	0.4089348×10^{-1}	0.4098645×10^{-1}

Now if, for fixed N and fixed k, we introduce the condition number corresponding to the restoration of the first k components

$$\alpha_N^{(k)}(d) = \alpha_{N,0}(d)/\alpha_{N,k-1}(d), \tag{5.4}$$

then, by using the expression (5.1) of the matrix $t_{nm}^{(N)}$ with $w_n = d$ and $p_n = d(n-1)$, it is easy to show that $\alpha_N^{(k)}(d) \to \infty$ both when $d \to 0$ and when $d \to \infty$. Therefore $\alpha_N^{(k)}(d)$ has certainly a minimum as a function of d (see also II).

The behaviour of the condition number (5.4) is quite similar to the behaviour of the condition numbers of the exponential sampling model computed in II. For instance, when $\gamma = 5$, $N = 64$ and $k = 3, 4, 5$, the minimum occurs approximately at $d_1 = 0.07$ both in the present case and in the case of the exponential sampling model (see table 1 in II). Furthermore the minimum is extremely flat.

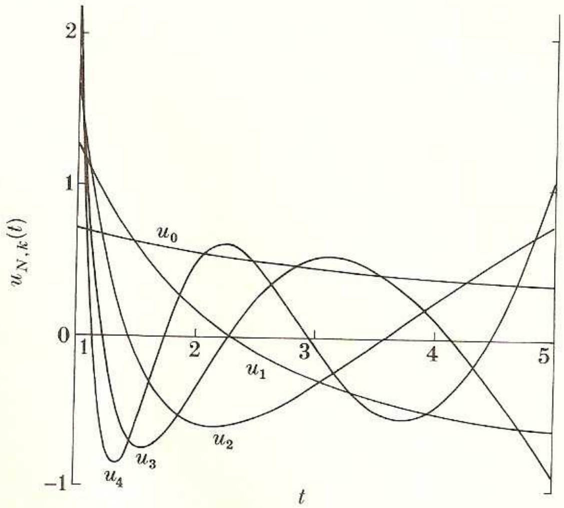

FIGURE 2. Singular functions $u_{N,k}(t)$, $1 \leqslant t \leqslant \gamma$, $k = 0, 1, \ldots, 4$ (denoted by u_k in the figure) in the case $\gamma = 5$, $N = 64$ and $d = 0.07$.

TABLE 2. SINGULAR VALUES α_k OF THE PROBLEM WITH CONTINUOUS DATA (SEE I) AND SINGULAR VALUES $\alpha_{N,k}(d)$ IN THE CASE $N = 64$, $d = 0.07$ ($\gamma = 5$ IN BOTH CASES)

(The relative percentage error of the singular value $\alpha_{N,k}(d)$ with respect to the singular value α_k is shown in brackets.)

k	α_k	$\alpha_{N,k}(d)$	
0	0.8751	0.9531	(8.9)
1	0.1935	0.2106	(8.8)
2	0.3827×10^{-1}	0.4135×10^{-1}	(8.0)
3	0.7434×10^{-2}	0.7942×10^{-2}	(6.0)
4	0.1435×10^{-2}	0.1482×10^{-2}	(3.3)

The five greatest singular values, in the case $\gamma = 5$, $N = 64$, $d = 0.07$, are given in table 2, the corresponding singular vectors $v_{N,k}$ are plotted in figure 1 while the singular functions $u_{N,k}(t)$ are plotted in figure 2. We notice that $u_{N,k}(t)$ has exactly k zeros inside the interval $[1, \gamma]$.

In figure 3 we give the behaviour of the condition number (5.4) in the case $k = 3$ (full line). The minimum is extremely flat: the value of the condition number does not change significantly in the interval from $d = 0.04$ to $d = 0.1$. Furthermore the value of the condition number is always greater than the condition number of the problem with continuous data, $\alpha^{(3)} = \alpha_0/\alpha_2 = 22.87$. The dotted line represents the function

$$e_{\text{rel}}^{(k)}(d) = \left\{\frac{1}{k}\sum_{j=0}^{k-1}\left[\frac{\alpha_{N,j}(d) - \alpha_j}{\alpha_j}\right]^2\right\}^{\frac{1}{2}}, \tag{5.5}$$

which is the relative mean square error on the first k singular values. As we see, $e_{\text{rel}}^{(k)}(d)$ also has a minimim, as a function of d, but at a value of d smaller than that corresponding to the minimum of the condition number and precisely $\bar{d}_1 = 0.035$.

