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# Intensity fluctuation distributions from photon counting distributions: a singular-system analysis of Poisson transform inversion

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**Abstract.** The ill-posed problem of inverting photon counting distributions to obtain distributions of classical light intensity fluctuations has been recently reconsidered by Byrne *et al* after many years of dormancy. They apply a weighted-space analysis with Tikhonov regularisation to obtain some approximate inversions. The inversion problem is one of a linear mapping from  $L^2(0, +\infty)$  into a Euclidean space and in this paper we calculate explicitly the singular system of the operator in order to quantify the information content of noisy data and provide a corresponding inversion method.

## 1. Introduction

The operation of an absorbing photodetector has for many years been described in a semiclassical approximation by the photon counting formula of Mandel [1],

$$p(n, T) = \int_0^{+\infty} \frac{\bar{n}^n}{n!} e^{-\bar{n}} P(\bar{n}) d\bar{n} \quad (1.1)$$

where  $p(n, T)$  is the probability of detecting  $n$  photoelectrons in time  $T$ ,  $\bar{n}$  is the mean number of counts which would be so detected if the incident light intensity were constant and is proportional to such a classical intensity and  $P(\bar{n})$  is the corresponding scaled distribution of classical intensity fluctuations.

This formula has a quantum mechanical generalisation in which the distribution function  $P(\bar{n})$  is replaced by the so called Glauber  $P$ -representation of the light field [2], which is then no longer a probability distribution, but may have negative values.

The most useful property of equation (1.1), which may be readily shown, is that the  $m$ th moment of the distribution  $P(\bar{n})$  is given exactly by the  $m$ th factorial moment of  $p(n, T)$  (see, for example, [3] p 9), defined as

$$\langle n^{[m]} \rangle = \sum_{n=m}^{+\infty} n(n-1) \dots (n-m+1) p(n). \quad (1.2)$$

This fact has essentially obviated the necessity to invert the Mandel formula, which, of course, cannot be done without additional knowledge because for any finite experimental

determination of  $p(n, T)$  there will be an infinite set of possible solutions. Theoretical predictions for various types of radiation field usually give the expected values of the lower moments and these, with sufficient care, may be determined quite accurately. In an early comparison of laser and thermal light, for example, moments up to the sixth order were used [4].

As an example of an ill-conditioned problem, however, it is of interest to apply recently developed techniques of quantification of the information content of the data, which we have called generalised information theory [5], and which apply quite generally to linear inverse problems with discrete data [6]. These techniques are based on the singular system of the operator to be inverted. For this purpose we give a more precise mathematical formulation of the problem. If we shift by one unit the index  $n$ , we denote  $p(n-1, T)$  by  $g_n$  and  $P(\bar{n})$  by  $f(x)$  (we maintain here the notation introduced in [6]), then the problem is: given the sequence of real numbers  $g_n$ , find a square-integrable function  $f(x)$  such that

$$g_n = \int_0^{+\infty} \frac{x^{n-1}}{(n-1)!} e^{-x} f(x) dx \quad n = 1, 2, 3, \dots \quad (1.3)$$

We assume  $f(x)$  square-integrable since, in such a case, the integrals in (1.3) may be interpreted as scalar products.

The map which transforms  $f(x)$  into the sequence  $g_n$  is called a *Poisson transform* ([3], p 79) and is injective. The condition  $g_n = 0$  implies indeed that  $f$  is orthogonal to all the functions

$$\varphi_n(x) = \frac{1}{(n-1)!} x^{n-1} e^{-x} \quad n = 1, 2, \dots \quad (1.4)$$

and from the completeness of Laguerre polynomials it follows that  $f(x) = 0$ .

