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Spectral properties of a differential operator related to the inversion of the finite Laplace transform

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Abstract. We investigate the spectrum of a differential operator whose eigenfunctions are the singular functions of the finite Laplace transform. We demonstrate a close connection of this operator with the Legendre operator and we give results of numerical computations of its eigenvalues and eigenfunctions. The latter are of great relevance in the problem of the finite Laplace transform inversion.

1. Introduction

In a previous paper [1] it was shown that the singular functions of the finite Laplace transform

$$(\mathscr{L}f)(p)=\int_1^{\gamma} \mathrm{e}^{-pt}f(t)\,\mathrm{d}t \qquad 0<p<\infty \tag{1.1}$$

are eigenfunctions of self-adjoint differential operators. This result provides a route for accurate and economical computations of such singular functions which are solutions of the coupled integral equations

$$\mathscr{L}u_k=\alpha_k v_k \qquad \mathscr{L}^* v_k=\alpha_k u_k \tag{1.2}$$

for $k=0, 1, \ldots$, where $\mathscr{L}^*$ is the adjoint operator

$$(\mathscr{L}^* g)(t)=\int_0^{+\infty} \mathrm{e}^{-pt}g(p)\,\mathrm{d}p \qquad 1<t<\gamma. \tag{1.3}$$

For details see [1, 2].

The singular functions u_k are eigenfunctions of the second-order self-adjoint positive definite differential operator

$$(\tilde{D}u)(t)=-[(t^2-1)(\gamma^2-t^2)u'(t)]'+2(t^2-1)u(t) \tag{1.4}$$

while the singular functions v_k are eigenfunctions of the fourth-order self-adjoint positive

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definite differential operator

$$(\hat{D}v)(p)=(p^2v''(p))''-(\gamma^2+1)(p^2v'(p))'+(\gamma^2p^2-2)v(p). \tag{1.5}$$

The operator $\tilde{D}$ commutes with the finite Stieltjes transform

$$(\mathscr{L}^*\mathscr{L}f)(t)=\int_1^\gamma \frac{f(s)}{t+s}\,\mathrm{d}s \tag{1.6}$$

and the operator $\hat{D}$ commutes with the integral operator

$$(\mathscr{L}\mathscr{L}^*g)(p)=\int_0^{+\infty} \frac{\mathrm{e}^{-(p+q)}-\mathrm{e}^{-\gamma(p+q)}}{p+q}\,g(q)\,\mathrm{d}q. \tag{1.7}$$

In this paper we investigate the eigenvalue problem of the differential operator (1.4) and we present a remarkable property of its eigenvalue spectrum. An interpretation of this property in terms of isospectral manifolds is suggested. Finally we give results of numerical computations of the eigenvalues and eigenfunctions of (1.4).

2. The limiting case $\gamma=1$

It is not hard to see what happens for $\gamma=1$. As in [1], the change of variables

$$t=\tfrac{1}{2}(\gamma-1)x+\tfrac{1}{2}(\gamma+1) \tag{2.1}$$

transforms (1.4) into

$$(Du)(x)=\tfrac{1}{4}[(x^2-1)(4+3\varepsilon+\varepsilon x)(4+\varepsilon+\varepsilon x)u'(x)]'+\tfrac{1}{2}\varepsilon(x+1)(4+\varepsilon+\varepsilon x)u(x) \tag{2.2}$$

with

$$\varepsilon=\gamma-1. \tag{2.3}$$

Thus, for $\varepsilon=0$ we get

$$(Du)(x)=4[(x^2-1)u'(x)]', \tag{2.4}$$

i.e. a scaled version of the well known Legendre operator.

If we denote by μ_k the eigenvalues of the operator (1.4),

$$\tilde{D}u_k=\mu_k u_k \tag{2.5}$$

for $k=0, 1, 2, \ldots$, the eigenvalues being ordered to form an increasing sequence, then we have

$$\lim_{\varepsilon\to 0}\mu_k=4k(k+1). \tag{2.6}$$

A remarkable property of the computed eigenvalues (see § 4) is that for moderate values of γ and k they are very well approximated by a formula similar to (2.6), i.e.

$$\bar{\mu}_k=\alpha_\gamma+\beta_\gamma k(k+1) \tag{2.7}$$

where α_γ and β_γ depend on γ, with $\alpha_\gamma\to 0$ and $\beta_\gamma\to 4$ when $\gamma\to 1$. If (2.7) were true for $\gamma\neq 1$, one would have

$$\rho_k\equiv 2\,\frac{\mu_k-\mu_0}{\mu_1-\mu_0}=k(k+1) \qquad k\geqslant 2. \tag{2.8}$$

