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Commuting differential operators for the finite Laplace transform

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Abstract. We prove that the singular functions of the finite Laplace transformation are eigenfunctions of self-adjoint differential operators. Several applications of this result to the problem of Laplace transform inversion are indicated.

1. Introduction

If $f(t) \in L^2(0, \infty)$ we denote its Laplace transform $g(p)$ by

$$(\mathscr{L}f)(p) \equiv g(p) \equiv \int_0^\infty \mathrm{e}^{-pt} f(t)\, \mathrm{d}t. \tag{1.1}$$

One can show that $\mathscr{L}$ is a bounded linear map from $L^2(0, \infty)$ into $L^2(0, \infty)$. This map is invertible but the inverse is not continuous, making the numerical inversion of the Laplace transform into an ill conditioned problem. These results are well known and fully explained in [1]. The reader should refer to [1] for a more detailed exposition of some points which are recalled in this paper.

The numerical inversion of the Laplace transform, or more precisely of *noisy* measurements of $g(p)$ for a *finite* number of values of p, is of great importance in several branches of experimental science concerned with exponential relaxation rates: NMR in chemistry and more recently in medical imaging, photon correlation spectroscopy, measurements of fluorescent decay and sedimentation equilibrium are a few examples.

Consider the problem of recovering $f(t)$, supported in $[a, b] \subset [0, \infty]$, from the knowledge of $g(p)$ for $p_1 \leqslant p \leqslant p_2$, $[p_1, p_2] \subset [0, \infty]$. We have a typical problem of attempting the recovery of f with *limited data* which is compensated by *a priori knowledge* of the support of f.

This problem is similar in spirit to the one considered by Slepian, Landau and Pollak in [6–10]. In their case a function $f(t)$ supported in $[-T, T]$ is to be recovered from values of its Fourier transform $g(\lambda)$ for $\lambda \in [-\Omega, \Omega]$. At the core of this remarkable series of papers lies the observation that for any value of T, Ω the integral operator acting in $L^2(-T, T)$ with kernel

$$\frac{\sin \Omega(t-s)}{\pi(t-s)} \tag{1.2}$$

admits a commuting second-order differential operator (with simple spectrum), namely

$$\frac{d}{dt}(T^2-t^2)\frac{d}{dt}-\Omega^2t^2. \tag{1.3}$$

This mathematical 'miracle' makes possible the accurate computation of the eigenfunctions of the integral operator. Since these are the 'singular functions' for the recovery problem in question, the effective computation of *many* of these functions for *several* values of T, Ω forms the basis of a number of 'regularisation methods'.

What makes this recovery problem into such a classical one is exactly the possibility of calculating the singular functions in an economic and accurate fashion by replacing the integral operator—a 'full matrix'—by a differential one—a 'sparse' one. The main point of this paper is to consider to what extent such a 'miracle' holds when the Fourier transform is replaced by the Laplace transform and the intervals $[-T, T]$ and $[-\Omega, \Omega]$ are replaced by $[a, b]$ and $[p_1, p_2]$ respectively, as indicated above.

We ignore all along the effects of sampling, i.e. in practice we only have $g(p_i)$ for a finite number of values of i.

2. Singular values and singular functions of the Laplace transform

Consider the linear map from $L^2(a, b)$ to $L^2(p_1, p_2)$ given by

$$(\mathscr{L}f)(p)=\int_a^b e^{-pt}f(t)\,dt \qquad p_1\leqslant p\leqslant p_2. \tag{2.1}$$

This map is injective and compact if $a>0$, $b<\infty$ and/or $p_1>0$, $p_2<\infty$. Its adjoint is given by

$$(\mathscr{L}^*g)(t)=\int_{p_1}^{p_2} e^{-tp}\,g(p)\,dp \qquad a\leqslant t\leqslant b \tag{2.2}$$

and $\mathscr{L}^*$ is again compact and injective, and the range of $\mathscr{L}$ is dense in $L^2(p, p_2)$.