We notice that the procedure outlined above of computing the moments of  $f(x)$  from the factorial moments of the sequence  $g_n$  does not seem convenient for the estimation of  $f(x)$ . In such a way the inversion problem is reduced to a Stieltjes moment problem, i.e. find a function  $f(x)$  given the sequence of its moments  $\mu_n$ ,

$$\mu_n = \int_0^{+\infty} x^n f(x) dx \quad n = 1, 2, 3, \dots \quad (1.5)$$

As is well known, the solution of such a problem is not unique in  $L^2(0, +\infty)$ . For example all the moments of the function

$$f(x) = \exp(-x^{1/4}) \sin(x^{1/4}) \quad (1.6)$$

are zero ([7], p 126).

An inversion formula for the problem (1.3) can be given in terms of Laguerre polynomials ([3], p 83). This formula can be easily implemented because the coefficient of order  $n$  in the expansion depends only on the  $g_m$  with  $m \leq n$  and, therefore, the coefficients can be computed recursively. However, this method masks the severe ill-conditioning of the problem and gives unacceptable results in the case of noisy data. In this paper we consider the expansion of the solution in an appropriate basis, the basis of the *singular functions* of the problem, which depend on the number  $N$  of measured  $g_n$  values. Therefore computations are heavier than with the previous method. However, one can give precise criteria for the truncation of this expansion [5] in order to find stable solutions. A clarification of the results obtained by means of regularisation techniques [8] can also be achieved.

In § 2 we consider the problem where only a finite number  $N$  of  $g_n$  are given and we introduce the singular system of the problem. We also present the results of some numerical computations and we indicate an interpretation of these results in terms of resolution limits and of size of the interval where the function can be recovered. In § 3 we consider the case where the function  $f(x)$  has an exponential decay at infinity. In such a case the problem with a finite number of data is the projection on a finite-dimensional subspace of the inversion of a compact operator. Finally, in § 4 we give the results of some numerical inversions.

**2. Poisson transform inversion with a finite number of data**

We assume now that we have only a finite number, say  $N$ , of values of  $g_n$ . Then equation (1.3) defines a linear operator, which we denote by  $L_N$ , from the Hilbert space  $X = L^2(0, +\infty)$ , into the  $N$ -dimensional Euclidean space of the vectors  $\mathbf{g} = \{g_n\}_{n=1}^N$ . We can write the operator  $L_N$  in the form

$$(L_N f)_n = (f, \varphi_n)_X \quad n = 1, \dots, N \tag{2.1}$$

where the scalar product is that of  $L^2(0, +\infty)$  and the functions  $\varphi_n$  are defined in equation (1.4). Then the adjoint operator  $L_N^*$  is given by [6]

$$(L_N^* \mathbf{g})(x) = \sum_{n=1}^N g_n \varphi_n(x). \tag{2.2}$$

The adjoint transforms a vector  $\mathbf{g}$  into a function of the subspace of  $X$  spanned by the functions  $\varphi_n$ .

The singular system  $\{\alpha_{N,k}; u_{N,k}, v_{N,k}\}_{k=1}^N$  of  $L_N$  is the set of the solutions of the coupled equations

$$L_N u_{N,k} = \alpha_{N,k} v_{N,k} \quad L_N^* v_{N,k} = \alpha_{N,k} u_{N,k}. \tag{2.3}$$

The singular vectors  $v_{N,k}$  are the eigenvectors, associated with the eigenvalues  $\alpha_{N,k}^2$ , of the operator  $L_N L_N^*$ , whose matrix is the Gram matrix of the functions  $\varphi_n$  [6]:

$$[L_N L_N^*]_{nm} = G_{nm} = (\varphi_n, \varphi_m)_X = \frac{(n+m-2)!}{(n-1)!(m-1)!} 2^{-(n+m-1)} \tag{2.4}$$

$n, m = 1, \dots, N.$

The eigenvalues and eigenvectors of the matrix  $[G]$  can be computed by standard methods of linear algebra. Notice, however, that usually we need only the largest eigenvalues, since when  $N$  is large (say  $N > 10$ ), the matrix  $[G]$  can be strongly ill-conditioned.