Table 1. Values of the ratios ρ_k for the first few values of k and for several values of γ.

k	$\gamma=2$	$\gamma=50$	$\gamma=100$	$\gamma=1000$	$\gamma=10\,000$	$\gamma=100\,000$
2	5.99987	5.94337	5.914	5.8036	5.6992	5.6156
3	11.9997	11.8656	11.795	11.513	11.228	10.984
4	19.9994	19.7636	19.639	19.134	18.610	18.146
5	29.9991	29.6365	29.445	28.663	27.842	27.105
6	41.9987	41.4842	41.213	40.098	38.924	37.862

In table 1 we give the values of the ratios ρ_k for several values of γ up to $\gamma=10^5$. It follows from table 1 that the ρ_k have a nice smooth behaviour: they are slowly decreasing functions of γ and their range is rather small. In fact, the total variation of ρ_k over quite a large interval is only a few per cent of its initial value, $k(k+1)$.

3. Isospectral manifolds for the Legendre operator

We now give a framework in which one can discuss the surprising fact that (2.8) is nearly true for moderate values of γ and k. We start with a brief description of the 'spectral' class $\boldsymbol{Q}$ of operators of the form

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2-1)\frac{\mathrm{d}}{\mathrm{d}x}+q(x) \qquad -1\leqslant x\leqslant 1 \tag{3.1}$$

having the same spectrum $\mu_k=\mu_k^0\equiv k(k+1)$ as Q^0, the Legendre operator corresponding to $q(x)=q_0(x)\equiv 0$.

It is useful to rewrite the operator above in a different form. Recall that by means of the Liouville transformation one reduces the Legendre operator in a two-step process to the equivalent form

$$-\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}-\frac{1}{4\sin^2\alpha}-\frac{1}{4} \qquad 0\leqslant\alpha\leqslant\pi. \tag{3.2}$$

This involves (*a*) setting $x=\cos\alpha$ and (*b*) conjugating the original operator by an appropriate function to get rid of the coefficient in $\mathrm{d}/\mathrm{d}\alpha$. We prefer to think of this as the Legendre operator and thus $q_0(\alpha)$ denotes $-1/4\sin^2\alpha-\frac{1}{4}$ from now on.

The Liouville transformation can also be used to reduce an operator of the form

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((x^2-1)a(x)\frac{\mathrm{d}}{\mathrm{d}x}\right)+b(x) \tag{3.3}$$

with $a(x)>0$ for $-1\leqslant x\leqslant 1$ to one of the form

$$-\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}+q(\alpha) \qquad 0\leqslant\alpha\leqslant\pi. \tag{3.4}$$

The first step would consist of choosing the new variable α such that

$$\mathrm{d}x=(1-x^2)^{1/2}a(x)^{1/2}\,\mathrm{d}\alpha. \tag{3.5}$$

To repeat: we consider the class $\boldsymbol{Q}$ of operators (3.4) having the same spectrum as Q^0 given by (3.2).

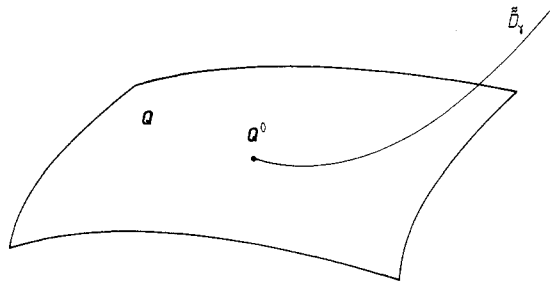

Figure 1. Two-dimensional caricature of the isospectral manifold for the Legendre operator.

One can picture $\boldsymbol{Q}$ as a smooth manifold of infinite dimension made up of potentials $q(\alpha)$ given by the expression

$$q(\alpha)=q_0(\alpha)-2\frac{\partial^2}{\partial\alpha^2}\ln\theta(\alpha, t_0, t_1, t_2, \ldots) \tag{3.6}$$

with

$$\theta=\det\left(\delta_{ij}+(\mathrm{e}^{t_i}-1)\int_\alpha^\pi P_i^0(\alpha)P_j^0(\alpha)\,\mathrm{d}\alpha\right) \qquad 0\leqslant i,j\leqslant\infty.$$

Here $P_i^0(\alpha)$ denote the normalised eigenfunctions of Q^0. This expression is of the Gelfand–Levitan type [3, 4], and at least when only a finite number of the free parameters $t_0, t_1, t_2, \ldots$ are non-zero one gets an effective construction for potentials in the class $\boldsymbol{Q}$. To get the whole manifold $\boldsymbol{Q}$ one has to impose rapid decay of the t_i.