The operator $\mathscr{L}$ then admits a 'singular system' $\{\alpha_k, u_k, v_k\}$, $k=0, 1, \ldots$, defined by the relations

$$\mathscr{L}u_k=\alpha_k v_k \qquad \mathscr{L}^*v_k=\alpha_k u_k. \tag{2.3}$$

We have $\alpha_k>0$ for all k, and the vectors u_k and v_k form orthogonal basis in $L^2(a, b)$ and $L^2(p_1, p_2)$ respectively.

The expressions for both $\mathscr{L}\mathscr{L}^*$ and $\mathscr{L}^*\mathscr{L}$ are important since their eigenvectors are the vectors v_k and u_k respectively. We obtain

$$(\mathscr{L}^*\mathscr{L}f)(t)=\int_a^b \frac{\exp[-p_1(t+s)]-\exp[-p_2(t+s)]}{t+s}f(s)\,ds \tag{2.4}$$

and

$$(\mathscr{L}\mathscr{L}^*g)(p)=\int_{p_1}^{p_2} \frac{\exp[-a(p+q)]-\exp[-b(p+q)]}{p+q}g(q)\,dg. \tag{2.5}$$

We notice that in the special case

$$p_1=0 \qquad p_2=\infty \tag{2.6}$$

the operator $\mathscr{L}^*\mathscr{L}$ is given in [1] and reduces to

$$(\mathscr{L}^*\mathscr{L}f)(t)=\int_a^b \frac{f(s)}{t+s}\,\mathrm{d}s. \tag{2.7}$$

In this case we have

$$(\mathscr{L}\mathscr{L}^*g)(p)=\int_0^\infty \frac{\exp[-a(p+q)]-\exp[-b(p+q)]}{p+q}\,g(q)\,\mathrm{d}q. \tag{2.8}$$

We also notice that this case is equivalent to the case of limited data and unlimited support,

$$a=0 \qquad b=\infty, \tag{2.9}$$

since one case can be obtained from the other by just exchanging the operators $\mathscr{L}^*\mathscr{L}$ and $\mathscr{L}\mathscr{L}^*$.

3. The case $p_1=0$, $p_2=\infty$

As we will see below this case leads to nice commuting differential operators. In terms of the original recovery problem its main drawback is that it exploits *a priori* knowledge about the unknown function but requires knowledge of its Laplace transform for all values of p, $0<p<\infty$.

From the remark at the end of the last section it follows that our result also holds when we have a partial knowledge of the Laplace transform but no *a priori* knowledge about the support of $f(t)$.

3.1. The operator $\mathscr{L}^*\mathscr{L}$

The integral operator (2.7) can be manipulated by introducing new variables

$$t=a+(b-a)x \qquad s=a+(b-a)y \tag{3.1}$$

and the parameter $\beta=2a/(b-a)$, so that the eigenvalue problem

$$(\mathscr{L}^*\mathscr{L}u_k)(t)=\alpha_k^2 u_k(t) \tag{3.2}$$

becomes

$$\int_0^1 \frac{\psi_k(y)}{x+y+\beta}\,\mathrm{d}y=\alpha_k^2\psi_k(x) \tag{3.3}$$

with

$$u_k(t)=\psi_k\left(\frac{t-a}{b-a}\right). \tag{3.4}$$

The kernel in (3.3) makes it likely that many properties enjoyed by the Hilbert matrix

$$H_{i,j}^{(n)}=\frac{1}{i+j+\theta} \qquad i,j=1,\ldots,n \tag{3.5}$$

(θ constant) should also apply to the integral operator. One such property is the existence of a commuting tridiagonal matrix [3].

Indeed, by an appropriate passage to the limit of the results in [3] one obtains the differential operator D appearing in the following theorem.

Theorem. The differential operator

$$D_x = \frac{d}{dx}\left(x(1-x)(\beta+x)(\beta+1+x)\frac{d}{dx}\right) - 2x(x+\beta) \tag{3.6}$$

commutes with the integral operator

$$f(x) \to \int_0^1 \frac{1}{x+y+\beta} f(y)\, dy \tag{3.7}$$

in $L^2(0, 1)$.