Once we have  $\alpha_{N,k}$  and  $v_{N,k}$ , the corresponding singular function  $u_{N,k}$  can be obtained by combining the second equation (2.3) with equation (2.2). The result is

$$\begin{aligned} u_{N,k}(x) &= \frac{1}{\alpha_{N,k}} \sum_{n=1}^N (v_{N,k})_n \varphi_n(x) \\ &= \frac{1}{\alpha_{N,k}} \left( \sum_{n=1}^N \frac{(v_{N,k})_n}{(n-1)!} x^{n-1} \right) e^{-x}. \end{aligned} \tag{2.5}$$

Clearly all the singular functions  $u_{N,k}$  belong to the subspace spanned by the functions  $\varphi_n$ .

**Table 1.** Singular values of the Poisson transform in the case of a finite number  $N$  of data values, for various values of  $N$ . Only the singular values  $\alpha_{N,1}/\alpha_{N,k} < 10^3$  are given.

$k$	$\alpha_{N,k}(N=10)$	$\alpha_{N,k}(N=20)$	$\alpha_{N,k}(N=30)$
1	0.945 16	0.970 19	0.979 39
2	0.743 15	0.852 73	0.896 13
3	0.482 90	0.676 50	0.764 00
4	0.260 00	0.484 98	0.607 23
5	0.116 30	0.314 74	0.450 30
6	$0.431\ 90 \times 10^{-1}$	0.185 33	0.311 92
7	$0.132\ 03 \times 10^{-1}$	$0.992\ 40 \times 10^{-1}$	0.202 10
8	$0.324\ 79 \times 10^{-2}$	$0.484\ 21 \times 10^{-1}$	0.122 65
9		$0.215\ 51 \times 10^{-1}$	$0.698\ 22 \times 10^{-1}$
10		$0.875\ 13 \times 10^{-2}$	$0.373\ 27 \times 10^{-1}$
11		$0.323\ 89 \times 10^{-2}$	$0.187\ 57 \times 10^{-1}$
12		$0.109\ 02 \times 10^{-2}$	$0.886\ 49 \times 10^{-2}$
13			$0.394\ 19 \times 10^{-2}$
14			$0.164\ 92 \times 10^{-2}$

We have computed the singular values  $\alpha_{N,k}$  for  $N=10, 20, 30$ . In table 1 we report only the singular values  $\alpha_{N,k}$  such that  $\alpha_{N,1}/\alpha_{N,k} < 10^3$ . We see that, for fixed  $k$ ,  $\alpha_{N,k}$  is an increasing function of  $N$ .

It is not difficult to explain this property if we consider that the  $\alpha_{N,k}^2$  are the eigenvalues of the finite-rank integral operator  $\tilde{L}_N = L_N^* L_N$ . If  $M > N$ , then

$$\tilde{L}_M = \tilde{L}_N + \tilde{R}_{NM} \quad (2.6)$$

where  $\tilde{R}_{NM}$  is an integral operator given by

$$(\tilde{R}_{NM}f)(x) = \sum_{n=N+1}^M \varphi_n(x) \int_0^{+\infty} \varphi_n(y) f(y) dy. \quad (2.7)$$

It is easy to see that the operator  $\tilde{R}_{NM}$  is positive, i.e.

