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Tomography with a finite set of projections: singular value decomposition and resolution

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Abstract. The problem of tomography with a finite set of projections has been the object of many investigations. In particular the formulae for the generalized solution and the singular values have been obtained in the case of equispaced projections. These results imply that the problem of determining the generalized solution is well-posed even if it may be ill-conditioned. In this paper we derive several properties of the singular values and singular functions both in the general case and in the case of equispaced projections. We use these results to identify the singular functions related to the aliasing effects in the generalized solution and to estimate the resolution achievable when these effects have been eliminated by means of a suitable filtering. It turns out that the resolution essentially depends on the number of projections and not on the noise, if the number of projections is smaller than a certain upper limit (depending on the noise), which can be quite large. In the case of equispaced projections, the resolution coincides with that estimated by the asymptotic theory even when the number of projections is rather small.

1. Introduction

The singular value decomposition of the Radon transform has been established by Davison [1] in the general case of functions of n variables, by assuming that the functions and the projections belong to suitable weighted L^2 spaces. In the particular case of functions of two variables, with support in the disc of radius one, $\mathcal{D} \subset \mathbb{R}^2$, the Radon transform is defined by

$$(Rf)(s, \boldsymbol{\theta}) = \int_{-w(s)}^{w(s)} f(s\boldsymbol{\theta} + t\boldsymbol{\theta}^{\perp})\,\mathrm{d}t \tag{1}$$

where $w(s) = (1 - s^2)^{1/2}, \boldsymbol{\theta} = \{\cos\phi, \sin\phi\} \in S^1, \boldsymbol{\theta}^{\perp} = \{-\sin\phi, \cos\phi\}$. Then the results of Davison imply that R is a compact operator from $L^2(\mathcal{D})$ into $L^2(Z; w^{-1}), Z = [-1, 1] \times S^1$. Its singular system is given by $(m = 0, 1, 2, \ldots)$

$$\sigma_{m,k} = \left(\frac{4\pi}{m+1}\right)^{\frac{1}{2}} \qquad k = 0, 1, \ldots, m \tag{2}$$

$$u_{m,k}(s, \boldsymbol{\theta}) = \sqrt{\frac{2}{\pi}}\, w(s) U_m(s) Y_{m-2k}(\boldsymbol{\theta}) \tag{3}$$

$$v_{m,k}(\boldsymbol{x}) = \sqrt{2m+2}\, Q_{m,|m-2k|}(|\boldsymbol{x}|) Y_{m-2k}\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right) \tag{4}$$

where $U_m(s)$ are the Chebyshev polynomials of the second kind

$$U_m(s) = \frac{\sin[(m+1)\arccos s]}{\sin(\arccos s)} \tag{5}$$

the radial functions $Q_{m,l}(r)$ are related to Jacobi polynomials $P_n^{\alpha,\beta}(t)$ by

$$Q_{m,l}(r) = r^l P^{0,l}_{\frac{1}{2}(m-l)}(2r^2-1) \tag{6}$$

and the angular functions $Y_l(\boldsymbol{\theta})$ are given by

$$Y_l(\boldsymbol{\theta}) = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\mathrm{i}l\phi}. \tag{7}$$

The case of a finite set of projections is a semi-discrete version of the Radon transform because only discretization of the angular variable ϕ is performed. This problem is investigated by Davison and Grünbaum [2] who give an explicit expression of the singular values in the case of equispaced directions. These singular values have a positive minimum so that the operator is not compact and its generalized inverse is continuous. Moreover, it turns out that the condition number is proportional to the square root of the number p of projections and therefore the problem of determining the generalized solution is not only well-posed but also well-conditioned if p is not too large. This property may be important in the case of a small number of projections and for this reason we have reconsidered the problem by focusing on those properties of the singular system which can be used for estimating the resolution achievable in practice.

Resolution in tomography is usually described in terms of the properties of the functions in the null space of the operator, i.e. of the functions whose projections in the given directions are zero. As first shown by Logan [3] in the case of p equispaced projections, the Fourier transform of a function in the null space is negligible inside a disc with a certain radius Ω_p. As a consequence, a function whose Fourier transform is concentrated within this disc can be reliably reconstructed from its p projections. When p is large one gets $\Omega_p \simeq p$ and this is the main result of the so-called asymptotic theory of resolution. Extensions of this result are given in [4, 5].

Since this theory does not take into account the ill-conditioning of the problem, it provides an estimate of the resolution which is noise independent. This is certainly not true when p is sufficiently large because the problem of the inversion of the Radon transform is ill-posed. It is obvious that more precise results could be deduced from the singular system of the problem with a finite set of projections.

By investigating the singular system we can show that, in the case of p equispaced projections, the result of the asymptotic theory is correct even for small values of p and remains correct for all the values of p smaller than a certain upper bound P which depends on the noise. For $p > P$ resolution is independent of p. However, even when the problem is well-conditioned, a suitable filtering of the singular function expansion of the generalized solution is required, for suppressing the aliasing effects caused by insufficient sampling of the Fourier transform of the unknown object at large frequencies [6].

The plan of the paper is as follows. In section 2 we consider the case of arbitrary directions, we derive a useful representation of the Fourier transform of the singular functions in object space and we show that the singular values accumulate to $(4\pi/p)^{1/2}$, a limit which depends on the number p of projections and does not depend on the directions of the projections. On the other hand, there exists a minimum singular value which can be much smaller than this asymptotic limit. This is true, in particular, if the distribution of the directions corresponds to a situation which can be described as a limited angle tomography, because it is known that this problem is extremely ill-conditioned [7, 8].

In section 3 we rediscuss the result of Davison and Grünbaum [2] for the case of equispaced directions. In particular, we observe that the relevant parameter for describing the behaviour of the singular system is the index m characterizing the subspaces identified by the Chebyshev polynomials. For $m = 0, 1, \dots, p-1$ the singular values and the singular functions in object space coincide with those of the complete Radon transform R, equation (1), and therefore are given by equation (2) and equation (4) respectively. For $m = p, p+1, \dots, 2p-2$ one still finds $p(p-1)/2$ singular functions in object space coinciding with those of R while the others already show a behaviour which is typical of all the remaining singular functions and can be described as follows: the Fourier transform of one of these functions is given by the product of the Fourier transform of a singular function of R and of an angular function which has maxima in correspondence to the p directions $\boldsymbol{\theta}_j$. These singular functions describe the aliasing effects mentioned above and are associated, in general, with singular values which are larger than the minimum one. It is clear that these singular functions can cause artifacts both in the generalized solution and in the regularized solutions, when regularization is needed.

Finally in section 4 we obtain a representation of the projection operator onto the orthogonal complement of the null space. This representation is given in terms of a space-variant transfer function. In this way the aliasing effects due to the finite set of projections are clearly identified.

The suppression of these effects requires a suitable filtering in frequency domain. From our analysis of the frequency content of the singular functions it follows that this filtering can be obtained by truncating the singular function expansion to values of m smaller than p, while it cannot be obtained by means of Tikhonov regularization or conjugate gradient or similar methods. The reason is that there exists an infinite set of singular values greater than some of those with $m < p$. A similar filtering is provided by the ART method with a small value of the relaxation parameter ω [5].