Proof. Instead of proceeding to the limit indicated above one just checks the result by a generous use of integration by parts. The crucial ingredient in the proof is the fact that

$$(a) \qquad D_x \frac{1}{x+y+\beta} = D_y \frac{1}{x+y+\beta}$$

and

$$(b) \qquad D_x = \frac{d}{dx} a(x) \frac{d}{dx} + b(x) \qquad \text{with } a(0) = a(1) = 0.$$

Property (a) is used in integrating by parts and property (b) in removing some boundary terms. This condition makes D_x into a self-adjoint operator.

Note. A special case of this theorem, for $\beta = 0$, is given in the last paragraph of [3].

In terms of the original variables we find that in $L^2(a, b)$ the operators $\mathscr{L}^*\mathscr{L}$ (see equation (2.7)) and

$$\tilde{D}_t = \frac{d}{dt}(t^2 - a^2)(b^2 - t^2)\frac{d}{dt} - 2(t^2 - a^2) \tag{3.8}$$

commute with each other.

3.2. *The operator* $\mathscr{L}\mathscr{L}^*$

Turning our attention now to $\mathscr{L}\mathscr{L}^*$ given by (2.8), we start from the result

$$\mathscr{L}^*\mathscr{L}\tilde{D} = \tilde{D}\mathscr{L}^*\mathscr{L} \tag{3.9}$$

which we have just proved. We set

$$\hat{D} = \mathscr{L}\tilde{D}\mathscr{L}^{-1} \tag{3.10}$$

and multiply (3.9) by $\mathscr{L}$ on the left and by $\mathscr{L}^{-1}$ on the right to get

$$\mathscr{L}\mathscr{L}^*\hat{D} = \hat{D}\mathscr{L}\mathscr{L}^*, \tag{3.11}$$

i.e. it turns out that $\hat{D}$ commutes with $\mathscr{L}\mathscr{L}^*$.

One can check that $\mathscr{L}\tilde{D}\mathscr{L}^{-1}$ gives the fourth-order differential operator

$$\hat{D}_p = -\frac{d^2}{dp^2} p^2 \frac{d^2}{dp^2} + (a^2 + b^2)\frac{d}{dp} p^2 \frac{d}{dp} - a^2 b^2 p^2 + 2a^2. \tag{3.12}$$

It is obvious that the singular functions u_k are eigenfunctions of the differential operator $\tilde{D}_t$, while the singular functions v_k are eigenfunctions of $\hat{D}_p$.

4. The case $0 < p_1, p_2 < \infty$

The kernel of $\mathscr{L}^*\mathscr{L}$ is given by the function

$$K(t, s) = \frac{\exp[-p_1(t+s)] - \exp[-p_2(t+s)]}{t+s}. \tag{4.1}$$

If one looks for a differential operator of the form

$$\tilde{D}_t = \sum_{j=0}^{m} \frac{d^j}{dt^j} C_j(t) \frac{d^j}{dt^j} \tag{4.2}$$

that commutes with $\mathscr{L}^*\mathscr{L}$, one has to satisfy the conditions

(*a*) $\tilde{D}_t K(t, s) = \tilde{D}_s K(t, s)$

(*b*) $C_j^{(l)}(a) = C_j^{(l)}(b) = 0 \qquad 0 \leqslant l < j \leqslant m.$

Assuming that the $C_j(t)$ are polynomials a lengthy computation shows that condition (*a*) forces the choice (see appendix for details)

$$C_j(t) = \gamma_j t^2 + \delta_j \tag{4.3}$$

with γ_j, δ_j constants. The constants δ_j are arbitrary while the constants γ_j have to satisfy the relations

$$\sum_{i=0}^{m} \gamma_i p_2^{2i} = 0 \tag{4.4}$$

$$\sum_{i=1}^{m} \gamma_i \sum_{j=0}^{i-1} p_2^{2i-2-2j} p_1^{2j} = 0. \tag{4.5}$$

This means that $m-1$ of the constants $\gamma_0, \ldots, \gamma_m$ can be chosen freely. However, condition (*b*) is impossible to satisfy with quadratic functions $C_j(t)$.