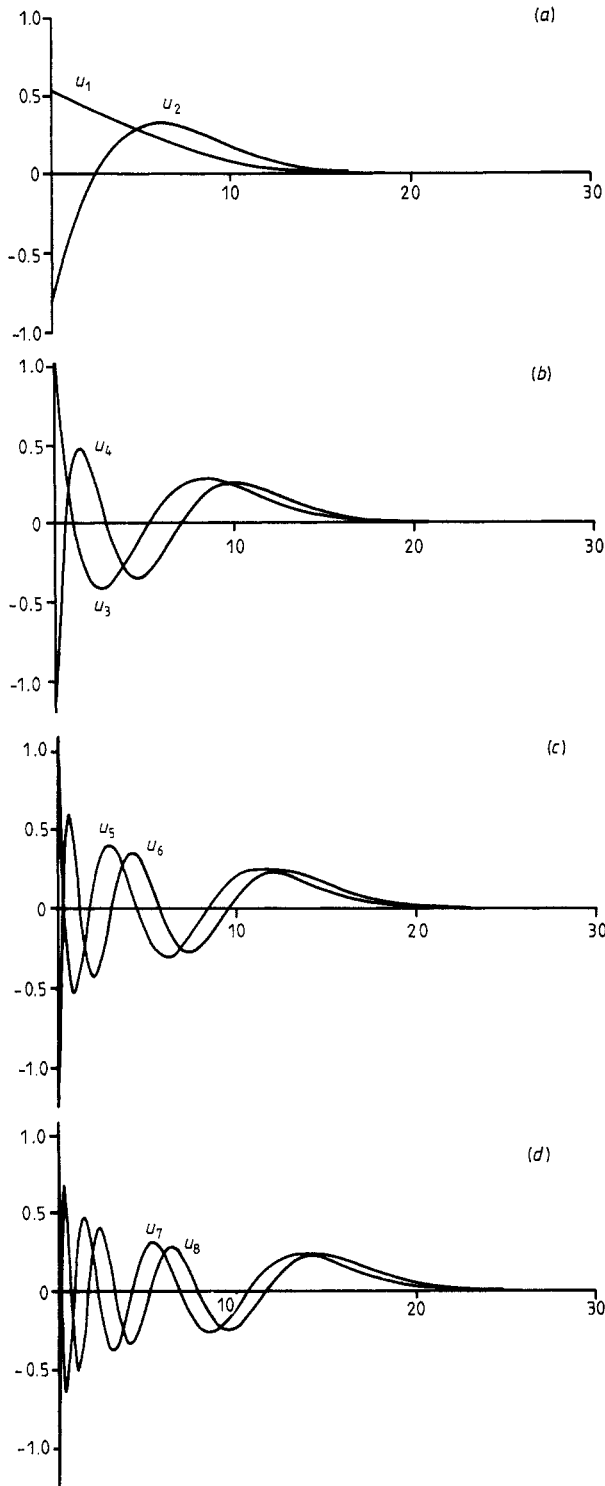
$$(\tilde{R}_{NM}f, f)_X \geq 0, \quad (2.8)$$

and therefore it follows from the Weyl–Courant lemma [9] that, for any  $k$ ,

$$\alpha_{M,k}^2 \geq \alpha_{N,k}^2 \quad M \geq N. \quad (2.9)$$

According to a standard terminology in optics, we call the *number of degrees of freedom* the number of singular values greater than some threshold value (given, for example, by the inverse signal-to-noise ratio) [5]. Furthermore, the number of degrees of freedom has a close connection with the resolution achievable in the retrieval problem.

Now, from table 1 it follows that the number of degrees of freedom grows if we increase  $N$ . The meaning of this property can be understood if we look at the singular functions  $u_{N,k}$ . The singular functions, corresponding to  $N=10$  and to the first eight singular values, are plotted in figure 1. These singular functions are significantly different from zero in the interval  $[0, 20]$  and therefore, using these functions, it is possible to recover  $f(x)$  at most in such an interval. We have also computed the singular functions in the cases  $N=20$  and  $N=30$  corresponding to the singular values of table 1 and we have found that they are significantly different from zero in longer intervals. Therefore, the increase in the number of degrees of freedom is presumably related to the increase in the



**Figure 1.** Singular functions  $u_{N,k}(x)$  in the case  $N=10$ : (a)  $k=1, 2$ ; (b)  $k=3, 4$ ; (c)  $k=5, 6$ ; (d)  $k=7, 8$ .