Finally, by using truncated singular function expansions and by computing the corresponding transfer functions, we show in a particular case ($p = 6$) that the asymptotic theory applies also to the case of a small number of equispaced projections.

2. Arbitrary directions

We consider a set of $2p$ directions, $\boldsymbol{\theta}_j = \{\cos\phi_j, \sin\phi_j\}$ ($j = 0, 1, \dots, 2p-1$) with $\phi_0 = 0, 0 < \phi_1 < \phi_2 < \cdots < \phi_{p-1} < \pi$ and $\boldsymbol{\theta}_{p+j} = -\boldsymbol{\theta}_j$. Let $\Theta_p \subset S^1$ be the set of these directions. The corresponding projections are redundant but this redundancy can be convenient for simplifying the analysis.

If f is a function with support in $\mathcal{D}$, we denote by $R_p f$ its Radon transform when restricted to the directions of Θ_p

$$(R_p f)(s, \theta_j) = \int_{-w(s)}^{w(s)} f(s\boldsymbol{\theta}_j + t\boldsymbol{\theta}_j^{\perp})\, \mathrm{d}t \qquad \boldsymbol{\theta}_j \in \Theta_p. \tag{8}$$

We put $\mathcal{X} = L^2(\mathcal{D})$ and $\mathcal{Y} = \{L^2([-1, 1], w^{-1})\}^{2p}$. More precisely the norm of an element $g \in \mathcal{Y}$, consisting of $2p$ projections, $\{g(s, \boldsymbol{\theta}_0), \dots, g(s, \boldsymbol{\theta}_{2p-1})\}$ is defined by

$$\|g\|_{\mathcal{Y}}^2 = \frac{\pi}{p} \sum_{j=0}^{2p-1} \int_{-1}^{1} |g(s, \boldsymbol{\theta}_j)|^2 \frac{\mathrm{d}s}{w(s)}. \tag{9}$$

The operator $R_p : \mathcal{X} \longrightarrow \mathcal{Y}$ is bounded and the functions in its range have the symmetry property

$$(R_p f)(s, \boldsymbol{\theta}_j) = (R_p f)(-s, \boldsymbol{\theta}_{p+j}). \tag{10}$$

Its adjoint operator R_p^* is given by

$$(R_p^* g)(\boldsymbol{x}) = \frac{\pi}{p} \sum_{j=0}^{2p-1} g(\boldsymbol{\theta}_j \cdot \boldsymbol{x}, \boldsymbol{\theta}_j) w^{-1}(\boldsymbol{\theta}_j \cdot \boldsymbol{x}). \tag{11}$$

Thanks to the completeness in $L^2([-1, 1], w)$ of the Chebyshev polynomials of the second kind, $U_m(s)$, which satisfy the following orthogonality and normalization conditions,

$$\int_{-1}^{1} w(s) U_m(s) U_{m'}(s) \, \mathrm{d}s = \frac{\pi}{2} \delta_{m,m'} \tag{12}$$

the space $\mathcal{Y}$ can be decomposed as the direct sum of the subspaces

$$\mathcal{Y}_m = \left\{ g \in \mathcal{Y} | g(s, \boldsymbol{\theta}_j) = \frac{\sqrt{2p}}{\pi} w(s) U_m(s) h(\boldsymbol{\theta}_j), \boldsymbol{\theta}_j \in \Omega_p \right\}. \tag{13}$$

Each subspace $\mathcal{Y}_m$ has dimension $2p$ because the numbers $h(\boldsymbol{\theta}_j)$ define a vector $\boldsymbol{h}$ of length $2p$. We use the notation $h(\boldsymbol{\theta}_j)$ for its components, in order to emphasize the correspondence between these components and the directions of the projections. We also observe that, if $g \in \mathcal{Y}_m$, then its norm coincides with the Euclidean norm of $\boldsymbol{h}$

$$\|g\|_{\mathcal{Y}}^2 = \|\boldsymbol{h}\|_2^2. \tag{14}$$

It is well known that each subspace $\mathcal{Y}_m$ reduces $R_p R_p^*$ and that the restriction of $R_p R_p^*$ to $\mathcal{Y}_m$ is given by a $2p \times 2p$ symmetric and positive semidefinite matrix $\mathbf{A}_m$. In fact, by means of straightforward computations one gets, for any given $g \in \mathcal{Y}_m$,

$$(R_p R_p^* g)(s, \boldsymbol{\theta}_j) = \frac{\sqrt{2p}}{\pi} w(s) U_m(s) (\mathbf{A}_m \boldsymbol{h})(\boldsymbol{\theta}_j) \tag{15}$$

where

$$(\mathbf{A}_m)_{j,j'} = \frac{2\pi}{p(m+1)} \frac{\sin[(m+1)(\phi_j - \phi_{j'})]}{\sin(\phi_j - \phi_{j'})}. \tag{16}$$

Moreover, it is easy to show that the following decomposition holds true

$$\mathbf{A}_m = \mathbf{B}_m \mathbf{B}_m^* \tag{17}$$

where $\mathbf{B}_m$ is the $2p \times (m+1)$ matrix given by

$$(\mathbf{B}_m)_{j,l} = \sqrt{\frac{4\pi}{m+1}} \left(\frac{\pi}{p} \right)^{\frac{1}{2}} Y_{m-2l}(\boldsymbol{\theta}_j) \qquad j = 0, 1, \ldots, 2p-1; l = 0, 1, \ldots, m. \tag{18}$$

From this decomposition it follows that the rank of $\mathbf{A}_m$ is given by

$$r(\mathbf{A}_m) = \min(m+1, p). \tag{19}$$

Let us denote by $\lambda_{m,k}$, $k = 0, 1, \ldots, r(\mathbf{A}_m) - 1$, the positive eigenvalues of $\mathbf{A}_m$, with $\lambda_{m,k} \geqslant \lambda_{m,k+1}$, and by $\boldsymbol{u}_{m,k}$ the corresponding eigenvectors (normalized in the Euclidean norm). Then the singular system of R_p is as follows:

- the singular values $\sigma_{m,k}^{(p)}$ are given by

$$\sigma_{m,k}^{(p)} = \sqrt{\lambda_{m,k}} \tag{20}$$

in particular, $\sigma_{0,0} = \sqrt{4\pi}$, as follows directly from equation (16);

- the singular functions in $\mathcal{Y}$ are given by

$$u^{(p)}_{m,k}(s,\boldsymbol{\theta}_j) = \frac{\sqrt{2p}}{\pi} w(s) U_m(s) \boldsymbol{u}_{m,k}(\boldsymbol{\theta}_j) \tag{21}$$

- the singular functions in $\mathcal{X}$ are given by

$$v^{(p)}_{m,k}(\boldsymbol{x}) = \frac{1}{\sigma^{(p)}_{m,k}} (R^*_p u^{(p)}_{m,k})(\boldsymbol{x}). \tag{22}$$

We prove now a few properties of the singular values and singular functions of R_p. As concerns the singular values we need the following lemma.