Notice that condition (4.5) is symmetric in (p_1, p_2) and that the same is true of (4.4) when (4.5) holds. Indeed if one multiplies (4.5) by p_2^2 one gets

$$\sum_{i=1}^{m} \gamma_i \sum_{j=0}^{i-1} p_2^{2i-2j} p_1^{2j} = \sum_{i=1}^{m} \gamma_i p_2^{2i} + \sum_{i=1}^{m} \gamma_i \sum_{j=1}^{i-1} p_2^{2i-2j} p_1^{2j} = 0 \tag{4.6}$$

and if one multiplies (4.5) by p_1^2 one gets

$$\sum_{i=1}^{m} \gamma_i p_1^{2i} + \sum_{i=1}^{m} \gamma_i \sum_{j=1}^{i-1} p_2^{2i-2j} p_1^{2j} = 0. \tag{4.7}$$

We conclude that (4.6) and (4.7) imply

$$\sum_{i=1}^{m} \gamma_i p_2^{2i} = \sum_{i=1}^{m} \gamma_i p_1^{2i} \tag{4.8}$$

and thus (4.4) holds with p_2 replaced by p_1.

In the next section we see that allowing $C_j(t)$ to be analytic at the origin does not give us any new examples, at least if $\tilde{D}_t$ has order four.

5. Hankel operators commuting with self-adjoint differential operators of order up to four

Since the kernel in the integral operators involved above is of the form

$$K(t, s) = k(t+s), \tag{5.1}$$

that is $\mathscr{L}^*\mathscr{L}$ and $\mathscr{L}\mathscr{L}^*$ are Hankel operators, it may be of interest to determine those functions $k(\xi)$ leading to an integral operator which admit a commuting self-adjoint differential operator.

We do not pursue this point here but we observe that if one considers the fourth-order differential operator

$$\tilde{D}_t = \frac{d^2}{dt^2} a(t) \frac{d^2}{dt^2} + \frac{d}{dt} b(t) \frac{d}{dt} + c(t) \tag{5.2}$$

then condition (a) in § 4, i.e. $\tilde{D}_t k(t+s) = \tilde{D}_s k(t+s)$, is equivalent to the functional equation

$$(a(t)-a(s))k^{\mathrm{IV}}(t+s) + 2(a'(t)-a'(s))k'''(t+s) + (a''(t)-a''(s)+b(t)-b(s))k''(t+s)$$
$$+ (b'(t)-b'(s))k'(t+s) + (c(t)-c(s))k(t+s) = 0. \tag{5.3}$$

This should be compared with a similar relation (see (3) in [4]) that obtains in the case of convolution integral operators. The situation for convolution operators is very restrictive: Morrison [5] proved that the analyticity of $k(\xi)$ at $\xi = 0$ forces

$$k(\xi) = \frac{\sin \Omega_1 \xi / \Omega_1 \xi}{\sin \Omega_2 \xi / \Omega_2 \xi} \tag{5.4}$$

if one wants a commuting second-order differential operator. Allowing differential operators of higher order does not seem to change the picture: this is proved in [4] for orders up to ten.

Returning to (5.3) and introducing the variables $x = t+s$, $y = t-s$, it is easy to see that if we assume that the functions a, b, c, k are analytic at the origin, then (5.3) is equivalent to the infinite set of conditions

$$a^{(2n+1)}(x)k^{\mathrm{IV}}(x) + 4a^{(2n+2)}(x)k'''(x) + (4a^{(2n+3)}(x) + b^{(2n+1)}(x))k''(x)$$
$$+ 2b^{(2n+2)}(x)k'(x) + c^{(2n+1)}(x)k(x) = 0. \tag{5.5}$$

Here $n \geqslant 0$ and $a^{(n)}, b^{(n)}, c^{(n)}$ stand for $d^n a/dx^n$, etc.

One can attempt to determine a, b, c directly from (5.3) or from the infinite set of equations (5.5). The function $c(x)$ is clearly determined, up to a constant, from (5.3). The determination of b and a from (5.3) is a bit harder.