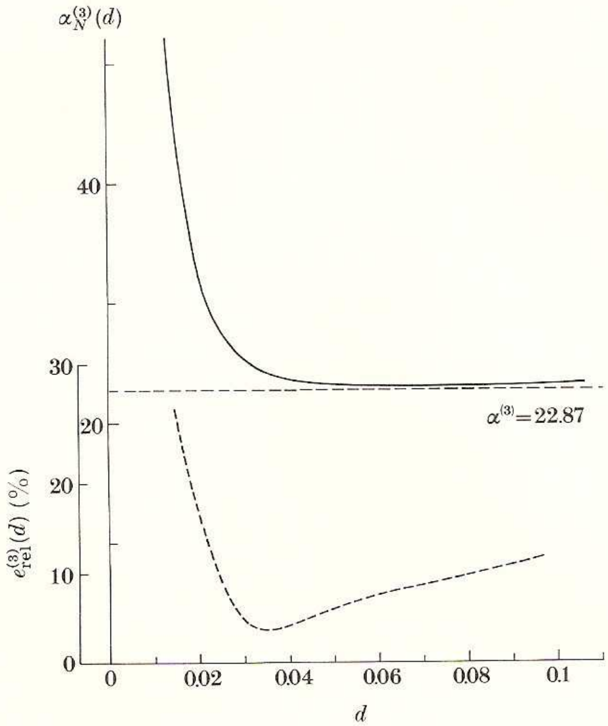

FIGURE 3. Behaviour of the condition number $\alpha_N^{(k)}(d)$ (full line) and of the average relative error $e_{\mathrm{rel}}^{(k)}(d)$ (broken line) as a function of d, in the case $\gamma = 5$, $N = 64$ and $k = 3$, $c = 0$; $\alpha^{(3)} = 22.87$ is the condition number of the problem with continuous data.

TABLE 3. SINGULAR VALUES CORRESPONDING TO THE MINIMUM OF THE RELATIVE MEAN SQUARE ERROR ($d = d_1 = 0.035$) AND TO THE MINIMUM OF THE CONDITION NUMBER ($d = d_1 = 0.07$)

(The relative percentage error with respect to the singular values of the problem with continuous data (first column of table 2) is shown in brackets.)

d	$k = 0$	1	2
0.035	0.9133 (4.4)	0.2010 (3.9)	0.3761×10^{-1} (1.7)
0.070	0.9531 (8.9)	0.2106 (8.8)	0.4135×10^{-1} (8.0)

The value of the condition number at $\bar{d}_1$ is $\alpha_N^{(3)}(\bar{d}_1) = 24.28$ while its value at the minimum point $d_1 = 0.07$ is $\alpha_N^{(3)}(d_1) = 23.05$ (condition number in the case of continuous data, $\alpha^{(3)} = 22.87$).

In table 3 we compare the first three singular values corresponding respectively to the minimum of the average relative error and to the minimum of the condition number.

As we see, for uniform sampling, using a rather large number of points ($N = 64$), it is impossible to reduce the average relative error on the first three singular values below 3.5%.

Concerning the singular functions a comparison can be done by comparing the zeros. If we consider the singular functions corresponding to the first three singular values, then we have only three zeros, since the first singular function has no zero, the second has only one zero, $t_{1,1}$, and the second has two zeros, $t_{1,2}$ and $t_{2,2}$ (we will denote by $t_{j,k}$ the position of the jth zero of the kth singular function). For the problem with continuous data and $\gamma = 5$ these zeros are (see the numerical method used in I)

$$t_{1,1} = 2.236 \quad t_{1,2} = 1.427 \quad t_{2,2} = 3.507. \tag{5.6}$$

In table 4 we give the zeros for uniformly sampled data with $N = 64$, both for the value of d corresponding to the minimum of the relative mean square error and for the value of d corresponding to the minimum of the condition number.