Lemma 2.1. The eigenvalues $\lambda_{m,k}$ of the matrix $\mathbf{A}_m$ satisfy the inequality

$$\left|\lambda_{m,k} - \frac{4\pi}{p}\right| \leqslant \frac{4\pi}{(m+1)\sin\Delta} \tag{23}$$

where Δ is the minimum angular distance between the directions $\boldsymbol{\theta}_j$

$$\Delta = \min_{j,j'} |\phi_j - \phi_{j'}|. \tag{24}$$

Proof. If we consider the $p \times p$ submatrix of $\mathbf{A}_m$ which is obtained by restricting the indices j, j' to the values $0, 1, \ldots, p-1$, its eigenvalues are the eigenvalues of the matrix $\mathbf{A}_m$ divided by two. Now, from equation (16), it follows that the diagonal elements of this submatrix are equal to $2\pi/p$ while the off-diagonal elements are bounded by

$$|(\mathbf{A}_m)_{j,j'}| \leqslant \frac{2\pi}{p(m+1)\sin\Delta} \qquad j, j' = 0, 1, \ldots, p-1. \tag{25}$$

Therefore the matrix $\mathbf{A}'_m$ defined by

$$(\mathbf{A}'_m)_{j,j'} = (\mathbf{A}_m)_{j,j'} - \frac{2\pi}{p}\delta_{j,j'} \qquad j, j' = 0, \ldots, p-1 \tag{26}$$

has diagonal elements equal to zero and off-diagonal elements satisfying the inequality (25). It follows that its norm is bounded by

$$\|\mathbf{A}_m\| \leqslant \left(\sum_{j,j'=0}^{p-1} |(\mathbf{A}'_m)_{j,j'}|^2\right)^{\frac{1}{2}} < \frac{2\pi}{(m+1)\sin\Delta}. \tag{27}$$

Since the eigenvalues of $\mathbf{A}'_m$ are given by $(\lambda_{m,k} - 4\pi/p)/2$ and since the eigenvalues are bounded by $\|\mathbf{A}'_m\|$, inequality (23) is proved. □

Theorem 2.1. The singular values $\sigma^{(p)}_{m,k}$ of R_p have a non-zero limit when $m \to \infty$ and precisely

$$\lim_{m\longrightarrow\infty} \sigma^{(p)}_{m,k} = \sqrt{\frac{4\pi}{p}}. \tag{28}$$

The singular functions $v^{(p)}_{m,k}$ are linear combinations of the singular functions $v_{m,l}$ of the complete Radon transform, equation (4), and are given by

$$v^{(p)}_{m,k}(\boldsymbol{x}) = \sum_{l=0}^{m} (\boldsymbol{v}_{m,k})_l v_{m,l}(\boldsymbol{x}) \tag{29}$$

the vectors $\boldsymbol{v}_{m,k}$ being related to the eigenvectors $\boldsymbol{u}_{m,k}$ of $\mathbf{A}_m$ by

$$\boldsymbol{v}_{m,k} = \frac{1}{\sigma^{(p)}_{m,k}} \mathbf{B}^*_m \boldsymbol{u}_{m,k} \qquad \|\boldsymbol{v}_{m,k}\|_2 = 1. \tag{30}$$

Moreover, their Fourier transforms are given by

$$\begin{aligned} \hat{v}^{(p)}_{m,k}(\boldsymbol{\xi}) &= 2\pi \left(\frac{2}{p}\right)^{1/2} \frac{\mathrm{i}^{-m}}{\sigma^{(p)}_{m,k}} |\xi|^{-1} J_{m+1}(|\boldsymbol{\xi}|) \Phi_{m,k}(\boldsymbol{\theta}) \\ \Phi_{m,k}(\boldsymbol{\theta}) &= \sum_{j=0}^{2p-1} (\boldsymbol{u}_{m,k})_j \frac{\sin[(m+1)(\phi - \phi_j)]}{\sin(\phi - \phi_j)} \end{aligned} \tag{31}$$

where $\boldsymbol{\theta} = \boldsymbol{\xi}/|\boldsymbol{\xi}| = \{\cos\phi, \sin\phi\}$ and J_{m+1} is the Bessel function of order $m+1$.

Proof. The limiting property of the singular values is a direct consequence of the lemma and of equation (20).

As concerns equation (29), first remark that the singular functions of the complete Radon transform R, equations (2)–(4), satisfy the relation

$$(R_p v_{m,k})(s, \boldsymbol{\theta}_j) = \left(\frac{4\pi}{m+1}\right)^{1/2} \left(\frac{2}{\pi}\right)^{1/2} w(s) U_m(s) Y_{m-2k}(\boldsymbol{\theta}_j) \tag{32}$$

which is obtained by restricting to Θ_p the equation $R v_{m,k} = \sigma_{m,k} u_{m,k}$. Then, from the definition (22) of the singular functions $v^{(p)}_{m,k}$ it follows that

$$(v^{(p)}_{m,k}, v_{m',l})_{\mathcal{X}} = \frac{1}{\sigma^{(p)}_{m,k}} (u^{(p)}_{m,k}, R_p v_{m',l})_{\mathcal{Y}}. \tag{33}$$

Since both $u^{(p)}_{m,k}$ and $R_p v_{m',l}$ contain a Chebyshev polynomial, from the orthogonality of these polynomials it follows that the scalar products (33) are zero except where $m = m'$. Then the completeness of the singular functions $v_{m,k}$ in $\mathcal{X}$ implies that $v^{(p)}_{m,k}$ is necessarily a linear combination of the $v_{m,l}$ having the same value of m. Since the $v_{m,l}$ are also normalized, the coefficients are given by the scalar products (33) with $m' = m$. From (32), by taking into account (12), we obtain

$$(v^{(p)}_{m,k}, v_{m,l})_{\mathcal{X}} = \frac{1}{\sigma^{(p)}_{m,k}} \left(\frac{4\pi}{m+1}\right)^{1/2} \left(\frac{\pi}{p}\right)^{1/2} \sum_{j=0}^{2p-1} \overline{Y_{m-2l}(\boldsymbol{\theta}_j)} (\boldsymbol{u}_{m,k})(\boldsymbol{\theta}_j) = \frac{1}{\sigma^{(p)}_{m,k}} (\mathbf{B}^*_m \boldsymbol{u}_{m,k})_l. \tag{34}$$

In the last step the definition (18) of the matrix $\mathbf{B}_m$ has been used. Equation (34) implies (29) and (30). The normalization of $\boldsymbol{v}_{m,k}$ comes out from the normalization of the eigenvectors $\boldsymbol{u}_{m,k}$.