The description given above for the 'spectral class' $\boldsymbol{Q}$ has been made quite rigorous in the case of regular Sturm–Liouville problems. The only singular case treated in the literature is the case of the harmonic oscillator, see [5].

Now we can depict $\boldsymbol{Q}$ as a nice smooth manifold with coordinates (if that turns out to be helpful) $(t_0, t_1, t_2, \ldots)$. We draw a two-dimensional caricature of $\boldsymbol{Q}$ in figure 1 and indicate the relative position of our family of operators $\tilde{D}_\gamma$ from (1.4)—or more correctly the one-parameter family of operators

$$\tilde{\tilde{D}}_\gamma=\frac{\tilde{D}_\gamma-\alpha_\gamma I}{\beta_\gamma}$$

obtained by shifting and scaling $\tilde{D}_\gamma$ to ensure that the first two eigenvalues of $\tilde{\tilde{D}}_\gamma$ are 0 and 2. The curve $\tilde{\tilde{D}}_\gamma$, $\gamma\geqslant 1$, does not belong to $\boldsymbol{Q}$ but has a 'high degree of contact' with $\boldsymbol{Q}$ at Q^0. One has to take γ fairly large before $\tilde{\tilde{D}}_\gamma$ is really far from $\boldsymbol{Q}$.

We expect to return to this point with a more quantitative form of this statement in a future publication.

4. The eigenvalue spectrum: numerical results

Solving the eigenvalue problem for the operator (1.4) is equivalent to looking for bounded solutions of the differential equation

$$-[(t^2-1)(\gamma^2-t^2)u'(t)]'+2(t^2-1)u(t)=\mu u(t). \tag{4.1}$$

The requirement of boundedness of the solution provides the following conditions at the end-points of the interval $[1, \gamma]$ (see [1]):

$$2(\gamma^2-1)u'(1)+\mu u(1)=0 \qquad -2\gamma(\gamma^2-1)u'(\gamma)+[\mu-2(\gamma^2-1)]u(\gamma)=0. \tag{4.2}$$

By introducing the variable

$$x=\frac{\ln(t/\sqrt{\gamma})}{\ln\sqrt{\gamma}} \tag{4.3}$$

and the function

$$\varphi(x)=\gamma^{x/4}u(\gamma^{(x+1)/2}) \tag{4.4}$$

(see [1], § 6), the differential equation (4.1) becomes

$$-\left(\frac{4}{(\ln\gamma)^2}(\gamma^2+1-2\gamma\cosh(x\ln\gamma))\varphi'(x)\right)' + [\tfrac{3}{2}\gamma\cosh(x\ln\gamma)+\tfrac{1}{4}(\gamma^2+1)-2]\varphi(x)=\mu\varphi(x) \tag{4.5}$$

with the following conditions at ± 1:

$$\begin{aligned} &4(\gamma^2-1)\varphi'(-1)+\ln\gamma[\mu-(\gamma^2-1)]\varphi(-1)=0\\ &-4(\gamma^2-1)\varphi'(1)+\ln\gamma[\mu-(\gamma^2-1)]\varphi(1)=0. \end{aligned} \tag{4.6}$$

Consider next the limiting behaviour when $\gamma\to 1$: notice that when $\varepsilon\equiv\gamma-1\to 0$,

$$\frac{4}{(\ln\gamma)^2}(\gamma^2+1-2\gamma\cosh(x\ln\gamma))=4(1-x^2)+O(\varepsilon) \tag{4.7}$$

and also

$$\tfrac{3}{2}\gamma\cosh(x\ln\gamma)+\tfrac{1}{4}(\gamma^2+1)-2=O(\varepsilon) \tag{4.8}$$

where the approximation is uniform in $x\in[-1, 1]$ in both cases. Therefore, in the limit $\varepsilon\to 0$ the differential equation (4.5) is approximated by the Legendre equation and the eigenvalues μ_k must converge to $4k(k+1)$ while the (normalised) eigenfunctions $\varphi_k(x)$ must converge to $(k+\frac{1}{2})^{1/2}P_k(x)$. This is, of course, the same remark as made earlier, but equation (4.5) is better than equation (4.1) since the coefficients are even functions of x.