We notice that, with regard to the zeros of the singular functions, the choices $d = d_1$ or $d = \bar{d}_1$ are almost equivalent.

TABLE 4. ZEROS OF THE SINGULAR FUNCTIONS $u_{N,1}$ (i.e. $t_{1,1}$) AND $u_{N,2}$ (i.e. $t_{1,2}$ AND $t_{2,2}$) IN THE CASE $N = 64$, $\gamma = 5$ AND FOR THE VALUES OF d CORRESPONDING TO THE MINIMUM OF THE RELATIVE MEAN-SQUARE ERROR AND OF THE CONDITION NUMBER

(The relative percentage error with respect to the values for continuous data (equation (5.6)) is given in brackets.)

d	$t_{1,1}$	$t_{1,2}$	$t_{2,2}$
0.035	2.285 (2.2)	1.454 (1.8)	3.568 (1.7)
0.070	2.313 (3.4)	1.442 (1)	3.563 (1.6)

5.2. *Geometrical sampling*

For the exponential sampling model, as shown in II, the geometrical sampling of data allows one to obtain better results than the uniform sampling, using a much smaller number of points. In the range $2 \leqslant \gamma \leqslant 8$ and for values of k up to $k = 4$, we need only 5 optimally placed data points. Similar results hold for the singular value method developed in this paper.

We have computed the eigenvalues of the matrix (5.1) in the case $N = 5$, $\gamma = 5$, the points p_n and the weights w_n being given by (1.10) and (4.1) respectively. Since we still have two parameters, c and Δ, we have done a detailed investigation only in the case $k = 3$, using the value of Δ suggested by the computations reported in II, namely $\Delta = 5.5$.

Using the notations of §4 we introduce the condition number

$$\alpha_N^{(k)}(c, \Delta) = \frac{\alpha_{N,0}(c, \Delta)}{\alpha_{N,k-1}(c, \Delta)}, \tag{5.7}$$

and the relative mean-square error with respect to the singular values of the problem with continuous data

$$e_{\mathrm{rel}}^{(k)}(c) = \left\{\frac{1}{k} \sum_{j=0}^{k-1} \left[\frac{\alpha_{N,j}(c, \Delta) - \alpha_j}{\alpha_j}\right]^2\right\}^{\frac{1}{2}}. \tag{5.8}$$

In the case $k = 3$, $N = 5$ and $\Delta = 5.5$, the function (5.7) is plotted in figure 4 (full line), while the dotted line, in the same figure, represents the function (5.8). As

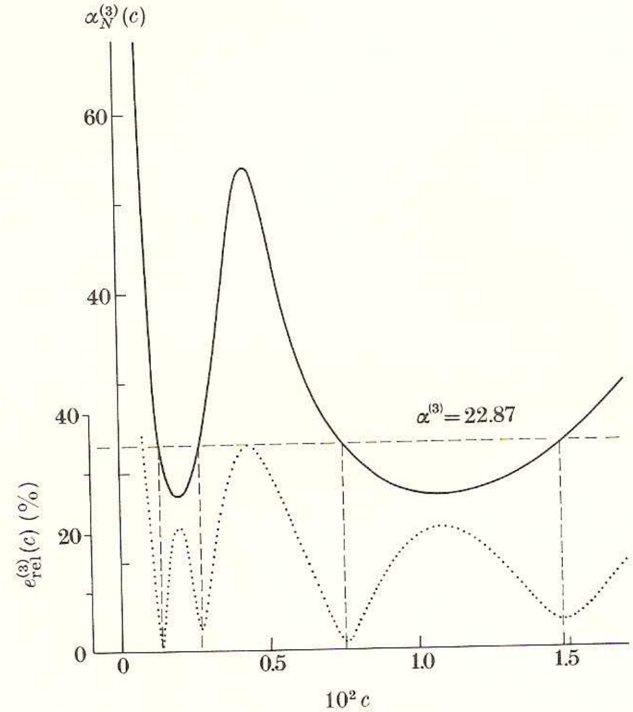

FIGURE 4. Behaviour of the condition number $\alpha_N^{(k)}(c)$ (full line) and of the average relative error $e_{\mathrm{rel}}^{(k)}(c)$ (broken line) in the case of geometrical sampling with $N = 5$. The values of the other parameters are $\gamma = 5$, $k = 3$ and $\Delta = 5.5$; $\alpha^{(3)} = 22.87$ is the condition number of the problem with continuous data.