Now, from the Fourier slice theorem and the formula for the Fourier transform of the Chebyshev polynomials (see, for instance, [5, p 185]), it follows that the Fourier transform of the singular functions $v_{m,l}$ of R is given by

$$\hat{v}_{m,l}(\boldsymbol{\xi}) = 2\pi (2m+2)^{1/2}\, \mathrm{i}^{-m} |\xi|^{-1} J_{m+1}(|\boldsymbol{\xi}|) Y_{m-2l}\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right) \tag{35}$$

so that from equation (29) one obtains

$$\hat{v}^{(p)}_{m,k}(\boldsymbol{\xi}) = 2\pi (2m+2)^{1/2}\, \mathrm{i}^{-m} |\xi|^{-1} J_{m+1}(|\boldsymbol{\xi}|) \sum_{l=0}^{m} Y_{m-2l}(\boldsymbol{\theta}) (\boldsymbol{v}_{m,k})_l \tag{36}$$

where $\boldsymbol{\theta} = \boldsymbol{\xi}/|\boldsymbol{\xi}| = \{\cos\phi, \sin\phi\}$. Finally, from equations (30) and (18), by summing with respect to l, one obtains (32). □

The results proved in the previous theorem have several implications. First of all, the fact that the singular values have a non-zero limit implies that the range of the operator R_p is closed. Therefore, the generalized inverse $R_p^\dagger$ is continuous.

The limit of the singular values does not depend on the distribution of the directions while it is well known that the well-conditioning or ill-conditioning of the problem strongly depends on this distribution. The explication resides in the fact that the limit value is not a lower bound for the singular values. For any distribution of directions there exists a minimum singular value which is smaller (and, in some cases, much smaller) than the limit value.

The existence of singular values smaller than $(4\pi/p)^{1/2}$ can be deduced from the observation that all the matrices $\mathbf{A}_m$ have exactly the same trace, i.e. 4π. Since for $m \geqslant p-1$, $\mathbf{A}_m$ has p eigenvalues different from zero and since its trace is just p times the limit value $4\pi/p$, it follows that these matrices must have eigenvalues both larger and smaller than $4\pi/p$. In the case of equispaced directions, the minimum eigenvalue can be computed (see [2] and also the next section) and is given by $[4\pi/(2p-1)]^{1/2}$.

As concerns the singular functions in object space, from the expression of their Fourier transform it follows that, when m is large, their Fourier transform is concentrated outside a disc, with centre at the origin and radius growing with m, and around the directions $\boldsymbol{\theta} = \boldsymbol{\theta}_j$, with an angular width of the order of $\pi/m+1$. According to the Fourier slice theorem, these are precisely the directions of the straight lines where data are available in the Fourier domain. Therefore, the contribution of the singular functions $v_{m,k}^{(p)}$, with large m, to the reconstruction of f is clear: they contribute to extrapolate the Fourier transform of f in an angular neighbourhood of the lines where its values are given by the data of the problem.

3. Equispaced directions

In the case of equispaced directions the angles ϕ_j are given by $\phi_j = \pi j/p$ $(j = 0, 1, \ldots, 2p-1)$. Then the matrices $\mathbf{A}_m$ are cyclic

$$(\mathbf{A}_m)_{j,j'} = \frac{2\pi}{p(m+1)} \frac{\sin[(\pi/p)(m+1)(j-j')]}{\sin[(\pi/p)(j-j')]}. \tag{37}$$

The eigenvalues of these matrices have a rather simple expression which has been established by Davison and Grünbaum [2]. A more convenient expression can be obtained by considering groups of p values of the index m. For the first group $(m = 0, 1, \ldots, p-1)$ each matrix $\mathbf{A}_m$ has only one non-zero eigenvalue with multiplicity $m+1$ and precisely

$$\lambda_{m,k} = \frac{4\pi}{m+1} \qquad k = 0, 1, \ldots, m. \tag{38}$$

For the other groups, characterized by

$$m = np + r \tag{39}$$

with $n = 1, 2, \ldots$ and $r = 0, 1, \ldots, p-1$, each matrix $\mathbf{A}_m$ has two non-zero eigenvalues, with multiplicity r and $p-r$, respectively, given by

$$\lambda_{m,k} = \frac{4\pi}{m+1} n_{m,k} \tag{40}$$

where

$$n_{m,k} = \begin{cases} n+1 & k = 0, 1, \ldots, r \\ n & k = r+1, \ldots, p-1. \end{cases} \tag{41}$$

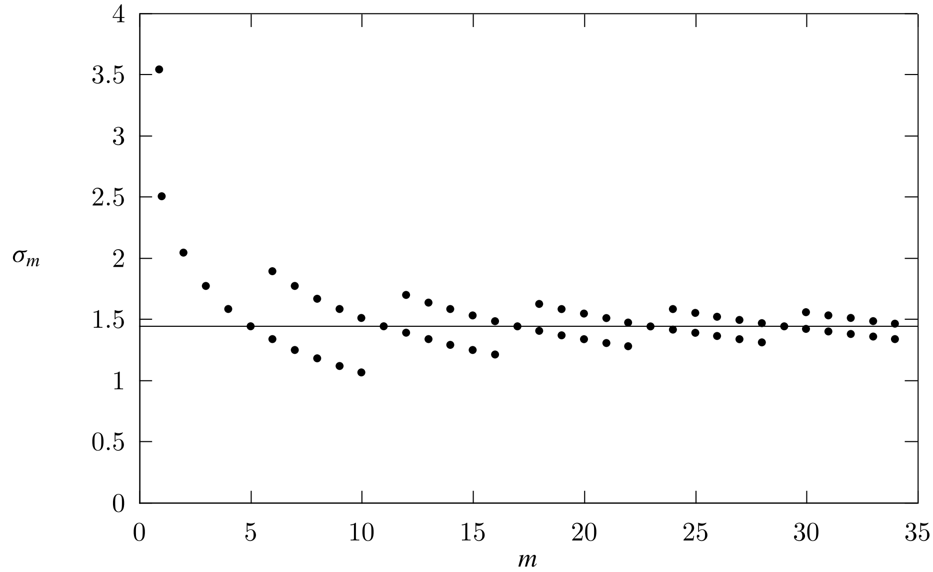

Figure 1. Plot of the singular values in the case $p = 6$. Only distinct singular values are considered for each value of m. The horizontal line corresponds to the asymptotic limit.

In figure 1 we give the plot of the singular values in the case $p = 6$.

The eigenvectors of A_m associated with the non-zero eigenvalues are given by

$$(\boldsymbol{u}_{m,k})(\boldsymbol{\theta}_j) = \frac{1}{\sqrt{2p}}\,\mathrm{e}^{-\mathrm{i}(\pi/p)(m-2k)j} = \sqrt{\frac{\pi}{p}}\,Y_{m-2k}(\boldsymbol{\theta}_j) \tag{42}$$

with k taking the values specified above.