As an example of the use of the conditions (5.5) we treat the case

$$k(\xi) = \frac{\exp(-p_1\xi) - \exp(-p_2\xi)}{\xi}. \tag{5.6}$$

The case $n = 0$ of (5.5) gives all the coefficients c_i, $i \geqslant 1$, in

$$c(x) = \sum_{i=0}^{\infty} c_i x^i. \tag{5.7}$$

If one then uses the case $n=1$ of (5.5) with the values of c_i already determined, one finds all the coefficients b_i, $i \geqslant 3$, in

$$b(x)=\sum_{i=0}^{\infty} b_i x^i. \tag{5.8}$$

Using these values for the case $n=2$ in (5.5) one finds all the coefficients a_i, $i \geqslant 5$, in

$$a(x)=\sum_{i=0}^{\infty} a_i x^i. \tag{5.9}$$

At this point the free parameters are the six quantities

$$b_1, b_2, a_1, a_2, a_3, a_4. \tag{5.10}$$

Notice that from (5.3) it is clear that a_0, b_0, c_0 are arbitrary.

Finally using (5.5) for $n=3$ and looking at the first six coefficients of the Taylor expansion at the origin we obtain a system of linear equations in the variables (5.10) with solution given by

$$\begin{array}{lll} a_1=0 & a_3=0 & a_4=0 \\ b_1=0 & b_2=-(p_1^2+p_2^2)a_2. & \end{array} \tag{5.11}$$

With these values one then finds

$$\begin{array}{ll} a_i=0 & i \geqslant 5 \\ b_i=0 & i \geqslant 3 \\ c_1=0 & \\ c_2=p_1^2 p_2^2 a_2 & \\ c_i=0 & i \geqslant 3 \end{array} \tag{5.12}$$

and the differential operator $\tilde{D}_t$ is given by

$$\frac{d^2}{dt^2}(a_0+a_2 t^2)\frac{d^2}{dt^2}+\frac{d}{dt}[b_0-(p_1^2+p_2^2)a_2 t^2]\frac{d}{dt}+(c_0+p_1^2 p_2^2 a_2 t^2). \tag{5.13}$$

6. Relation with Toeplitz operators

Here we exploit the results in § 2.1 to get a new pair of commuting operators.

Recall that $\mathscr{L}^*\mathscr{L}$ was given by (2.7). If we make the change of variables

$$x=\ln(t/\sqrt{ab}) \tag{6.1}$$

and we introduce the quantity

$$\gamma=b/a \tag{6.2}$$

then the operator—acting now in $L^2(-\ln\sqrt{\gamma}, \ln\sqrt{\gamma})$—is given by

$$(\mathscr{L}^*\mathscr{L}h)(x)=\int_{-\ln\sqrt{\gamma}}^{\ln\sqrt{\gamma}} \frac{h(y)}{e^x+e^y}\, e^y\, dy. \tag{6.3}$$

Therefore if we conjugate $\mathscr{L}^*\mathscr{L}$ by the function $e^{-x/2}$, i.e. consider the operator

$$T=e^{x/2}\mathscr{L}^*\mathscr{L}\,e^{-x/2}, \tag{6.4}$$

we obtain

$$(Th)(x)=\tfrac{1}{2}\int_{-\ln\sqrt{\gamma}}^{\ln\sqrt{\gamma}}\frac{h(y)}{\cosh[\tfrac{1}{2}(x-y)]}\,dy. \tag{6.5}$$

Notice that T is a finite convolution integral operator.

On the other hand, using the change of variables given above and conjugating the operator $\tilde{D}_t$ from § 2.1 we obtain a second-order differential operator D_x, given by

$$D_x=\frac{d}{dx}(\gamma^2+1-2\gamma\cosh 2x)\frac{d}{dx}-(\tfrac{3}{2}\gamma\cosh 2x+\tfrac{1}{4}\gamma^2-\tfrac{7}{4}). \tag{6.6}$$

Therefore our result from § 2.1 appears as a special case of the result of Morrison if we put $\Omega_2=2\Omega_1=i$ in equation (5.4).