in the case of the exponential sampling model investigated in II, the condition number $\alpha^{(k)}(c)$ has two minima at the points

$$c_1 = 0.2 \times 10^{-2}, \quad c_2 = 0.109 \times 10^{-1}, \tag{5.9}$$

where it takes approximately the same value, i.e. 17.3, which is smaller than the value 22.87 of the condition number of the problem with continuous data. As a consequence there are four points where the condition number $\alpha_N^{(k)}(c)$ takes the value of $\alpha^{(k)}$. In a neighbourhood of each of these points there is a minimum of the relative mean-square error (5.8) and precisely at the points

$$\left.\begin{aligned} \bar{c}_1 &= 0.1363 \times 10^{-2}, \quad \bar{c}_2 = 0.270 \times 10^{-2}, \\ \bar{c}_3 &= 0.755 \times 10^{-2}, \quad\quad \bar{c} = 0.149 \times 10^{-1}. \end{aligned}\right\} \tag{5.10}$$

The smallest value of the relative mean-square error is 0.64 % and is taken at $\bar{c}_1$ (recall that for uniform sampling, with 64 points, the minimum relative mean-square error was approximately 3.5 %). The values at $\bar{c}_2$, $\bar{c}_3$ and $\bar{c}_4$ are approximately 3.8 %, 1.2 %, and 4.7 % respectively. The absolute minimum at $\bar{c}_1$ is, however, rather sharp so that a small variation in c can induce a rather large variation in the relative mean-square error.

In table 5 we give the first three singular values corresponding to the values of c given in (5.9) and (5.10). As we see the greatest singular value is not very sensitive to c while the second singular value and especially the third singular value are much more sensitive.

TABLE 5. SINGULAR VALUES CORRESPONDING TO THE MINIMA OF THE RELATIVE MEAN-SQUARE ERROR (GIVEN IN (5.10)) AND TO THE MINIMA OF THE CONDITION NUMBER (GIVEN IN (5.9))

(The relative percentage error with respect to the singular values of the problem with continuous data (first column of table 2) are shown in brackets.)

	$k = 0$	1	2
$\bar{c}_1$	0.8742 (0.1)	0.1915 (1.0)	0.3815×10^{-1} (0.3)
$\bar{c}_2$	0.8761 (0.1)	0.1810 (6.5)	0.3767×10^{-1} (1.6)
$\bar{c}_3$	0.8692 (0.6)	0.1897 (2.0)	0.3802×10^{-1} (0.6)
$\bar{c}_4$	0.8663 (1.0)	0.1782 (7.9)	0.3751×10^{-1} (2.0)
c_1	0.8790 (4.5)	0.1667 (13.8)	0.5088×10^{-1} (32.9)
c_2	0.8717 (3.9)	0.1648 (14.8)	0.5058×10^{-1} (32.2)

Concerning the singular functions we have computed the zeros and give the results in table 6. We notice that the zeros are rather stable with respect to variations of c.

TABLE 6. ZEROS OF THE SINGULAR FUNCTIONS $u_{N,1}$ AND $u_{N,2}$ IN THE CASE $N = 5$, $\varDelta = 5.5$ ($\gamma = 5$), FOR THE VALUES OF c CORRESPONDING TO THE MINIMA OF THE RELATIVE MEAN-SQUARE ERROR AND OF THE CONDITION NUMBER

(The relative percentage error with respect to the values given in (5.6) is given in brackets.)