As concerns the singular functions in object space, for $m = 0, 1, \ldots, p-1$, one derives from equation (30) that $(\boldsymbol{v}_{m,k})_k = 1$ while the other components are zero, so that from equation (29) one gets

$$v_{m,k}^{(p)}(\boldsymbol{x}) = v_{m,k}(\boldsymbol{x}). \tag{43}$$

For the other groups of values of m, from equation (30) one now derives

$$(\boldsymbol{v}_{m,k})_l = \begin{cases} (n_{m,k})^{-1/2} & l = k\,(\mathrm{mod}\,p) \\ 0 & l \neq k\,(\mathrm{mod}\,p) \end{cases} \tag{44}$$

for $l = 0, 1, \ldots, m$. The non-zero components of the vector correspond to $l = k, k + p, \ldots, k + (n_{m,k} - 1)p$ and therefore the number of these components is just $n_{m,k}$. From equation (29) one gets

$$v_{m,k}^{(p)}(\boldsymbol{x}) = \frac{1}{(n_{m,k})^{1/2}} \sum_{l=k\,(\mathrm{mod}\,p)} v_{m,l}(\boldsymbol{x}). \tag{45}$$

Finally, from equation (36), by computing the sum with respect to l one obtains

$$\hat{v}_{m,k}^{(p)}(\boldsymbol{\xi}) = \hat{v}_{m,k}(\boldsymbol{\xi})\,\mathrm{e}^{\mathrm{i}p(n_{m,k}-1)\phi}\,\frac{\sin(pn_{m,k}\phi)}{(n_{m,k})^{1/2}\sin(p\phi)} \tag{46}$$

where ϕ is the angle defined by $\boldsymbol{\xi}/|\boldsymbol{\xi}| = \{\cos\phi, \sin\phi\}$.

A few remarks about these results. For the first group of values of m, i.e. $m = 0, 1, \ldots, p-1$, the singular values of R_p coincide with the singular values of the complete Radon transform R and have the same multiplicity. Moreover, also the singular functions in object space are the same. The total number of these singular values is $p(p+1)/2$. The minimum singular value in this group corresponds to $m = p-1$ and therefore it is given by $(4\pi/p)^{1/2}$, which is just the asymptotic limit of the singular values, theorem 3.1.

In the second group of values of m, i.e. $m = p, p+1, \ldots, 2p-1$, we still find singular values and singular functions coinciding with those of the complete Radon transform R, even if the multiplicity is different. This result follows from equations (40) and (46), if we observe that $n_{m,k} = 1$ for $k = (m-p)+1, \ldots, p-1$ and $m = p, \ldots, 2p-2$. The total number of these eigenvalues is $p(p-1)/2$. The remaining $p(p+1)/2$ singular values are greater than the corresponding singular values of R by a factor $\sqrt{2}$. The Fourier transforms of their singular functions in object space are characterized by an angular modulation of the Fourier transforms of the corresponding singular functions of R, as follows from equation (46) with $n_{m,k} = 2$. Therefore, these singular functions already describe aliasing effects due to the finite number of projections [6].

The minimum singular value of R_p is reached in this second group; it corresponds to $m = 2p-2$ and $k = p-1 (n_{m,k} = 1)$ and therefore is given by

$$\sigma_{\min} = \left(\frac{4\pi}{2p-1}\right)^{1/2}. \tag{47}$$

It is smaller than the asymptotic limit by a factor of the order $\sqrt{2}$.

In the subsequent groups of values of m all the singular values of R_p are greater than the corresponding singular values of R. From equation (40) it follows that the asymptotic limit $(4\pi/p)^{1/2}$ is also a singular value of R_p with infinite multiplicity. This property clearly appears in figure 1.

As concerns the Fourier transforms of the singular functions in object space, they show the angular modulation described by equation (46) and therefore they are associated with the already mentioned aliasing effects. In order to avoid these effects one must keep only the singular values corresponding to values of m from 0 to $p-1$.

Now we shortly discuss the ill-conditioned nature of the problem and the use of regularization methods. Since the largest singular value of R_p is $\sigma_{\max} = \sigma_{0,0} = (4\pi)^{1/2}$, the condition number of R_p is given by

$$\mathrm{cond}(R_p) = \frac{\sigma_{\max}}{\sigma_{\min}} = (2p-1)^{1/2}. \tag{48}$$

The problem of determining the generalized solution is not only well-posed but also well-conditioned when the number of projections is small. For instance, $\mathrm{cond}(R_p) = 3.32$ for $p = 6$ and $\mathrm{cond}(R_p) = 18.9$ for $p = 180$, a moderate ill-conditioning. The situation is even better if we consider only the singular values with $m \leqslant p-1$, i.e. a truncated version of R_p, which we can denote by $R_p^{(t)}$. In such a case the minimum singular value coincides with the asymptotic limit and therefore

$$\mathrm{cond}(R_p^{(t)}) = \sqrt{p}. \tag{49}$$

In such a case we have $\mathrm{cond}(R_p^{(t)}) = 2.45$ for $p = 6$ and $\mathrm{cond}(R_p^{(t)}) = 13.4$ for $p = 180$.

In order to see when regularization is needed, we can consider the method based on truncated singular function expansions, with truncation controlled by noise. According to Miller [9] one can proceed as follows. Let us assume to have an estimate ϵ of the norm, in $L^2([-1, 1], w^{-1})$, of the noise affecting each projection, so that the estimate of the $\mathcal{Y}$-norm of the noise is $\sqrt{2\pi}\epsilon$. Moreover, let us assume to have an estimate E of the norm of f. Then, an approximate solution compatible with this *a priori* information is provided by any $f \in \mathcal{X}$ such that

$$\|R_p f - g\|_{\mathcal{Y}} \leqslant \sqrt{2\pi}\epsilon \qquad \|f\|_{\mathcal{X}} \leqslant E \tag{50}$$

so that one must look for a truncated singular function expansion satisfying these constraints. This is obtained by keeping all the singular values satisfying the inequality

$$\sigma_{m,k}^{(p)} \geqslant \sqrt{2\pi}\frac{\epsilon}{E} \tag{51}$$

i.e. all the singular values greater than the ratio between the norm of the noise and the norm of the object to be restored.

Now, if σ_{min} satisfies this inequality, then all the singular values are acceptable and therefore no regularization is needed. From equation (47) it follows that this situation occurs if

$$p \leqslant \left(\frac{E}{\epsilon}\right)^2 + 1. \tag{52}$$

If p does not satisfy this condition, then truncation is needed. However, if we proceed in such a way, we keep singular functions related to the aliasing effects already mentioned, as one clearly understand by looking at figure 1 and remembering the discussion above. A way already indicated for suppressing these effects is to keep only singular values with $m \leqslant p - 1$. Then, if all these singular values satisfy condition (51) a further truncation is not needed. This situation occurs when the smallest singular value, which coincides with the asymptotic limit, satisfies condition (51) and this is true if

$$p \leqslant 2\left(\frac{E}{\epsilon}\right)^2. \tag{53}$$

If this condition is not satisfied then a further truncation, controlled by condition (51), is required. Since the singular values in the first group depend only on m, from equation (38) one derives the following condition on m:

$$m + 1 \leqslant 2\left(\frac{E}{\epsilon}\right)^2. \tag{54}$$

From the properties of the Fourier transform of the singular functions in object space it follows that, if one keeps all the singular values up to a certain value m, then one can correctly restore the Fourier transform of the unknown function in a disc of radius $\Omega = m$. From the previous analysis we conclude that, for a given number p of projections, this radius is given by

$$\begin{aligned} \Omega_p &= p && \text{if } p \leqslant 2\left(\frac{E}{\epsilon}\right)^2 \\ \Omega_p &= 2\left(\frac{E}{\epsilon}\right)^2 && \text{if } p > \left(\frac{E}{\epsilon}\right)^2. \end{aligned}$$

This point will be further considered in the next section.