7. Some properties of the singular functions

We consider the case $p_1=0$, $p_2=\infty$ to exploit the fact that the eigenfunctions $u_k(t)$ of $\mathscr{L}^*\mathscr{L}$ are those of the differential operator $\tilde{D}_t$ (equation (3.8)). We have been unable to find this operator among the classical ones in the mathematical physics literature. Therefore in contrast to the case of the finite Fourier transform, where the eigenfunctions of the sinc kernel are recognised as the 'prolate spheroidal wavefunctions', the eigenfunctions u_k will have to go without a name.

As remarked in [1], the eigenvalues $\lambda_k=\alpha_k^2$ of $\mathscr{L}^*\mathscr{L}$ depend on a, b through their ratio $\gamma=b/a$. The same property holds for the eigenvalues $-\mu_k$ of the differential operator $\tilde{D}_t$. Therefore it is not restrictive to consider the case of functions supported in the interval $[1,\gamma]$ and in this case the functions u_k are simultaneous solutions of the eigenvalue problems

$$\int_1^{\gamma}\frac{u_k(s)}{t+s}\,ds=\lambda_k u_k(t) \tag{7.1}$$

$$-[(t^2-1)(\gamma^2-t^2)u_k'(t)]'+2(t^2-1)u_k(t)=\mu_k u_k(t). \tag{7.2}$$

If the eigenvalues μ_k are ordered to form an increasing sequence, then the usual properties of eigenfunctions of a second-order differential operator apply here: u_k, $k=0,1,2,\ldots$, has exactly k zeros in the interval $[1,\gamma]$.

We notice that the differential equation (7.2) has five ordinary singularities at the points $t=\pm 1$, $\pm\gamma$, ∞. In particular the indicial equations of the points $t=1$ and $t=\gamma$ have the root $\nu=0$ with multiplicity 2. That means that in the neighbourhood of both singularities there exists only one regular solution, while the others have a logarithmic singularity and therefore are unbounded. If we require that $u_k(t)$ is regular both at $t=1$ and at $t=\gamma$, then from equation (7.2) it follows that

$$u_k'(1)=-\frac{\mu_k}{2(\gamma^2-1)}u_k(1) \tag{7.3}$$

$$u_k'(\gamma)=\left(\frac{\mu_k}{2\gamma(\gamma^2-1)}-\frac{1}{\gamma}\right)u_k(\gamma). \tag{7.4}$$

From (7.3) and (7.4) we can derive the following results.

(*a*) No eigenfunction u_k can be zero at 1 or γ; it is identically zero otherwise.

(*b*) If u_k is normalised by $u_k(1)=1$, then $u_k'(1)$ is always negative and tends to $-\infty$ when $k\to\infty$. It follows that the first zero of $u_k(t)$ tends to 1 when $k\to\infty$.

The differential equation (7.2) is invariant with respect to the change of variables $t\to\gamma/t$, $u_k\to(\gamma/t)u_k(\gamma/t)$. Since the eigenvalues μ_k have multiplicity 1, it follows that the singular functions $u_k(t)$ must be alternately 'even' and 'odd' in the sense

$$u_k(\gamma/t)=(-1)^k(t/\sqrt{\gamma})u_k(t). \tag{7.5}$$

Since $u_0(t)$ has no zero, it must be 'even'. All the 'odd' eigenfunctions have a zero at $\sqrt{\gamma}$, the geometric mean of 1 and γ. It is easy to recognise that the eigenfunctions of the operators introduced in § 6 are even and odd in the usual sense.

Another interesting remark concerns the behaviour of the singular functions when $\gamma\to 1$. If we put

$$t=\tfrac{1}{2}(\gamma-1)x+\tfrac{1}{2}(\gamma+1) \qquad \varepsilon=\gamma-1 \tag{7.6}$$

then the interval $[1,\gamma]$ is transformed into the interval $[-1, 1]$ and the differential operator $\tilde{D}_t$ becomes

$$\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}x}(1-x^2)(4+3\varepsilon+\varepsilon x)(4+\varepsilon+\varepsilon x)\frac{\mathrm{d}}{\mathrm{d}x}-\tfrac{1}{2}\varepsilon(x+1)(4+\varepsilon+\varepsilon x) \tag{7.7}$$

and for $\varepsilon=0$ we get four times the Legendre operator

$$4\frac{\mathrm{d}}{\mathrm{d}x}(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}. \tag{7.8}$$

Therefore, for $\varepsilon\to 0$, $u_k(t)$ approaches the Legendre polynomial $cP_k[2(t-1)/\varepsilon-1]$, where c is a suitable constant.