	$t_{1,1}$	$t_{1,2}$	$t_{2,2}$
$\bar{c}_1$	2.122 (5.1)	1.434 (0.5)	3.533 (0.7)
$\bar{c}_2$	2.348 (5.0)	1.397 (2.1)	3.425 (2.3)
$\bar{c}_3$	2.116 (5.4)	1.431 (0.3)	3.526 (0.5)
$\bar{c}_4$	2.338 (4.6)	1.396 (2.2)	3.419 (2.5)
c_1	2.222 (0.6)	1.417 (0.7)	3.482 (0.7)
c_2	2.204 (1.4)	1.416 (0.8)	3.452 (1.6)

6. Resolution limits

The problem of resolution in the Laplace transform inversion was one of the main points discussed in I, where a resolution ratio δ_s was defined by assuming a geometric distribution of the zeros of the singular functions $u_k(t)$. This property is not exact but it is approximately satisfied, as illustrated by the following example where the zeros of the singular functions corresponding to $\gamma = 5$ and to 64 uniformly sampled data points, at a distance $d = 0.07$, are given:

$$\begin{aligned}
&t_{1,1} = 2.313,\\
&t_{1,2} = 1.442,\quad t_{2,2} = 3.563,\\
&t_{1,3} = 1.221,\quad t_{2,3} = 2.267,\quad t_{3,3} = 4.152,\\
&t_{1,4} = 1.135,\quad t_{2,4} = 1.741,\quad t_{3,4} = 2.944,\quad t_{4,4} = 4.453.
\end{aligned} \tag{6.1}$$

We have the following values for the ratios between adjacent zeros:

$$\begin{aligned} &t_{2,2}/t_{1,2} = 2.471, \\ &t_{2,3}/t_{1,3} = 1.857, \quad t_{3,3}/t_{2,3} = 1.831. \\ &t_{2,4}/t_{1,4} = 1.534, \quad t_{3,4}/t_{2,4} = 1.691, \quad t_{4,4}/t_{3,4} = 1.513, \end{aligned} \tag{6.2}$$

and therefore the zeros of the two last singular functions form approximately a geometric progression. We also notice that the values of δ_s estimated in I for the case $\gamma = 5$ were: $\delta_s = 1.736$ for a signal-to-noise ratio $E/\epsilon = 10^2$ and $\delta_s = 1.445$ for a signal-to-noise ratio $E/\epsilon = 10^3$. These values are smaller than the ratios between adjacent zeros given in (6.2) (the values to be compared are respectively those contained in the second line and in the third line). But if we take an average of the ratios between adjacent zeros, including also the ratio between the first zero and the lower bound of the support and the ratio between the upper bound of the support and the last zero, we find $\delta_{av} = 1.772$ for restorations using 3 singular functions (which corresponds to $E/\epsilon = 10^2$) and $\delta_{av} = 1.399$ for restorations using 5 singular functions (which corresponds to $E/\epsilon = 10^3$). This result suggests that the resolution ratios computed in I can also be interpreted as average resolution ratios in the sense specified above.

We can have more insight into the problem of resolution limits in Laplace (and similar) transform inversion if we look at the relation between the restored and the true solution. In the absence of noise and by using the singular value method such a relation is given by (1.7) and (1.8) or by (2.9) and (2.10), and the smoothing kernel $M(t, s)$ that appears in these equations has the following interpretation: if we assume that the unknown function is a δ-function concentrated at the point $t = a$, then the restoration provided by the truncated singular function expansion in the absence of noise is

$$\tilde{f}_a(t) = M(t, a) = \sum_{k=0}^{J-1} u_k(t)\, u_k(a). \tag{6.3}$$

This equation can be used to illustrate the improvement in resolution due to knowledge of the support of the solution. To do this we must derive the equation corresponding to (6.3) in the case where we have no constraint on the support of the unknown function. In such a case we can use a truncated inverse Mellin transform or a truncated eigenfunction expansion (McWhirter & Pike 1978) given by

$$\tilde{f}^{(0)}(t) = \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} \frac{\tilde{g}(-\omega)}{\Gamma(\frac{1}{2} - \mathrm{i}\omega)}\, t^{-(\frac{1}{2}+\mathrm{i}\omega)}\, \mathrm{d}\omega, \tag{6.4}$$

where $\tilde{g}(\omega)$ is the Mellin transform of $g(p)$ and ω_0 is given by the condition (see I and also McWhirter & Pike (1978))

$$|\Gamma(\tfrac{1}{2} + \mathrm{i}\omega_0)| = \left(\frac{\pi}{\cosh \omega_0}\right)^{\frac{1}{2}} = \frac{\epsilon}{E}, \tag{6.5}$$