4. Resolution limits

As we already remarked, the previous results imply that, for any finite set of directions $\boldsymbol{\theta}_j$, the generalized inverse $R_p^\dagger$ is continuous so that the problem of determining the generalized solution $f^\dagger$ is well-posed. An explicit formula of $f^\dagger$, in the case of equispaced projections, is given by Logan and Shepp [10]. In general, $f^\dagger$ can be represented by a singular function expansion

$$f^\dagger = \sum_{m,k}^{\infty} \frac{1}{\sigma_{m,k}^{(p)}} (g, u_{m,k}^{(p)})_{\mathcal{Y}} v_{m,k}^{(p)} \tag{55}$$

where $g \in \mathcal{Y}$ is the given set of projections and summation is extended to $m = 1, 2, \ldots$ and $k = 0, 1, \ldots, r(\mathbf{A}_m) - 1$.

If g is in the range of the operator R_p, then there exist functions $f \in \mathcal{X}$ such that $g = R_p f$. By inserting this relation in equation (55), one finds the following relationship between $f^\dagger$ and f

$$f^\dagger = \sum_{m,k}^{\infty} (f, v_{m,k}^{(p)})_{\mathcal{X}} v_{m,k}^{(p)} \tag{56}$$

showing that $f^\dagger$ is the orthogonal projection of f onto the orthogonal complement of the null space of R_p. We have the following representation for this projection.

Theorem 4.1. For any $f \in \mathcal{X}$, its component $f^\dagger$ orthogonal to the null space of R_p is given by

$$f^\dagger(\boldsymbol{x}) = \frac{p}{4\pi} \left(R_p^* R_p f\right)(\boldsymbol{x}) + \int_{\mathcal{D}} H_p(\boldsymbol{x}, \boldsymbol{x}') f(\boldsymbol{x}')\, \mathrm{d}\boldsymbol{x}' \tag{57}$$

where $H_p(\boldsymbol{x}, \boldsymbol{x}')$ is a square integrable kernel, $H_p \in L^2(\mathcal{D} \times \mathcal{D})$.

Proof. Let us write equation (56) as follows,

$$f^\dagger = \frac{p}{4\pi} \sum_{m,k}^{\infty} \lambda_{m,k} (f, v_{m,k}^{(p)})_{\mathcal{X}} v_{m,k}^{(p)} + \frac{p}{4\pi} \sum_{m,k}^{\infty} \left(\frac{4\pi}{p} - \lambda_{m,k} \right) (f, v_{m,k}^{(p)})_{\mathcal{X}} v_{m,k}^{(p)} \tag{58}$$

where the $\lambda_{m,k}$ are the eigenvalues of the matrices $\mathbf{A}_m$. Since the $\lambda_{m,k}$ are just the eigenvalues of $R_p^* R_p$ and the $v_{m,k}^{(p)}$ are the eigenfunctions associated with these eigenvalues, the first term on the right-hand side of equation (58) is the spectral representation of $R_p^* R_p$ and therefore it coincides with the first term on the right-hand side of equation (57). Moreover, from lemma 2.1 it follows that the series

$$H_p(\boldsymbol{x}, \boldsymbol{x}') = \frac{p}{4\pi} \sum_{m,k}^{\infty} \left(\frac{4\pi}{p} - \lambda_{m,k} \right) v_{m,k}^{(p)}(\boldsymbol{x}) \overline{v}_{m,k}^{(p)}(\boldsymbol{x}') \tag{59}$$

is convergent in $L^2(\mathcal{D} \times \mathcal{D})$ so that it defines a square integrable function $H_p(\boldsymbol{x}, \boldsymbol{x}')$. This is just the second term in equation (57). □

In the following we consider f and $f^\dagger$ as functions which are defined everywhere in $\mathbb{R}^2$, being zero outside the disc $\mathcal{D}$. Analogously we consider the kernel H_p as defined in $\mathbb{R}^2 \times \mathbb{R}^2$, being zero outside $\mathcal{D} \times \mathcal{D}$. Then we can write equation (57) in the following form

$$f^\dagger(\boldsymbol{x}) = \int_{\mathbb{R}^2} K_p(\boldsymbol{x}, \boldsymbol{x}') f(\boldsymbol{x}')\, \mathrm{d}\boldsymbol{x}' \tag{60}$$

where $K_p(\boldsymbol{x}, \boldsymbol{x}')$ is a distribution given by ($\chi_{\mathcal{D}}$ = characteristic function of the disc)

$$K_p(\boldsymbol{x}, \boldsymbol{x}') = \frac{1}{2}\chi_{\mathcal{D}}(\boldsymbol{x})\chi_{\mathcal{D}}(\boldsymbol{x}')\sum_{j=0}^{p-1} w^{-1}(\boldsymbol{\theta}_j \cdot \boldsymbol{x})\delta[\boldsymbol{\theta}_j \cdot (\boldsymbol{x} - \boldsymbol{x}')] + H_p(\boldsymbol{x}, \boldsymbol{x}'). \tag{61}$$

The distribution $K_p(\boldsymbol{x}, \boldsymbol{x}')$ is the point spread function (PSF) of the tomographic process. It contains a singular part which, for fixed $\boldsymbol{x}'$, consists of delta distributions concentrated on the straight lines passing through the point $\boldsymbol{x}'$ and orthogonal to the directions $\boldsymbol{\theta}_j$. Superimposed onto this star-shaped PSF there is also a smooth one, given by $H_p(\boldsymbol{x}, \boldsymbol{x}')$.

The singular part of $K_p(\boldsymbol{x}, \boldsymbol{x}')$ describes the aliasing effects already mentioned, as can be shown by introducing a space-variant transfer function defined by

$$T_p(\boldsymbol{x}, \boldsymbol{\xi}) = \mathrm{e}^{-\mathrm{i}\boldsymbol{x}\cdot\boldsymbol{\xi}}\overline{\overline{\hat{K}}_p(\boldsymbol{x}, \boldsymbol{\xi})} \tag{62}$$

where

$$\overline{\hat{K}}_p(\boldsymbol{x}, \boldsymbol{\xi}) = \int_{\mathbb{R}^2} \overline{K_p(\boldsymbol{x}, \boldsymbol{x}')}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{x}'\cdot\boldsymbol{\xi}}\,\mathrm{d}\boldsymbol{x}'. \tag{63}$$

This definition is motivated by the following formal relationship between $f^\dagger$, f and T_p, which follows from Parseval equality if $K_p(\boldsymbol{x}, \boldsymbol{x}')$ is a square integrable function

$$f^\dagger(\boldsymbol{x}) = \frac{1}{(2\pi)^2}\int T_p(\boldsymbol{x}, \boldsymbol{\xi})\widehat{f(\boldsymbol{\xi})}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{x}\cdot\boldsymbol{\xi}}\,\mathrm{d}\boldsymbol{\xi}. \tag{64}$$

This relationship shows that $T_p(\boldsymbol{x}, \boldsymbol{\xi})$ acts as an $\boldsymbol{x}$-dependent filter on the Fourier transform of f.