Then, by means of arguments very similar to those used in [6], we can prove the following result which was conjectured in [1] and used to establish the improvement in resolution due to *a priori* knowledge of the support $f(t)$ in the Laplace transform inversion.

Theorem. For each $\gamma>1$, the eigenvalues of $\mathscr{L}^*\mathscr{L}$ have the properties

(i) any λ_k has multiplicity 1;

(ii) the λ_k form a decreasing sequence.

Proof. (i) Assume that two linearly independent eigenfunctions u_1 and u_2 are associated with the same eigenvalue λ of (7.1) and first consider the case where u_1 is 'even' and u_2 is 'odd'. Then, from (7.5) one gets $\gamma u_1(\gamma)u_2(\gamma)=u_1(1)u_2(1)$, so that

$$\int_1^\gamma (tu_1u_2'+tu_1'u_2+u_1u_2)\,\mathrm{d}t=2u_1(1)u_2(1). \tag{7.9}$$

On the other hand, by differentiating the integral equation (7.1) for u_1 and u_2, we obtain

$$\lambda\int_1^\gamma (tu_1u_2'+tu_1'u_2+u_1u_2)\,\mathrm{d}t$$

$$=-\int_1^\gamma \mathrm{d}t\int_1^\gamma \mathrm{d}s\left(\frac{tu_1(t)u_2(s)}{(t+s)^2}+\frac{su_2(s)u_1(t)}{(t+s)^2}-\frac{u_1(t)u_2(s)}{t+s}\right)=0 \tag{7.10}$$

and therefore $u_1(1)u_2(1)=0$, in contradiction with property (*a*) above.

Consider now the case where both u_1 and u_2 are 'even' or 'odd'. From (7.5) and from the relation obtained by differentiating (7.5), it follows that

$$\gamma^2(u_1(\gamma)u_2'(\gamma)-u_1'(\gamma)u_2(\gamma))=u_1(1)u_2'(1)-u_1'(1)u_2(1). \tag{7.11}$$

Furthermore, by differentiating twice the integral equation for u_1 and u_2, one obtains

$$\lambda\left(\frac{1}{\gamma^2}-1\right)(u_1(1)u_2'(1)-u_2(1)u_1'(1))=\lambda\int_1^\gamma (u_1u_2''-u_1''u_2)\,\mathrm{d}t$$
$$=2\int_1^\gamma \mathrm{d}t\int_1^\gamma \mathrm{d}s\left(\frac{u(t)v(s)}{(t+s)^3}-\frac{u(s)v(t)}{(t+s)^3}\right)=0. \tag{7.12}$$

Since $u_1(1)$ and $u_2(1)\neq 0$

$$\frac{u_1'(1)}{u_1(1)}=\frac{u_2'(1)}{u_2(1)} \tag{7.13}$$

and the 'boundary condition' (7.3) implies that u_1, u_2 belong to the same eigenvalue of the differential operator.

(ii) The eigenvalues λ_k are continuous functions of ε. Therefore, if $\lambda_k>\lambda_{k+1}$ for a given k and some ε, then the inequality must be satisfied for any ε, due to (i).

The result can be proved in the limit $\varepsilon\to 0$. We multiply the integral equation for u_k' by $\lambda_{k+1}u_{k+1}$, the integral equation for u_{k+1}' by $\lambda_k u_k$, subtract and integrate over $[1,\gamma]$:

$$\lambda_k\lambda_{k+1}\int_1^\gamma (u_k'u_{k+1}-u_ku_{k+1}')\,\mathrm{d}t$$
$$=(\lambda_k-\lambda_{k+1})\int_1^\gamma \mathrm{d}t\int_1^\gamma \mathrm{d}s\,\frac{u_k(t)u_{k+1}(s)}{(t+s)^2}$$
$$=-\lambda_{k+1}(\lambda_k-\lambda_{k+1})\int_1^\gamma u_ku_{k+1}'\,\mathrm{d}t \tag{7.14}$$

so that

$$\lambda_k-\lambda_{k+1}=\lambda_k\left(1-\frac{\int_1^\gamma u_k'u_{k+1}\,\mathrm{d}t}{\int_1^\gamma u_ku_{k+1}'\,\mathrm{d}t}\right). \tag{7.15}$$