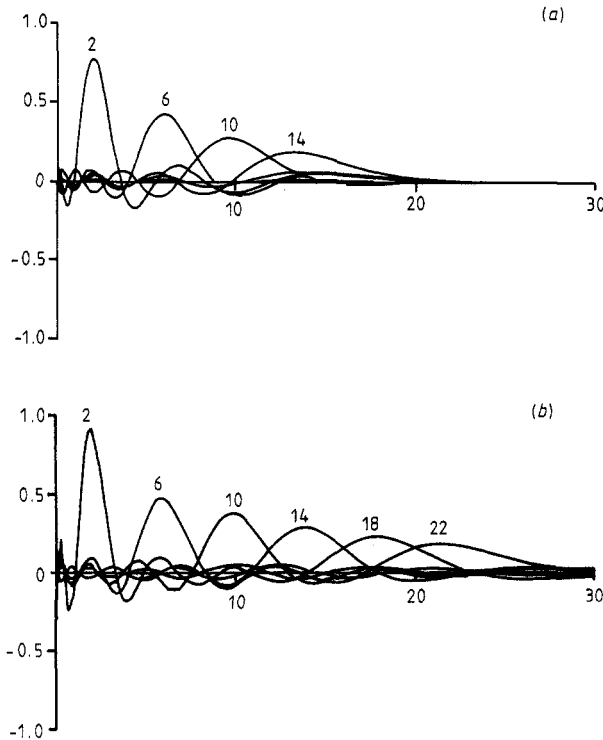
interval where the function can be recovered, the number of degrees of freedom for unit length being approximately constant.

We can give evidence of this phenomenon by considering the 'impulse response' of the retrieval algorithm (see § 4). This is given by

$$A(x, y) = \sum_{k=1}^{N_0} u_{N,k}(x)u_{N,k}(y) \quad (2.10)$$

where  $N_0$  is the number of degrees of freedom. The function  $A(x, y)$ , with  $y$  fixed, is an 'approximation' of the  $\delta$  function and has a main lobe around  $x=y$ . The width of this lobe can be taken as a measure of the resolution achievable in the inversion algorithm which uses  $N_0$  terms in the singular function expansion for  $f(x)$ .

In figure 2 we plot the function  $A(x, y)$  in the cases  $N=10$  and  $N=20$ , using in both cases the number of singular values given in table 1 as the value of  $N_0$ . This choice corresponds to assuming that the signal-to-noise ratio is roughly the same in the case of 10 data points and in the case of 20 data points. We have considered the values  $y=2, 6, 10, 14, 18, 22$ . In the case of  $N=10$  and  $y > 14$ ,  $A(x, y)$  is quite small and is not plotted in the figure; on the other hand, in the case  $N=20$ ,  $A(x, y)$  takes significant values up to  $y=22$ , although the maximum value of  $A(x, y)$  decreases for increasing  $y$ . Furthermore, the width of the peaks corresponding to the same  $y$  is approximately the same in the two cases. As a conclusion we can say that resolution depends essentially on the signal-to-noise



**Figure 2.** Plot of the impulse functions  $A(x, y)$  for various values of  $y$  ( $y=2, 6, 10, 14, 18, 22$ ) for (a)  $N=10$ ,  $N_0=8$  and (b)  $N=20$ ,  $N_0=12$ . The singular functions used for the computation of  $A(x, y)$  correspond to the singular values given in table 1.

ratio (and is a function of the distance from the origin) while the interval where the function can be recovered depends on the number of data points.

### 3. Spaces of exponentially decreasing functions

If we know that the function  $f$  has an exponential tail at infinity, then we can use such *a priori* knowledge in the inversion procedure just by changing the Hilbert space  $X$  of the unknown functions [6]. For this reason in this section we assume that the scalar product in  $X$  is defined by

$$(f, \varphi)_X = \int_0^{+\infty} e^{2\beta x} f(x)\varphi(x) dx \tag{3.1}$$

where  $\beta$  is a lower limit for the decay constant of  $f(x)$ , i.e.  $f(x)$  must decrease faster than  $\exp(-\beta x)$ .