From equations (61) and (59) one gets

$$T_p(\boldsymbol{x}, \boldsymbol{\xi}) = \sum_{p=0}^{p-1} \mathrm{sinc}[w(\boldsymbol{\theta}_j \cdot \boldsymbol{x})(\boldsymbol{\theta}_j \cdot \boldsymbol{\xi})]\,\mathrm{e}^{-\mathrm{i}(\boldsymbol{\theta}_j^\perp\cdot\boldsymbol{x})(\boldsymbol{\theta}_j^\perp\cdot\boldsymbol{\xi})} + \frac{p}{4\pi}\sum_{m,k}^{\infty}\left(\frac{4\pi}{p} - \lambda_{m,k}\right) v_{m,k}^{(p)}(\boldsymbol{x})\overline{\hat{v}}_{m,k}^{(p)}(\boldsymbol{\xi}) \tag{65}$$

the sinc-function being defined by $\mathrm{sinc}(t) = t^{-1}\sin(t)$.

We do not give a complete discussion of the contribution of the delta distributions, i.e. the first term in the expression of $T_p(\boldsymbol{x}, \boldsymbol{\xi})$. We only observe that, in the case $\boldsymbol{x} = 0$, it is the sum of functions $\eta_j(\boldsymbol{\xi}) = \mathrm{sinc}(\boldsymbol{\theta}_j^\perp \cdot \boldsymbol{\xi})$, one for each direction. If we consider the straight line L_j passing through the origin with direction $\boldsymbol{\theta}_j$, the function $\eta_j(\boldsymbol{\xi})$ is constant over the straight lines parallel to L_j while it behaves as a sinc-function on the straight lines orthogonal to L_j, the first zero occurring at distance π from L_j (notice that π is just the sampling distance of the Fourier transforms of the projections, since they have support in $[-1, 1]$). In conclusion, for each direction $\boldsymbol{\theta}_j$ there exists a strip in frequency domain, centered on L_j and having a width of the order of π, such that $T_p(\mathbf{0}, \boldsymbol{\xi})$ is close to 1 on this strip, at sufficiently high frequencies. Since at high frequencies these strips are disjoint, this is a description of the incomplete information on $\hat{f}$ at these frequencies. A similar analysis can be performed for $\boldsymbol{x} \neq 0$.

In the previous section we conjectured that the aliasing effects can be suppressed by considering truncated singular function expansions, the truncation being based on a suitable choice of the maximum value of the index m. Let us denote by M this maximum value and by $f^{(M)}$ the corresponding truncated solution. In the case $g = R_p f$, the relationship between $f^{(M)}$ and f is given by

$$f^{(M)} = \sum_{m,k}^{M} (f, v_{m,k}^{(p)})_{\mathcal{X}} v_{m,k}^{(p)} \tag{66}$$

the summation being extended to $m = 0, 1, \ldots, M$ and $k = 0, 1, \ldots, r(\mathbf{A}_m) - 1$. The corresponding PSF, which is now a well behaved function, is given by

$$K_p^{(M)}(\boldsymbol{x}, \boldsymbol{x}') = \sum_{m,k}^{M} v_{m,k}^{(p)}(\boldsymbol{x}) \overline{v}_{m,k}^{(p)}(\boldsymbol{x}') \tag{67}$$

and this is just the kernel of the orthogonal projection onto the subspace spanned by the singular functions $v_{m,k}^{(p)}$ with $m \leqslant M$. Analogously the transfer function is obtained from

$$T_p^{(M)}(\boldsymbol{x}, \boldsymbol{\xi}) = \mathrm{e}^{-\mathrm{i}\boldsymbol{x}\cdot\boldsymbol{\xi}} \sum_{m,k}^{M} v_{m,k}^{(p)}(\boldsymbol{x}) \overline{\hat{v}}_{m,k}^{(p)}(\boldsymbol{\xi}) \tag{68}$$

and the relationship between $f^{(M)}$, $\hat{f}$ and $T_p^{(M)}$ is the same as the relationship between $f^\dagger$, $\hat{f}$ and T_p, equation (64).

From this relationship it follows that, if $T_p^{(M)}(\boldsymbol{x}, \boldsymbol{\xi})$ is approximately equal to 1 for any $\boldsymbol{x} \in \mathcal{D}$ and any $\boldsymbol{\xi}$ in a domain $\mathcal{B}$ of the frequency plane, then $f^{(M)}(\boldsymbol{x})$ is a reliable approximation of any function f whose Fourier transform is significantly different from zero only inside $\mathcal{B}$. Otherwise, a suitable filtering of $f^{(M)}$ can provide a $\mathcal{B}$-bandlimited approximation of f. Given a set Θ_p of directions, the existence of such a set $\mathcal{B}$ can be established by computing numerically the transfer function $T_p^{(M)}(\boldsymbol{x}, \boldsymbol{\xi})$. In the case of equispaced directions one can obtain asymptotic results for large p. If p is small, numerical computations are also required.

The analysis of section 3 shows that, in the equispaced case, it is convenient to consider groups of p values of the index M, so that we take $M = p - 1, 2p - 1, 3p - 1, \ldots$. The most interesting are the first two groups because they contain singular functions in the object space $\mathcal{X}$ which coincide with singular functions of the complete Radon transform.

The expression of $T_p^{(M)}$ is rather simple in the case $\boldsymbol{x} = 0$. From properties of the Jacobi polynomials (see [11], equations 22.4.1 and 22.2.1), one obtains for the singular functions of R, equation (4)

$$v_{m,k}(\mathbf{0}) = \begin{cases} (-1)^k \left(\dfrac{2k+1}{\pi}\right)^{1/2} & m \text{ even, } k = m/2 \\ 0 & \text{otherwise.} \end{cases} \tag{69}$$

It follows that, for $M = p - 1$ and $M = 2p - 1$, only the singular functions which coincide with those of R contribute to $T_p(\mathbf{0}, \boldsymbol{\xi})$. The result is

$$T_p^{(M)}(\mathbf{0}, \boldsymbol{\xi}) = 2 \sum_{k=0}^{K} (2k+1) |\boldsymbol{\xi}|^{-1} J_{2k+1}(|\boldsymbol{\xi}|) \tag{70}$$

where, for $M = p - 1$, we have $K = p/2 - 1$ if p is even and $K = (p-1)/2$ if p is odd while, for $M = 2p - 1$, we have $K = p - 1$.