E/ϵ being the signal-to-noise ratio. Now, if we use the approximate solution (6.4) in the case of noise-free data, namely $\tilde{g}(\omega) = \Gamma(\frac{1}{2}+\mathrm{i}\omega)f(-\omega)$, by elementary computations we get

$$\tilde{f}^{(0)}(t) = \int_0^{+\infty} M_0(t,s)f(s)\,\mathrm{d}s, \tag{6.6}$$

where

$$M_0(t,s) = \frac{\sin[\omega_0 \ln(t/s)]}{\pi(ts)^{\frac{1}{2}}\ln(t/s)}. \tag{6.7}$$

Again the kernel (6.7) has the following interpretation: for a δ-function concentrated at the point $t = a$ and in the absence of noise, the restoration provided by (6.4) is

$$\tilde{f}_a^{(0)}(t) = M_0(t,a) = \frac{\sin[\omega_0 \ln(t/a)]}{\pi(at)^{\frac{1}{2}}\ln(t/a)}. \tag{6.8}$$

The principal maximum of $\tilde{f}_a^{(0)}(t)$ is approximately at $t = a$ (in fact it is slightly to the left of this point) and the first zero to the right of $t = a$ is at $t_1 = a\exp(\pi/\omega_0)$, so that, using the Rayleigh criterion, we find the resolution ratio $\delta_0 = \exp(\pi/\omega_0)$.

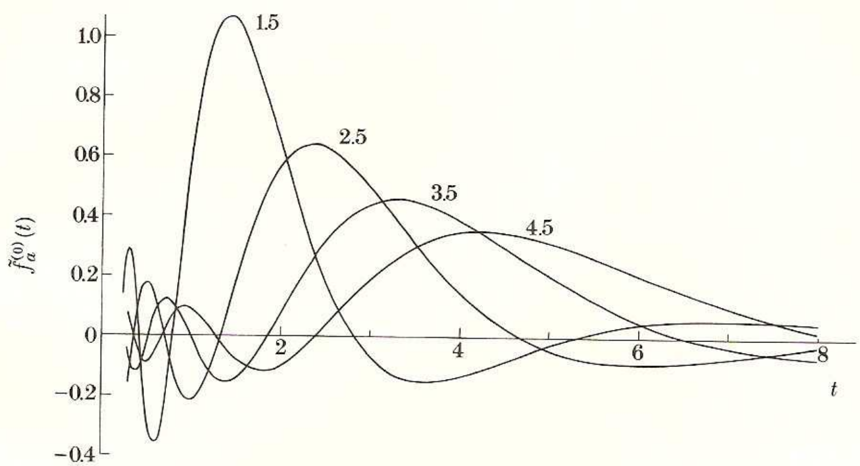

FIGURE 5. Behaviour of the function $\tilde{f}_a^{(0)}(t)$ (equation (6.8)), for different values of a. The value of ω_0 taken in these computations is 4.98, corresponding to $E/\epsilon = 10^3$ in (6.5).

The function (6.8), for various values of a, is plotted in figure 5. The typical behaviour is that the principal peak becomes broader and lower as a increases. Furthermore the ratio between the peak values corresponding to two different positions of the δ-function, say a_1 and a_2, is approximately a_2/a_1 and it is independent of ω_0 (in other words it is independent of the signal-to-noise ratio). We also notice that the position of the principal maximum is not exactly at $t = a$ but at $t = \bar{a} < a$. A comparison between a and $\bar{a}$ is given in table 7. As we see, the relative error decreases for increasing a.