Both problems (4.1) and (4.5) have the standard form

$$-(A(x)f'(x))'+B(x)f(x)=\mu f(x) \tag{4.9}$$

with $A(x)$ zero at the end-points of a bounded interval $[a, b]$ and both the conditions (4.2) and (4.6) have the form

$$A'(a)f'(a)+(\mu-B(a))f(a)=0 \qquad A'(b)f'(b)+(\mu-B(b))f(b)=0. \tag{4.10}$$

This problem can be discretised by the method of centred differences so that the approximation is of the order of h^2.

If we put $x_n=a+nh$ and also

$$A_n=A(x_n-\tfrac{1}{2}h) \qquad f_n=f(x_n) \qquad B_n=B(x_n), \tag{4.11}$$

then we may use the standard finite difference operators

$$\Delta f_n = f_{n+1} - f_n \qquad \nabla f_n = f_n - f_{n-1} \tag{4.12}$$

to get

$$(A(x)f'(x))'|_{x=x_n} = \frac{1}{h^2}\,\Delta(A_n \nabla f_n) + O(h^2), \tag{4.13}$$

and equation (4.9) is replaced by

$$\frac{1}{h^2}\,\Delta(A_n \nabla f_n) + B_n f_n = \mu f_n. \tag{4.14}$$

This equation is written for $n = 0, 1, \ldots, N$ $(x_N = b)$; then f_{-1} and f_{N+1} are eliminated from the first and the last equation respectively using the conditions

$$\begin{aligned} &\frac{1}{2h}\,A'(x_0)(f_1 - f_{-1}) + (\mu - B(x_0))f_0 = 0 \\ &\frac{1}{2h}\,A'(x_N)(f_{N+1} - f_{N-1}) + (\mu - B(x_N))f_N = 0. \end{aligned} \tag{4.15}$$

The eigenvalues have been computed using 1000 and 2000 points and applying the deferred correction method, or Richardson extrapolation, in order to improve the

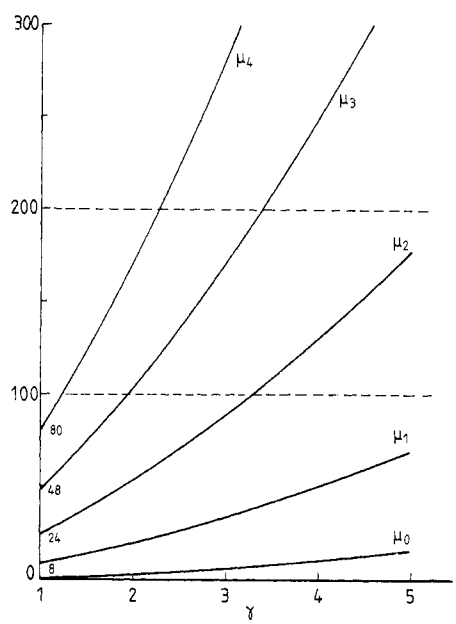

Figure 2. Behaviour of the eigenvalues of the differential operator (2.1) as a function of γ. The limiting values for $\gamma = 1$ are also indicated.

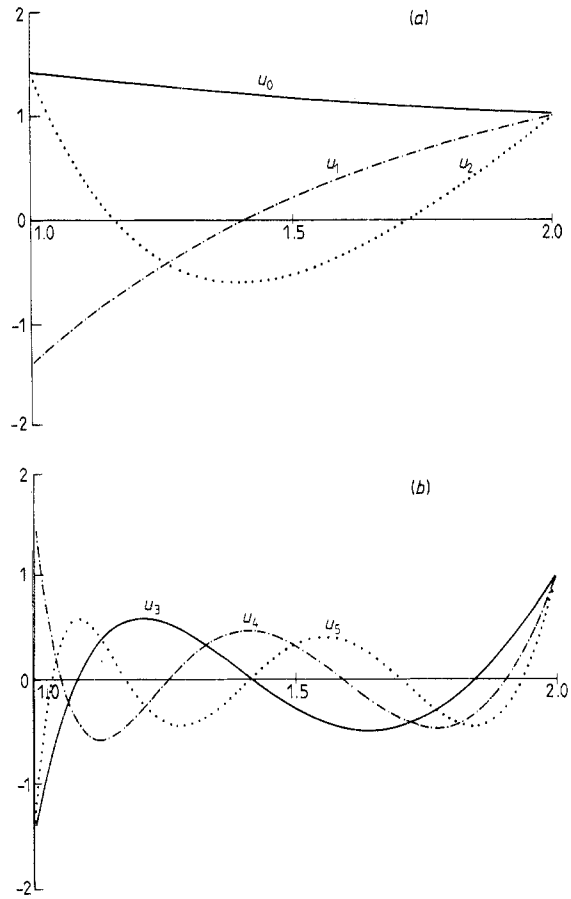

Figure 3. Eigenfunctions of the differential operator (4.1) in the case $\gamma=2$. (*a*) Eigenfunctions u_0, u_1, u_2; (*b*) eigenfunctions u_3, u_4, u_5. These eigenfunctions are normalised by the condition $u_k(\gamma)=1$.

approximation. The behaviour of the first five eigenvalues as a function of γ, for values of γ up to 5, is shown in figure 2.