Thanks to the completeness of the singular functions of R, the limit of $T_p(\mathbf{0}, \boldsymbol{\xi})$ is 1, for $p \to \infty$, so that we can write equation (70) as follows

$$T_p^{(M)}(\mathbf{0}, \boldsymbol{\xi}) = 1 - 2 \sum_{k=K+1}^{\infty} (2k+1) |\boldsymbol{\xi}|^{-1} J_{2k+1}(|\boldsymbol{\xi}|). \tag{71}$$

Then, from the asymptotic estimate of the Bessel functions for large values of the index [5] one can derive that the second term on the right-hand side of equation (71) is negligible for $|\boldsymbol{\xi}| < p$ if $M = p - 1$ and for $|\boldsymbol{\xi}| < 2p$ if $M = 2p - 1$. The first result is just that provided by the asymptotic theory.

The previous estimates provide a picture of the situation for $\boldsymbol{x} = 0$ and $M = p-1, 2p-1$. For other values of $\boldsymbol{x}$ and other values of M one must compute $T_p^{(M)}(\boldsymbol{x}, \boldsymbol{\xi})$ as given by equation (68). We have performed these computations in the case $p = 6$ and the results are shown in figure 2.

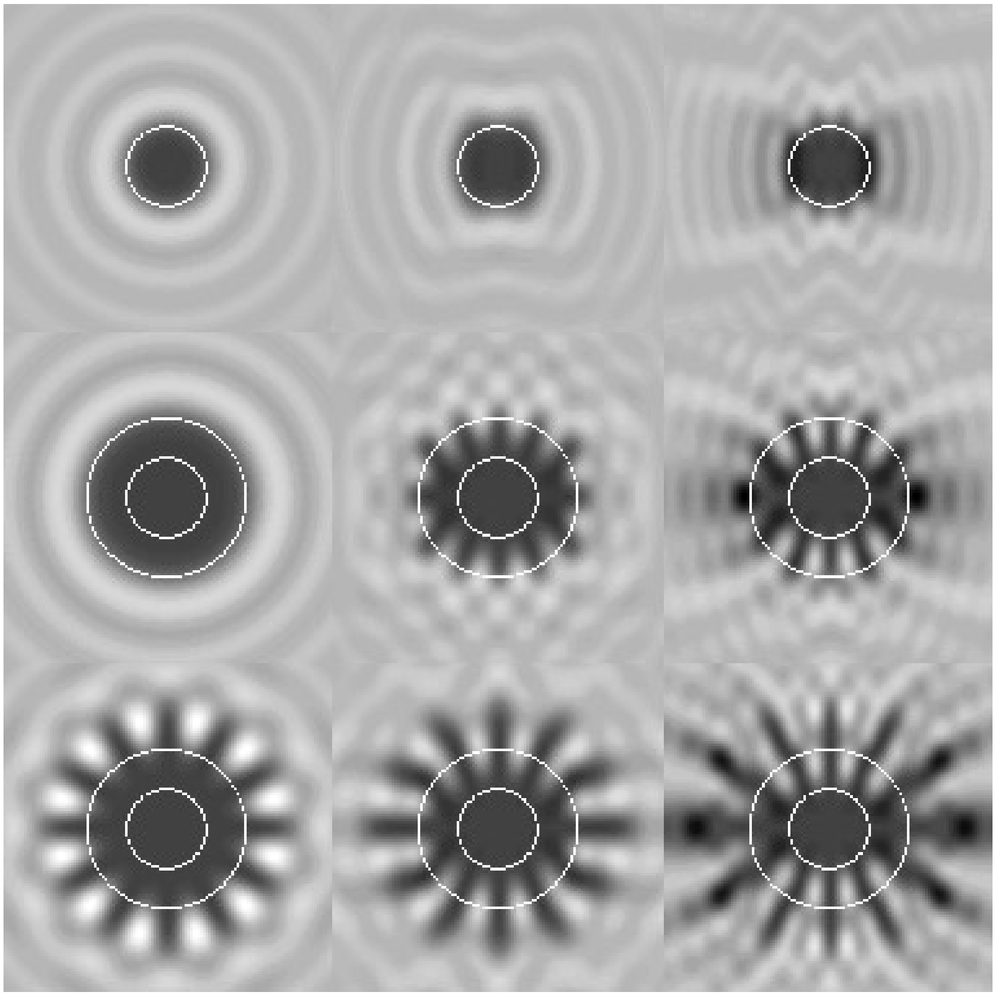

Figure 2. Pictures of the real part of the transfer function $T_p^{(M)}(\boldsymbol{x}, \boldsymbol{\xi})$, defined in equation (68), in the case $p = 6$. The pictures of the first row correspond to $M = p-1 = 5$ and to points $\boldsymbol{x}$ at a distance from the origin 0, 0.4 and 0.8. The black level corresponds to the value 1 of the transfer function. The white circle, with radius $p = 6$ indicates the band of the asymptotic theory. The pictures of the second and third row correspond to $M = 2p-1 = 11$ and $M = 3p-1 = 17$, respectively, the transfer functions being computed at the same points of the first row. The largest white circle corresponds to a band which is twice the asymptotic band.

The pictures of this figure are grey-level representations of the real part of $T_p^{(M)}(\boldsymbol{x}, \boldsymbol{\xi})$, where the black level corresponds to the value 1. The white circles have radius 6 and 12, respectively. The three rows correspond to different values of M and precisely $M = 5$, 11 and 17, while the three columns correspond to different values of $\boldsymbol{x}$ and precisely to points at distances 0, 0.4 and 0.8 from the origin in the direction $\boldsymbol{\theta}_0 = \{1, 0\}$.

The pictures of the first row show that $T_p^{(M)}(\boldsymbol{x},\boldsymbol{\xi})$, with $M = p-1$, is essentially equal to 1 inside the disc of radius p at any point $\boldsymbol{x}$. Since p is rather small in this example ($p = 6$) this result shows that the asymptotic theory applies also to this case and that the resolution is rather uniform over the disc. A filtering in Fourier domain can improve the uniformity.

The pictures of the second row correspond to $M = 2p - 1$. We find that for $\boldsymbol{x} = \mathbf{0}$ the transfer function is equal to 1 over a disc of radius $2p$ as shown before. However, as the point $\boldsymbol{x}$ goes away from the origin, the radius of the disc where the transfer function is 1 decreases and is close to p for points $\boldsymbol{x}$ at the boundary of $\mathcal{D}$. One could describe this situation by saying that if we use all the singular functions with $m \leqslant 2p - 1$ we have an $\boldsymbol{x}$-dependent resolution, which is twice that given by asymptotic theory at the centre of $\mathcal{D}$ and approximately equal to the asymptotic one at the boundary of $\mathcal{D}$.

Finally, in the third row, we give pictures of the transfer function in the case $M = 3p-1$. In this case the structure of the transfer function responsible for the aliasing effects is evident at any point $\boldsymbol{x}$.

The conclusion which can be derived from this computation is that the results of asymptotic theory are reliable also when the number of projections is small.

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