Now equation (1.3) defines a map  $L$  of  $X$  into the Hilbert space  $Y$  of the square-summable sequences, with a scalar product given by

$$(g, h)_Y = \sum_{n=1}^{\infty} g_n h_n. \tag{3.2}$$

We will prove that *the operator  $L$  is compact* or, more precisely, that  $L$  is of the Hilbert–Schmidt class since  $LL^*$  is of the trace class.

First notice that the operator  $L^*: Y \rightarrow X$  is given by

$$(L^*g)(x) = \left( \sum_{n=1}^{+\infty} \frac{x^{n-1}}{(n-1)!} g_n \right) \exp[-(1+2\beta)x]. \tag{3.3}$$

Then the operator  $LL^*$  can be represented by an infinite-dimensional matrix in  $Y$ , the matrix elements being given by (the functions  $\varphi_n$  are those defined in equation (1.4))

$$\begin{aligned} [LL^*]_{nm} &= \int_0^{+\infty} \varphi_n(x)\varphi_m(x) e^{-2\beta x} dx \\ &= \frac{(n+m-2)!}{(n-1)!(m-1)!} (2+2\beta)^{-(n+m-1)}. \end{aligned} \tag{3.4}$$

From Stirling’s formula we have, for large  $n$ ,

$$[LL^*]_{nn} \simeq \frac{1}{\sqrt{4\pi n}} (1+\beta)^{-2n+1} \tag{3.5}$$

and therefore

$$\text{Tr}(LL^*) = \sum_{n=1}^{\infty} [LL^*]_{nn} < +\infty. \tag{3.6}$$

Since the operator  $L$  is compact, it admits a singular system  $\{\alpha_k; u_k, v_k\}_{k=1}^{\infty}$ . The  $u_k$  are singular functions in  $X$  and the  $v_k$  are singular sequences in  $Y$ .

The operator  $L_N$ , corresponding to the problem with  $N$  data, is just the projection on a subspace of dimension  $N$  of the operator  $L$ . Therefore it is obvious (and it can be easily proved) that the singular system  $\{\alpha_{N,k}; u_{N,k}, v_{N,k}\}_{k=1}^N$  of  $L_N$  approaches the singular system of  $L$  when  $N \rightarrow \infty$ . It follows that, provided  $N$  is sufficiently large, the number of

**Table 2.** Singular values of the Poisson transform in a weighted space with exponential weight  $\exp(2\beta x)$ , for various values of  $\beta$ .

$k$	$\alpha_k(\beta=1)$	$\alpha_k(\beta=0.5)$	$\alpha_k(\beta=0.25)$
1	0.517 64	0.618 03	0.707 11
2	0.138 70	0.236 07	0.353 55
3	$0.371\ 65 \times 10^{-1}$	$0.901\ 70 \times 10^{-1}$	0.176 78
4	$0.995\ 83 \times 10^{-2}$	$0.344\ 42 \times 10^{-1}$	$0.883\ 88 \times 10^{-1}$
5	$0.266\ 83 \times 10^{-2}$	$0.131\ 56 \times 10^{-1}$	$0.441\ 94 \times 10^{-1}$
6	$0.714\ 97 \times 10^{-3}$	$0.502\ 50 \times 10^{-2}$	$0.220\ 97 \times 10^{-1}$
7		$0.191\ 94 \times 10^{-2}$	$0.110\ 49 \times 10^{-1}$
8		$0.733\ 14 \times 10^{-3}$	$0.552\ 43 \times 10^{-2}$
9			$0.276\ 21 \times 10^{-2}$
10			$0.138\ 11 \times 10^{-2}$

degrees of freedom does not depend on  $N$  but depends only on the signal-to-noise ratio. The intuitive reason for this fact is that the functions in  $X$  have an approximately bounded support and therefore it is not required to recover  $f(x)$  on a large interval.