By using the singular functions computed by means of the methods developed in this paper it is easy to illustrate the effect of the knowledge of the support of the unknown function on the resolution limit. In figure 6 we give the restoration of a δ-function concentrated at $t = 2$ using the unconstrained restoration (6.8) (full

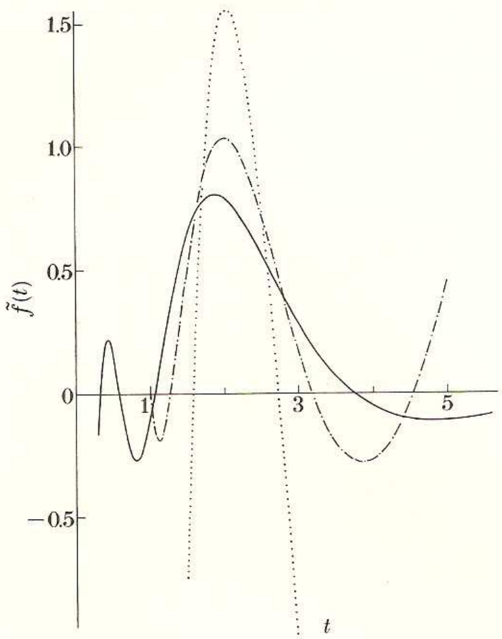

FIGURE 6. Restoration of a δ-function concentrated at $t = 2$, with use of a truncated Mellin transform inversion (full line) and truncated singular function expansions with $\gamma = 5$ (chained line) and $\gamma = 2$ (dotted line). In all cases we have used truncations corresponding to $E/\epsilon = 10^3$.

TABLE 7. POSITION $\bar{a}$ OF THE ABSOLUTE MAXIMUM OF THE FUNCTION $\tilde{f}_a^{(0)}(t)$ (EQUATION (6.8)), FOR VARIOUS VALUES OF a, TOGETHER WITH THE RELATIVE ERROR

a	1.5	2.5	3.5	4.5
$\bar{a}$	1.4	2.35	3.3	4.25
$(a-\bar{a})/a$ (%)	6.6	6	5.7	5.5

line) with $\omega_0 = 4.98$ (corresponding to $\delta_0 = 1.88$ and $E/\epsilon = 10^3$) and also the constrained restoration (6.3), in one case using the interval $[1, 5]$ as a support of the unknown function (chained line) and in another case using the interval $[1.5, 3]$, i.e. $\gamma = 2$ (dotted line). In the case $\gamma = 5$ we have taken 5 terms in (6.3) and three terms for $\gamma = 2$, namely we have used the singular functions corresponding to singular values greater than 10^{-3}. As we see, the maximum occurs approximately at the same point in the three cases, but the position of the first zero to the right of the principal maximum shifts to the left as γ decreases. Precisely in the three cases we have: $t_1 = 3.76$ for $\gamma = \infty$, $t_1 = 3.20$ for $\gamma = 5$ and $t_1 = 2.73$ for $\gamma = 2$. The corresponding resolution ratios are: $\delta = 1.88$ for $\gamma = \infty$, $\delta = 1.6$ for $\gamma = 5$ and $\delta = 1.36$ for $\gamma = 2$. The undesirable non-zero values of the reconstructions at the limits of support may be removed by reconstruction in a weighted L^2 space (Bertero *et al.* 1985).

For practical applications it is interesting to know not only the resolution limits