We give a numerical example in the case $\gamma=2$:

$\mu_0=2.49509$	$\mu_1=19.4735$	$\mu_2=53.4292$
$\mu_3=104.363$	$\mu_4=172.274$	$\mu_5=257.164$
$\mu_6=359.031$	$\mu_7=477.877$	$\mu_8=613.700$
$\mu_9=766.501$	$\mu_{10}=936.280$	$\mu_{11}=1123.04$
$\mu_{12}=1326.72$	$\mu_{13}=1547.49$	$\mu_{14}=1785.18.$

In figure 3 we give the first six eigenfunctions of equation (4.1) and in figure 4 the first nine eigenfunctions of equation (4.5), both in the case $\gamma=2$. Notice that they look very similar to Legendre polynomials even if the value of γ is still rather far from 1.

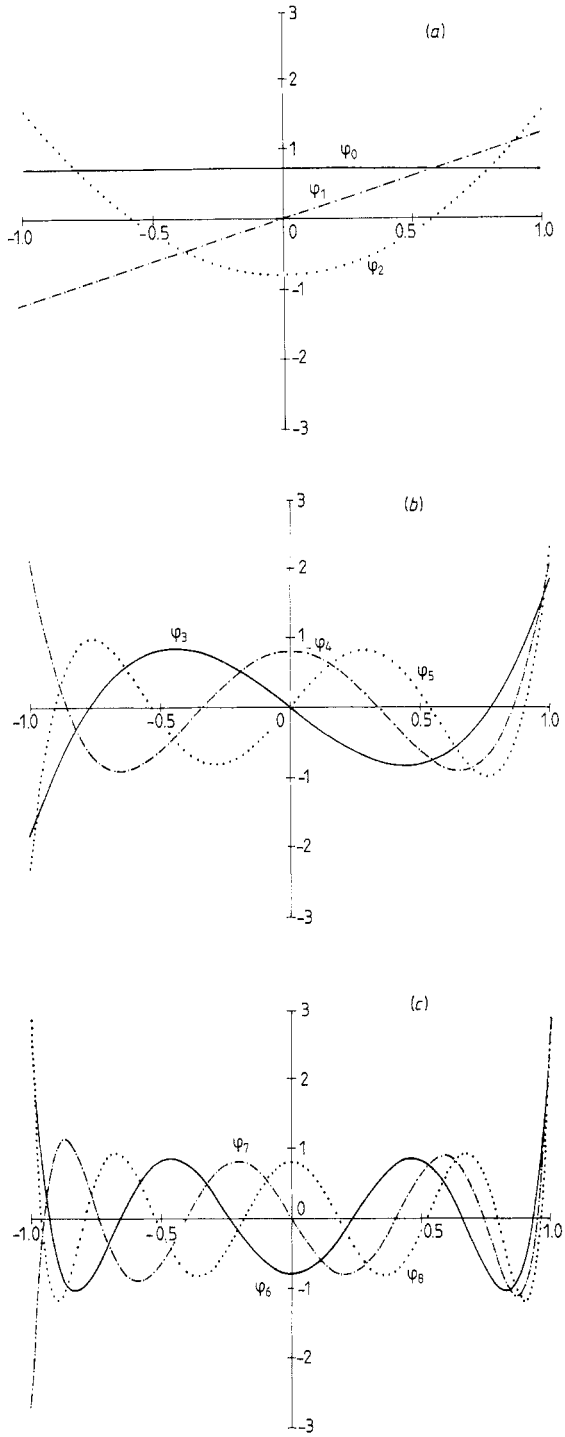

Figure 4. Eigenfunctions of the differential operator (4.5) in the case $\gamma=2$. (*a*) Eigenfunctions φ_0, φ_1, φ_2; (*b*) eigenfunctions φ_3, φ_4, φ_5; (*c*) eigenfunctions φ_6, φ_7, φ_8. The eigenfunctions are normalised with respect to the norm of $L^2(-1, 1)$.

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