In table 2 we give the largest singular values of  $L$ , i.e. the  $\alpha_k$  such that  $\alpha_1/\alpha_k < 10^3$ , for some values of  $\beta$ . These singular values have been computed with a finite value of  $N$  but  $N$  was chosen in such a way that an increase of  $N$  does not change the first five digits of  $\alpha_k$ . The minimum values of  $N$  we have used are  $N=20$  for  $\beta=1$ ,  $N=30$  for  $\beta=0.5$  and  $N=40$  for  $\beta=0.25$ .

As  $\beta$  decreases, the number of degrees of freedom increases, as follows from the discussion of § 2, since a smaller value of  $\beta$  means a larger ‘support’ of  $f(x)$ . It should not be difficult to prove that  $\alpha_k$  is a decreasing function of  $\beta$  for fixed  $k$ .

**4. Reconstruction algorithm**

In the case of a finite number of data, the solution of problem (1.3) is not unique. Given a particular solution one can add an arbitrary function orthogonal to the subspace spanned by the  $\varphi_n(x)$  ( $n=1, \dots, N$ ) without changing the values of the data. However, since the functions  $\varphi_n(x)$  are linearly independent, there always exists a unique solution  $f^+(x)$  of minimal norm. This can be expressed in terms of the singular system of the operator  $L_N$  as follows [6]:

$$f^+(x) = \sum_{k=1}^N \frac{1}{\alpha_{N,k}} (\mathbf{g}, \mathbf{v}_{N,k})_Y u_{N,k}(x). \tag{4.1}$$

The problem of the computation of  $f^+(x)$  can be extremely ill-conditioned when  $N$  is large. The ratio  $\alpha_{N,1}/\alpha_{N,N}$  tends to infinity when  $N \rightarrow \infty$ . Then one can introduce an approximate, stable solution by a very simple ‘filtering’ technique, i.e. by taking only the first  $N_0$  terms in equation (4.1), where  $N_0$  is the number of degrees of freedom [5]. We denote such a ‘truncated’ expansion by  $\tilde{f}(x)$ .

It is interesting to express  $\tilde{f}(x)$  in terms of the impulse response when the data are free of noise. In such a case we have  $\mathbf{g} = L_N f$ , where  $f$  is the ‘true’ unknown function. Observing that

$$(\mathbf{g}, \mathbf{v}_{N,k})_Y = (L_N f, \mathbf{v}_{N,k})_Y = (f, L_N^* \mathbf{v}_{N,k})_X = \alpha_{N,k}(f, u_{N,k})_X,$$

we conclude that  $\tilde{f}(x)$  can be written as

$$\tilde{f}(x) = \int_0^{+\infty} A(x, y) f(y) dy, \tag{4.2}$$

the kernel  $A(x, y)$  being given by equation (2.10). Therefore, in the absence of noise,  $\tilde{f}(x)$  is a 'blurred' copy of  $f(x)$ ; in the case of noisy data one must also add an error term to equation (4.2). The computation of  $\tilde{f}(x)$  can be very easy and fast if the singular system of  $L_N$  has been precomputed and stored.

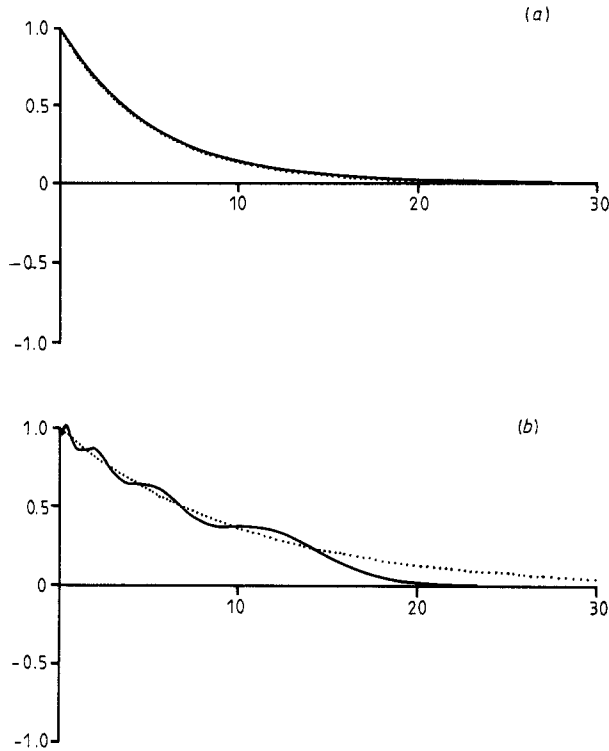
In order to give some examples of numerical inversions, we have considered the case of an exponential function