Now, in the limit $\varepsilon\to 0$,

$$\frac{\int_1^\gamma u_k'u_{k+1}\,\mathrm{d}t}{\int_1^\gamma u_ku_{k+1}'\,\mathrm{d}t}\to\frac{\int_{-1}^1 P_k'P_{k+1}\,\mathrm{d}x}{\int_{-1}^1 P_kP_{k+1}'\,\mathrm{d}x}=\frac{0}{2} \tag{7.16}$$

and therefore, for ε small, $\lambda_k>\lambda_{k+1}$.

Finally we recall the possibility of using our result for the accurate computation of many singular functions. A few functions for some values of γ have already been computed using the integral equation (7.1) directly and the results have been presented in [2]. These computations can now be economically improved and extended and applied to various 'regularisation methods' for the finite Laplace transform inversion. Some preliminary computations have already been performed. Extended computations of eigenvalues and eigenfunctions, together with the discussion of some remarkable spectral properties of the differential operator (7.2), will be presented in a subsequent paper.

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Appendix

With a kernel

$$K(x, y)=\int_{p_1}^{p_2} \exp[-p(x+y)]\,\mathrm{d}p \tag{A.1}$$

as in (4.1), consider the problem of determining

$$D_x=\sum_{j=0}^{m} \frac{\mathrm{d}^j}{\mathrm{d}x^j}\, C_j(x) \frac{\mathrm{d}^j}{\mathrm{d}x^j} \tag{A.2}$$

such that

$$D_x K(x, y)=D_y K(x, y). \tag{A.3}$$

We have

$$D_x \exp[-p(x+y)]=\sum_{j=0}^{m} \sum_{k=0}^{j} \binom{j}{k}\left(\frac{\mathrm{d}^{j-k}}{\mathrm{d}x^{j-k}}\, C_j(x)\right)(-p)^{j+k} \exp[-p(x+y)]. \tag{A.4}$$

Therefore, recalling that

$$\int p^n\, \mathrm{e}^{ap}\, \mathrm{d}p=(ap)^n-n(ap)^{n-1}+n(n-1)(ap)^{n-2}-n(n-1)(n-2)(ap)^{n-3}+\ldots, \tag{A.5}$$

we obtain

$$D_x K(x+y)=A(p_2, x, y) \exp[-p_2(x+y)]-A(p_1, x, y) \exp[-p_1(x+y)]. \tag{A.6}$$

Notice that the choice of the functions C_j which, together with complicated contributions from their derivatives, make up the function $A(p, x, y)$ could depend on p. In particular we cannot assume that $A(p, x, y)$ is a polynomial in p and, since p_1 and p_2 are fixed, it is better to put

$$D_x K(x+y)=F(x, y) \exp[-p_2(x+y)]-G(x, y) \exp[-p_1(x, y)]. \tag{A.7}$$

If this function is symmetric in x, y we conclude that

$$\exp(-p_2 \xi) R(\xi, \eta)-\exp(-p_1 \xi) Q(\xi, \eta)=0 \tag{A.8}$$

where $\xi=x+y$, $\eta=x-y$ and R and Q are the functions $F(x, y)-F(y, x)$ and $G(x, y)-G(y, x)$ in terms of ξ, η.

We find

$$\exp[(p_1-p_2)\xi]=Q(\xi, \eta)/R(\xi, \eta) \tag{A.9}$$

which under minimal assumptions gives $R \equiv Q \equiv 0$, i.e. both $F(x, y)$ and $G(x, y)$ symmetric.

Therefore, in determining $C_j(x)$ such that (A.3) is satisfied, we should express $D_x K(x+y)$ as in (A.7) and then impose the symmetry of both $F(x, y)$ and $G(x, y)$.

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