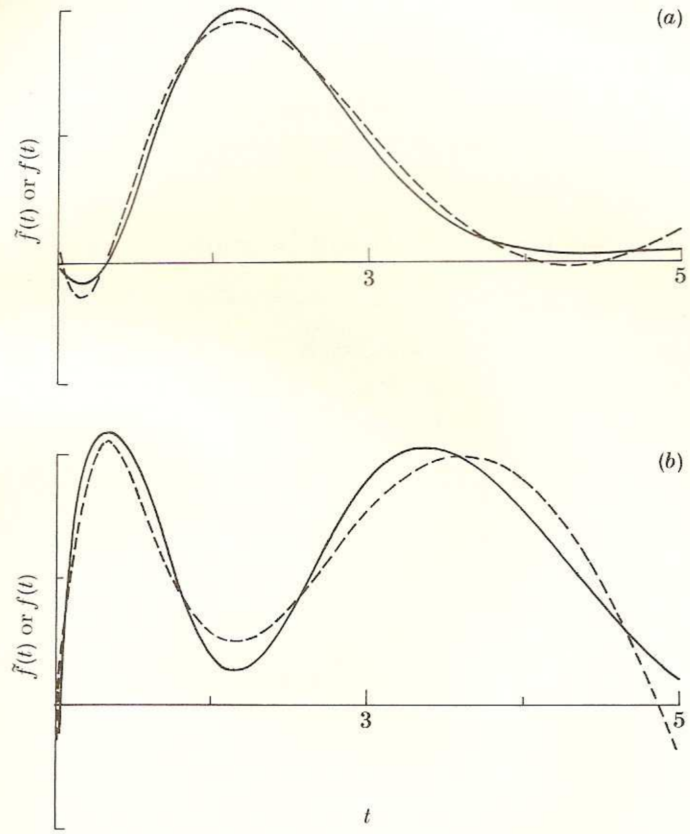

FIGURE 7. Restoration (broken lines) of two functions of the class (6.9) (solid lines) by using five singular functions in the truncated expansion for $\gamma = 5$. The values of the parameters for the two functions are specified in the text.

achievable in Laplace transform inversion, but also the class of the functions that can be accurately restored. The answer to this question is trivially given by the methods described in our papers: once the 'number of degrees of freedom' J has been determined for a given signal-to-noise ratio E/ϵ, then it is possible to restore accurately only those functions whose projection on the orthogonal complement of the subspace spanned by the singular functions $u_0, u_1, \ldots, u_{J-1}$ is small; namely the functions that are well approximated by a linear combination of these singular functions. It is interesting to note that, if we consider a function of the type

$$f(t) = \sum_{k=1}^{K} c_k \left(\frac{t_k}{t}\right)^{\frac{1}{2}} \frac{\sin\left[\Omega \ln (t/t_k)\right]}{\Omega \ln (t/t_k)}, \tag{6.9}$$

where

$$t_k = t_1 \delta^{k-1}, \quad \delta = \exp(\pi/\Omega), \tag{6.10}$$

then, from considerations founded on the sampling theorem (Ostrowsky *et al.* 1981; Pike *et al.* 1983) it is expected that such a function is well reproduced by a truncated singular function expansion with K terms when δ is approximately equal to the resolution ratio δ_s.

In figure 7 we give restorations of two functions of this type in the case $\gamma = 5$.

In both cases we have $K = 4$, $\delta = 1.495$ and $t_1 = \delta$, $\Omega = 7.812$ and $\delta^4 = 4.995$. The restoration uses 5 singular functions, corresponding to a resolution ratio $\delta_s = 1.4$ to 1.5. The values of the coefficients c_k are: $c_1 = 0.5$, $c_2 = 2$, $c_3 = 0.5$, $c_4 = 0.1$ in the case (*a*); $c_1 = 2$, $c_2 = 0.3$, $c_3 = 2$, $c_4 = 0.2$ in the case (*b*). If the singular values, the singular vectors and the singular functions are precomputed and stored, then the computation of the truncated solution (2.13) is extremely fast.

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References

Bertero, M., Boccacci, P. & Pike, E. R. 1982 *Proc. R. Soc. Lond.* A **383**, 15.

Bertero, M., Boccacci, P. & Pike, E. R. 1984 *Proc. R. Soc. Lond.* A **393**, 51.

Bertero, M., Brianzi, P. & Pike, E. R. 1985 *Inverse Problems* **1**. (In the Press.)

Cummins, H. Z. & Pike, E. R. (eds) 1974 *Photon correlation and light beating spectroscopy*. New York: Plenum Press.

Doetsch, G. 1943 *Laplace transformation*. New York: Dover.

Kato, T. 1966 *Perturbation theory for linear operators*. Berlin: Springer Verlag.

McWhirter, J. G. & Pike, E. R. 1978 *J. Phys.* A **11**, 1729.

Ostrowsky, N., Sornette, D., Parker, P. & Pike, E. R. 1981 *Optica Acta* **28**, 1059.

Papoulis, A. 1956 *Q. appl. Math.* **14**, 405.

Pike, E. R., Watson, D. & McNeil Watson, F. 1983 *Measurement of suspended particles by quasi-elastic light scattering* (ed. B. Dahneke), p. 107. New York: Wiley.

Riesz, F. & Sz. Nagy, B. 1955 *Functional analysis*. New York: Ungar.