$$f(x) = \exp(-\gamma x) \tag{4.3}$$

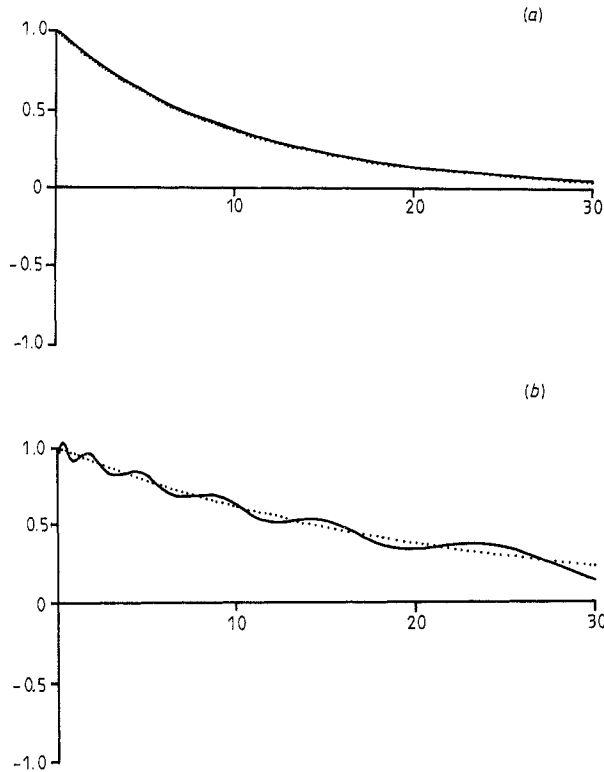
whose Poisson transform is

$$g_n = (1 + \gamma)^{-n} \quad n = 1, 2, \dots \tag{4.4}$$

We have computed the reconstruction of (4.3) for various values of  $\gamma$  in the cases of 10 and 20 data points. The number of terms used in the expansion of  $\tilde{f}(x)$  was taken from table 1, i.e. eight terms in the case  $N=10$  and twelve terms in the case  $N=20$ . Some results are shown in figures 3 and 4.



**Figure 3.** Recovery of exponential functions  $f(x) = \exp(-\gamma x)$  using 10 data points and the corresponding singular values of table 1: (a)  $\gamma = 0.2$ ; (b)  $\gamma = 0.1$ . The dotted curve is the exponential, the full curve is the reconstructed exponential.



**Figure 4.** Recovery of exponential functions  $f(x) = \exp(-\gamma x)$  using 20 data points and the corresponding singular values of table 1: (a)  $\gamma = 0.1$ ; (b)  $\gamma = 0.05$ . The dotted curve is the exponential, the full curve is the recovered exponential.

From the analysis of § 2 we expect that, for a given value of  $N$ , the reconstruction must be good if  $\gamma$  is sufficiently large and poor if  $\gamma$  is too small. Furthermore, if we increase the number of data points, we must be able to recover exponentials with smaller values of  $\gamma$ . These conclusions have been confirmed by the numerical simulations. With  $N = 10$  it is possible to have a satisfactory reconstruction of the exponential in the case  $\gamma = 0.2$  (and even better when  $\gamma > 0.2$ ), but not in the case  $\gamma = 0.1$ . The latter case can be recovered using 20 data points.

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